

On special values of zeta functions
associated with a self-dual cone

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以下に提げるのは松島与三氏還暦記念論文集 (Birkhauser) のための原稿の一部である。京都の研究集会ではこの後半についてお話ししたので、その要約を提出する予定であったが、都合上原稿 (の原稿) のまゝ出させて頂くことにした。本文で説明した通り、こゝに述べる手法は本質的に故新谷氏 [11] のアイディアによるものである。 $r=2$ (circular cone) の場合にはより精密な計算をすることができ、栗原氏も独立に結果を得ておられるが、これについてはまた別の機会に觸れたいと思う。

§ 1. Introduction

To explain the main idea of this paper, and also to fix some notations, we start with reviewing the classical case of Riemann zeta function. As usual, we set

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (\operatorname{Re} s > 1),$$

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx \quad (\operatorname{Re} s > 0).$$

Then, for $\operatorname{Re} s > 1$, one obtains

$$\begin{aligned} \Gamma(s) \zeta(s) &= \sum_{n=1}^{\infty} n^{-s} \int_0^{\infty} x^{s-1} e^{-x} dx \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} x^{s-1} e^{-nx} dx' \quad (x = nx') \\ &= \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx. \end{aligned}$$

We put

$$b(x, y) = \frac{e^{xy}}{e^x - 1} = \sum_{v=0}^{\infty} \frac{B_v(y)}{v!} x^{v-1} \quad (|x| < 2\pi),$$

where

$$B_v(y) = \sum_{\mu=0}^v \binom{v}{\mu} b_{\mu} y^{v-\mu}$$

is the Bernoulli polynomial, in which the b_{μ} are the Bernoulli numbers:

$$\begin{aligned} b_0 &= 1, \quad b_1 = -\frac{1}{2}, \\ b_v &= \begin{cases} (-1)^{\frac{v}{2}-1} B_{\frac{v}{2}} & (v \text{ even}, \geq 2), \\ 0 & (v \text{ odd}, \geq 3). \end{cases} \end{aligned}$$

Then the above integral can be transformed into a contour integral of the form

$$(1.1) \quad \Gamma(s) \zeta(s) = (e^{2\pi i s} - 1)^{-1} \int_{I(\varepsilon, \infty)} x^{s-1} b(x, 0) dx,$$

where $I(\varepsilon, \infty)$ denotes the contour consisting of the half-line $[\varepsilon, \infty)$ taken twice in opposite directions and of a (small) circle of radius ε

about the origin taken in the counterclockwise direction. The contour integral is absolutely convergent for all $s \in \mathbb{C}$, so that the function $\Gamma(s) \zeta(s)$ can be analytically continued to a meromorphic function on $\mathbb{C}$. Moreover, in virtue of the functional equation of the gamma function:

$$(1.2) \quad \Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin \pi s} = 2\pi i \frac{e^{\pi i s}}{e^{2\pi i s} - 1},$$

one obtains

$$(1.3) \quad \zeta(s) = e^{-\pi i s} \Gamma(1-s) \cdot \frac{1}{2\pi i} \int_{I(\varepsilon, \infty)} x^{s-1} b(x, 0) dx.$$

This shows that $\zeta(s)$ is holomorphic for $\operatorname{Re} s < 1$. In particular, for $s = 1 - m$, $m \in \mathbb{Z}^+$ (positive integers), the contour integral reduces to the residue of $x^{-m} b(x, 0)$ at $x = 0$, i.e., $b_m/m!$. Hence one obtains

$$(1.4) \quad \zeta(1-m) = (-1)^{m-1} (m-1)! \frac{b_m}{m!} = (-1)^{m-1} \frac{b_m}{m}.$$

Thus $\zeta(1-m)$ ($m \in \mathbb{Z}^+$) is rational. In particular,

$$\begin{aligned} \zeta(0) &= -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12}, \\ \zeta(-2\mu) &= 0, \quad \zeta(1-2\mu) = (-1)^\mu \frac{B_\mu}{2\mu} \quad (\mu \geq 1). \end{aligned}$$

This result has been generalized by Hecke, Klingen and Siegel [13] to the case of Dedekind zeta functions of totally real number fields. More recently, Shintani [11] gave a proof based on a direct generalization of the classical method explained above. Zeta functions attached to self-dual homogeneous cones have been studied by Siegel [13] in a special case of quadratic cones, and by Sato-Shintani [8] in a more general context of "prehomogeneous spaces". (Cf. also Shintani [9], [10].) On the other hand, the gamma functions attached to self-dual homogeneous cones were studied by Koecher [5], Gindikin [3] and others (cf. e.g., Resnikoff [6]). In this paper, we try to extend Shintani's method (i.e., the classical method) to examine the rationality of the special values of zeta functions attached to self-dual homogeneous cones.

§2. The gamma function of a self-dual homogeneous cone

2.1. Let U be a real vector space of dimension n , endowed with a positive definite inner product $\langle \cdot, \cdot \rangle$. By a "cone" in U we always mean a non-degenerate open convex cone in U with vertex at the origin, i.e., a non-empty open set $\mathcal{D}$ in U such that

$$x, y \in \mathcal{D}, \lambda, \mu \in \mathbb{R}^+ \Rightarrow \lambda x + \mu y \in \mathcal{D}$$

and such that $\mathcal{D}$ does not contain any straight line. A cone $\mathcal{D}$ in U is called homogeneous if the group of linear automorphisms

$$G(\mathcal{D}) = \{g \in GL(U) \mid g(\mathcal{D}) = \mathcal{D}\}$$

is transitive on $\mathcal{D}$; and $\mathcal{D}$ is called self-dual if the "dual" of

$$\mathcal{D}^* = \{x \in U \mid \langle x, y \rangle > 0 \text{ for all } y \in \mathcal{D} - \{0\}\}$$

coincides with $\mathcal{D}$.

Let $\mathcal{D}$ be a self-dual homogeneous cone in U and $G = G(\mathcal{D})^\circ$. Then it is well-known (e.g., Satake [7]) that the Zariski closure of G (in $GL(U)$) is a reductive algebraic group, containing $G(\mathcal{D})$ as a subgroup of finite index, and $g \mapsto {}^t g^{-1}$ is a Cartan involution of G ; the corresponding maximal compact subgroup $K = G \cap O(U)$ coincides with the isotropy subgroup of G at a "base point" $e \in \mathcal{D}$ (which is not unique, but will be fixed once and for all). Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the corresponding Cartan decomposition of $\mathfrak{g} = \text{Lie } G$. Then $\mathfrak{k} = \text{Lie } K$ and one has for $T \in \mathfrak{p}$

$$(2.1) \quad T \in \mathfrak{k} \iff {}^t T = -T \iff Te = 0.$$

It follows that, for each $u \in U$, there exists a uniquely determined element $T_u \in \mathfrak{p}$ such that $T_u e = u$. It is well-known that the vector space U endowed with a product

$$u \circ u' = T_u u' \quad (u, u' \in U)$$

becomes a formally real Jordan algebra (cf. Braun-Koecher [2], or Satake [7]).

We define the (regular) trace on U by

$$(2.2) \quad \tau(u) = \text{tr}(T_u).$$

For the given $(\mathcal{L}, e)$, one may assume (by Schur's lemma) that the inner product $\langle \rangle$ is so normalized that one has

$$(2.3) \quad \langle u, u' \rangle = \tau(u \circ u') \quad (u, u' \in U).$$

Next, let $u \in \mathcal{L}$. Then, since G is transitive on $\mathcal{L}$, there exists $g_1 \in G$ such that $u = g_1 e$. We define the (regular) norm $N(u)$ by

$$N(u) = \det(g_1),$$

which is clearly independent of the choice of g_1 . There exists a unique element $u_1 \in U$ such that $u = \exp u_1$ (which is defined to be $(\exp T_{u_1})e$); then by definition one has

$$(2.4) \quad N(u) = \det(\exp T_{u_1}) = e^{\tau(u_1)}.$$

In terms of the "quadratic multiplication" $P(u) = 2 T_u^2 - T_{u^2}$, one can also write $N(u) = \det(P(u))^{\frac{1}{2}}$. By the definition, it is clear that

$$(2.5) \quad N(e) = 1, \quad N(gu) = \det(g) N(u) \quad (g \in G(\mathcal{L}), u \in \mathcal{L}),$$

which characterizes the norm uniquely. Denoting the Euclidean measure on U by du , we see that $d_{\mathcal{L}}(u) = N(u)^{-1} du$ is an invariant measure on $\mathcal{L}$.

Example. Let $U = \text{Sym}_r(\mathbb{R})$ (the space of real symmetric matrices of degree r) and $\mathcal{L} = \mathcal{P}_r(\mathbb{R})$ (the cone of positive definite elements in U). Then one has

$$T_u(u') = \frac{1}{2} (uu' + u'u)$$

and so

$$\tau(u) = \frac{r+1}{2} \text{tr}(u), \quad N(u) = \det(u)^{\frac{r+1}{2}}.$$

2.2. We define the gamma function of the cone $\mathcal{L}$ by

$$(2.6) \quad \Gamma_{\mathcal{L}}(s) = \int_{\mathcal{L}} N(u)^{s-1} e^{-\tau(u)} du$$

which converges absolutely for $\operatorname{Re} s$ sufficiently large (actually for $\operatorname{Re} s > 1 - \frac{r}{n}$ as we will see later).

LEMMA 2.1. Suppose that the inner product $\langle \rangle$ is normalized by (2.3).

Then one has for any $v \in \mathcal{L}$

$$(2.7) \quad \int_{\mathcal{L}} N(u)^{s-1} e^{-\langle u, v \rangle} du = \Gamma_{\mathcal{L}}(s) N(v)^{-s}.$$

Proof. Let $v = g_1 e$ with $g_1 \in G$ and put $u' = {}^t g_1 u$. Then one has

$$\langle u, v \rangle = \langle u, g_1 e \rangle = \langle u', e \rangle = \tau(u').$$

Hence by (2.5) the left-hand side of (2.7) is equal to

$$\begin{aligned} & \int_{\mathcal{L}} N(u)^s e^{-\langle u, v \rangle} d_{\mathcal{L}}(u) \\ &= \int_{\mathcal{L}} (\det(g_1)^{-1} N(u'))^s e^{-\tau(u')} d_{\mathcal{L}}(u') \\ &= N(v)^{-s} \Gamma_{\mathcal{L}}(s), \text{ q.e.d.} \end{aligned}$$

It is known that the function $\Gamma_{\mathcal{L}}(s)$ can be expressed as a product of ordinary gamma functions (cf. e.g., Resnikoff loc. cit.). For the sake of completeness, we sketch a proof. First, it is clear that, if

$$\mathcal{L} = \mathcal{L}_1 \times \dots \times \mathcal{L}_m$$

is the decomposition of $\mathcal{L}$ into the direct product of irreducible (self-dual homogeneous) cones, then one has

$$\Gamma_{\mathcal{L}}(s) = \Gamma_{\mathcal{L}_1}(s) \dots \Gamma_{\mathcal{L}_m}(s).$$

Hence, for our purpose, we may assume that $\mathcal{L}$ is irreducible.

We need the root structure of $\mathfrak{g}$, which can be determined as follows.

Let

$$(2.8) \quad e = \sum_{i=1}^r e_i, \quad e_i e_j = \delta_{ij} e_i$$

be a decomposition of e (in the Jordan algebra U) into the sum of mutually orthogonal primitive idempotents. ("Primitive" means that each e_i can not be decomposed into the sum of mutually orthogonal idempotents any more.) Then we obtain the following decomposition of U into the direct sum of subspaces ("Peirce decomposition").

$$(2.9) \quad U = \bigoplus_{1 \leq i \leq j \leq r} U_{ij},$$

where

$$U_{ii} = \{ u \in U \mid e_i u = u \},$$

$$U_{ij} = \{ u \in U \mid e_i u = e_j u = \frac{1}{2} u \} \quad (i \neq j).$$

Then one has $e_k u = 0$ for $u \in U_{ij}$, $k \neq i, j$. Moreover

$$(2.10) \quad \dim U_{ii} = 1, \quad \dim U_{ij} = d \quad (i \neq j),$$

where d is a positive integer depending on the irreducible cone $\mathcal{L}$. (For instance, one has $d = 1$ for $\mathcal{L} = \mathcal{P}_r(R)$.) From (2.9), (2.10) one has the relation

$$(2.11) \quad n = r + \frac{1}{2} r(r-1)d, \quad \text{i.e.,} \quad d = \frac{2(n-r)}{r(r-1)}.$$

It follows that

$$(2.12) \quad \tau(e_i) = \text{tr}(T_{e_i}) = 1 + \frac{1}{2} (r-1)d = \frac{n}{r}.$$

Put

$$(2.13) \quad \mathcal{O} = \{ T_{e_i} \mid (1 \leq i \leq r) \}.$$

Then $\mathcal{O}$ is an abelian subalgebra of $\mathcal{V}$ of dimension r contained in $\mathcal{J}$.

We denote by (λ_i) the basis of $\mathcal{O}^*$ (the dual space of $\mathcal{O}$) dual to (T_{e_i}) , i.e., one has the relation

$$T = \sum_{i=1}^r \lambda_i(T) T_{e_i} \quad (T \in \mathcal{O}).$$

We put $\alpha_{ij} = \frac{1}{2} (\lambda_i - \lambda_j) \quad (i \neq j)$.

PROPOSITION 1. The root system of $\mathfrak{g}$ relative to $\mathfrak{n}$ is given by $\Phi = \{ \alpha_{ij} \ (i \neq j) \}$. The root space $\mathfrak{g}(\alpha_{ij})$ corresponding to α_{ij} is given by

$$(2.14) \quad \mathfrak{g}(\alpha_{ij}) = \{ T_u + [T_{e_i - e_j}, T_u] \mid u \in U_{ij} \}.$$

This can be verified by a straightforward computation; see e.g., Ash et al. [1] Ch. II, §3. Proposition 1 implies that the R-rank of $\mathfrak{g}$ is equal to r and the root system Φ is of type (A_{r-1}) .

2.3. Next we determine the Haar measure of G . Put

$$\mathfrak{u} = \sum_{i < j} \mathfrak{g}(\alpha_{ij})$$

and let A, N be the analytic subgroups of G corresponding to $\mathfrak{n}, \mathfrak{u}$, respectively. Then one has an Iwasawa decomposition $G = NA \cdot K (\approx N \times A \times K)$, which gives rise to the following formula for (the volume element of) a (biinvariant) Haar measure on G :

$$(2.15) \quad dg = c_1 e^{-2\rho(\log a)} d\mathfrak{n} da dk$$

for $g = \mathfrak{n}ak$ with $\mathfrak{n} \in N, a \in A, k \in K$, where $d\mathfrak{n}, da, dk$ denote Haar measures on N, A, K , respectively, c_1 is a positive constant depending on the normalization of the Haar measures, and ρ is a linear form on $\mathfrak{n}$ defined by

$$\rho(T) = \frac{1}{2} \operatorname{tr}(\operatorname{ad} T|_{\mathfrak{u}}) \quad (T \in \mathfrak{n});$$

by Proposition 1 one has

$$(2.16) \quad \rho = \frac{d}{2} \sum_{i < j} \alpha_{ij} = \frac{d}{2} \sum_{i=1}^r (r - 2i + 1) \lambda_i.$$

The Haar measure of K is always normalized by $\int_K dk = 1$. We make an identification $A = (R^+)^r$ by the correspondence $a \longleftrightarrow (t_i)$ defined by the relation $a = \exp(\sum \lambda_i T_{e_i})$, $t_i = e^{\lambda_i}$; then one has $da = \prod (dt_i/t_i)$.

Moreover one has

$$(2.17) \quad \begin{aligned} \det(a) &= e^{\tau(\sum \lambda_i e_i)} = e^{\frac{n}{r} \sum \lambda_i} = \left(\prod_{i=1}^r t_i \right)^{\frac{n}{r}}, \\ a \cdot e &= \sum e^{\lambda_i} e_i = \sum_{i=1}^r t_i e_i, \end{aligned}$$

$$e^{2\rho(\log a)} = \prod_{i=1}^r t_i^{\frac{d}{2}(r-2i+1)}.$$

Since $\mathcal{Q} = G/K$, we can normalize the Haar measure of G by the relation $dg = d_{\mathcal{Q}}(u) \cdot dk$ where $u = ge$. Then by (2.15), (2.16), (2.17) one has

$$\begin{aligned} \Gamma_{\mathcal{Q}}(s) &= \int_G N(ge)^s e^{-\tau(ge)} dg \\ (2.18) \quad &= c_1 \int_A \det(a)^s e^{-2\rho(\log a)} da \int_N e^{-\tau(\underline{n} \cdot ae)} d\underline{n} \\ &= c_1 \int_0^\infty \dots \int_0^\infty \prod_{i=1}^r (t_i^{\frac{n}{r}s - \frac{d}{2}(r-2i+1)-1} dt_i) \\ &\quad \times \int_N e^{-\tau(\underline{n}(\sum t_i e_i))} d\underline{n} \end{aligned}$$

To compute the integral over N , we introduce some notations. For $u =$

$\sum_{i < j} u_{ij} \in U$ with $u_{ij} \in U_{ij}$, we put

$$\begin{aligned} T_u^{(+)} &= \frac{1}{2} (T_u + \sum_{i < j} [T_{e_i - e_j}, T_{u_{ij}}]), \\ (2.19) \quad \mathcal{E}^{(+)}(u) &= \sum_{i < j} \sum_{v=1}^\infty \frac{1}{v!} \sum_{i < k_1 < \dots < k_v \leq j} u_{ik_1} u_{k_1 k_2} \dots u_{k_{v-1} j}. \end{aligned}$$

Then one has The U_{ij} -component of $\mathcal{E}^{(+)}(u)$ is denoted by $\varepsilon_{ij}^{(+)}(u)$.

LEMMA 2. The notation being as above, one has

$$\begin{aligned} (2.20) \quad (\exp T_u^{(+)})(\prod_{i=1}^r t_i e_i) &= \sum_{i=1}^r (t_i + \frac{1}{4} \sum_{k > i} t_k \varepsilon_{ik}^{(+)}(u)^2) e_i \\ &\quad + \frac{1}{2} \sum_{i < j} (t_j \varepsilon_{ij}^{(+)}(u) + \sum_{k > j} t_k \varepsilon_{ik}^{(+)}(u) \varepsilon_{jk}^{(+)}(u)) \end{aligned}$$

This may be regarded as a generalization of the so-called "Jacobi transformation". The proof is again straightforward. It follows that, if $\underline{n} = \exp T_u^{(+)} (u \in \sum_{i < j} U_{ij})$, one has

$$(2.21) \quad \tau(\underline{n}(\sum_i t_i e_i)) = \frac{n}{r} \sum_i t_i + \frac{1}{8} \sum_{i < k} \tau(\varepsilon_{ik}^{(+)}(u)^2) t_k.$$

We denote the Euclidean measure on U_{ij} ($i < j$) (relative to the inner product $\langle \rangle$) by du_{ij} and define the Haar measure on N by

$$d\underline{n} = \prod_{i < j} du_{ij} \quad \text{for } \underline{n} = \exp T_u^{(+)}.$$

Since the map $\mathcal{E}^{(+)}$ is a bijection of $\sum_{i < j} U_{ij}$ onto itself with jacobian

equal to one, one has

$$du = \prod_{i < j} du_{ij} = \prod_{i < j} du'_{ij},$$

where $u' = \xi^{(+)}(u)$. Hence by (2.21) one has

$$\begin{aligned} \int_N e^{-\tau(\underline{n} \geq t_i e_i)} d\underline{n} &= e^{-\frac{n}{r} \geq t_i} \prod_{i < j} \int_{U_j} e^{-\frac{t_j}{8} \tau(u_j'^2)} du'_{ij} \\ &= e^{-\frac{n}{r} \geq t_i} \prod_{i < j} \left(\frac{8\pi}{t_j}\right)^{\frac{d}{2}} \\ &= (8\pi)^{\frac{n-r}{2}} \prod_j (t_j^{-\frac{d}{2}(j-1)} e^{-\frac{n}{r} t_j}). \end{aligned}$$

Inserting this in (2.18), one obtains

$$\begin{aligned} \Gamma_{\mathcal{Q}}(s) &= c_1 (8\pi)^{\frac{n-r}{2}} \prod_{j=1}^r \left(\int_0^\infty t_j^{\frac{n}{r}s - \frac{d}{2}(r-j)-1} e^{-\frac{n}{r} t_j} dt_j \right) \\ &= c_1 (8\pi)^{\frac{n-r}{2}} \prod_{j=1}^r \left(\frac{n}{r}\right)^{-\frac{n}{r}s + \frac{d}{2}(r-j)} \Gamma\left(\frac{n}{r}s - \frac{d}{2}(r-j)\right) \\ &= c_1 (8\pi)^{\frac{n-r}{2}} \left(\frac{n}{r}\right)^{-ns + \frac{n-r}{2}} \prod_{j=1}^r \Gamma\left(\frac{n}{r}s - \frac{d}{2}(j-1)\right). \end{aligned}$$

The constant c_1 can be determined by the following observation. We set

$$U_0 = \sum_{i=1}^r U_{ii} = \{e_1, \dots, e_r\}_R$$

and denote by du_0 the Euclidean measure on U_0 (relative to $\langle \rangle$). Then, since $\langle e_i, e_j \rangle = \frac{n}{r} \delta_{ij}$, the bijection $A \rightarrow U_0$ defined by $a = \exp T_{u_0}$, or equivalently by $ae = \exp u_0$, gives the relation

$$du_0 = \left(\frac{n}{r}\right)^{\frac{r}{2}} da.$$

Hence, when

$$u = (\underline{n}a)e = \underline{n} \left(\sum_i t_i e_i \right),$$

$$\underline{n} = \exp T_{\underline{x}}^{(+)} \quad \underline{x} \in \sum_{i < j} U_{ij}, \quad \underline{x}' = \xi^{(+)}(\underline{x}),$$

one has by Lemma 2

$$\frac{\partial(u)}{\partial(t, x)} = \frac{\partial(u_0, u_{ij})}{\partial(t_i, x'_{ij})} = \left(\frac{n}{r}\right)^{\frac{r}{2}} \prod_{j=1}^r \left(\frac{t_j}{2}\right)^{(j-1)d}$$

$$= 2^{r-n} \left(\frac{n}{r}\right)^{\frac{r}{2}} \prod_{j=1}^r t_j^{(j-1)d}.$$

It follows that

$$d_{\mathcal{Q}}(u) = 2^{r-n} \left(\frac{n}{r}\right)^{\frac{r}{2}} \prod_{j=1}^r (t_j^{(j-1)d - \frac{n}{r}} dt_j) dx,$$

which, in view of (2.11) and (2.16), implies (2.15) and the relation

$$(2.22) \quad c_1 = 2^{r-n} \left(\frac{n}{r}\right)^{\frac{r}{2}}.$$

Thus we obtain the formula

$$(2.23) \quad \Gamma_{\mathcal{Q}}(s) = (2\pi)^{\frac{n-r}{2}} \left(\frac{n}{r}\right)^{n(\frac{1}{2}-s)} \prod_{j=1}^r \Gamma\left(\frac{n}{r}s - \frac{d}{2}(j-1)\right).$$

Our computation also shows that the integral for $\Gamma_{\mathcal{Q}}(s)$ converges absolutely for $\text{Re } s > 1 - \frac{r}{n}$.

From the relation (1.2) one obtains

$$\begin{aligned} \Gamma_{\mathcal{Q}}(s) \Gamma_{\mathcal{Q}}(1-s) &= (2\pi)^{n-r} \prod_{j=1}^r \Gamma\left(\frac{n}{r}s - \frac{d}{2}(j-1)\right) \Gamma\left(\frac{n}{r}(1-s) - \frac{d}{2}(r-j)\right) \\ &= (2\pi)^{n-r} (2\pi i)^r \prod_{j=1}^r \frac{e^{-\pi i(\frac{n}{r}s - \frac{d}{2}(j-1))}}{e^{2\pi i(\frac{n}{r}s - \frac{d}{2}(j-1))} - 1} \end{aligned}$$

Since one has by (2.11)

$$n - r = d \frac{r(r-1)}{2} \equiv \begin{cases} 0 \pmod{2} & \text{for } d \text{ even} \\ \left[\frac{r}{2}\right] \pmod{2} & \text{for } d \text{ odd,} \end{cases}$$

one has

$$\prod_{j=1}^r e^{-\pi i \frac{d}{2}(j-1)} = (-i)^d \frac{r(r-1)}{2} = \begin{cases} i^{n-r} & \text{for } d \text{ even} \\ i^{n-r} (-1)^{\left[\frac{r}{2}\right]} & \text{for } d \text{ odd.} \end{cases}$$

Hence one obtains the following functional equation:

$$(2.24) \quad \Gamma_{\mathcal{Q}}(s) \Gamma_{\mathcal{Q}}(1-s) = (2\pi i)^n e^{n\pi i s} \begin{cases} (e^{2\pi i \frac{n}{r}s} - 1)^{-r} & (d \text{ even}) \\ (e^{2\pi i \frac{n}{r}s} - 1)^{-\left[\frac{r+1}{2}\right]} (e^{2\pi i \frac{n}{r}s+1})^{-\left[\frac{r}{2}\right]} & (d \text{ odd}). \end{cases}$$

§3. Zeta functions of a self-dual homogeneous cone.

3.1. We fix a $\mathbb{Q}$ -structure on U and assume that (the Zariski closure of) G is defined over $\mathbb{Q}$ and $e \in U_{\mathbb{Q}}$; then (the Zariski closure of) K is also defined over $\mathbb{Q}$. We also fix a lattice L in U compatible with that $\mathbb{Q}$ -structure, i.e., such that $U_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$, and an arithmetic subgroup Γ fixing L , i.e., a subgroup of $G_L = \{g \in G \mid gL = L\}$ of finite index; for simplicity we assume that Γ has no fixed point in $\mathcal{D}$. We then define the zeta function associated with $\mathcal{D}$, Γ , L as follows:

$$(3.1) \quad \zeta_{\mathcal{D}}(s; \Gamma, L) = \sum_{u \in \Gamma \backslash \mathcal{D} \cap L} N(u)^{-s},$$

the summation being taken over a complete set of representatives of $\mathcal{D} \cap L$ modulo Γ . It can be shown easily that this series is absolutely convergent for $\operatorname{Re} s > 1$.

By the reduction theory, Γ has a fundamental domain in $\mathcal{D}$ which is a rational polyhedral cone. More precisely, there exists a finite set of simplicial cones

$$\begin{aligned} C^{(i)} &= \{v_1^{(i)}, \dots, v_{l_i}^{(i)}\}_{\mathbb{R}_+} \\ &= \left\{ \sum_{j=1}^{l_i} \lambda_j v_j^{(i)} \mid \lambda_j \in \mathbb{R}_+ \right\} \quad (1 \leq i \leq m), \end{aligned}$$

where $v_1^{(i)}, \dots, v_{l_i}^{(i)}$ are linearly independent elements in $\overline{\mathcal{D}} \cap L$, such that

$$\mathcal{D} = \bigsqcup_{\substack{\gamma \in \Gamma \\ 1 \leq i \leq m}} \gamma C^{(i)}.$$

It follows that

$$\zeta(s; \Gamma, L) = \sum_{i=1}^m \sum_{u \in C^{(i)} \cap L} N(u)^{-s}.$$

For a set of linearly independent vectors $v_1, \dots, v_{l_i} \in L$, we put

$$R((v_j), L) = \left\{ \sum_{j=1}^{l_i} \lambda_j v_j \mid 0 < \lambda_j \leq 1 \right\} \cap L,$$

which is finite. Then $u \in C^{(i)} \cap L$ can be written uniquely in the form

$$u = v_0 + \sum_{j=1}^{l_i} m_j v_j^{(i)}, \quad v_0 \in R((v_j^{(i)}), L), \quad m_j \in \mathbb{Z}, \quad m_j \geq 0.$$

For a set of linearly independent vectors $v_1, \dots, v_l \in \bar{L} \cap V_Q$ and $v_0 = \sum_{j=1}^l \alpha_j v_j$ ($\alpha_j \in \mathbb{Q}_+$), we define a "partial zeta function" by

$$(3.2) \quad \zeta_{\mathcal{L}}(s; (v_j), v_0) = \sum_{m_j \geq 0} N(v_0 + \sum_{j=1}^l m_j v_j)^{-s},$$

which will also be written as $\zeta_{\mathcal{L}}(s; (v_j), (\alpha_j))$. Then the zeta function

(3.1) can be written as a finite sum of partial zeta functions as follows:

$$(3.3) \quad \zeta_{\mathcal{L}}(s; \Gamma, L) = \sum_{i=1}^m \sum_{v_i \in R((v_j^{(i)}), L)} \zeta_{\mathcal{L}}(s; (v_j^{(i)}), v_0).$$

Hence the study of special values of $\zeta_{\mathcal{L}}(s; \Gamma, L)$ is reduced to that of the partial zeta functions of the form (3.2).

3.2. Let (v_j) and v_0 be as above. Then by (2.7) one obtains

$$\begin{aligned} \Gamma_{\mathcal{L}}(s) \zeta_{\mathcal{L}}(s; (v_j), v_0) &= \sum_{\substack{m_j \geq 0 \\ 1 \leq j \leq l}} \Gamma_{\mathcal{L}}(s) N(v_0 + \sum_{j=1}^l m_j v_j)^{-s} \\ &= \sum_{\substack{m_j \geq 0 \\ 1 \leq j \leq l}} \int_{\mathcal{L}} N(u)^{s-1} e^{-\sum_{j=1}^l (\alpha_j + m_j) \langle v_j, u \rangle} du \\ &= \int_{\mathcal{L}} N(u)^s \prod_{j=1}^l b(\langle v_j, u \rangle, 1 - \alpha_j) d_{\mathcal{L}}(u) \\ &= \int_G \det(g)^s \prod_{j=1}^l b(\langle v_j, ge \rangle, 1 - \alpha_j) dg. \end{aligned}$$

In the notation of § 2, but this time using the decomposition $G = KAK$, one has

$$(3.4) \quad dg = c \Delta(a) dk \cdot da \cdot dk'$$

for $g = kak'$, $k, k' \in K$, $a \in A$. Here c is a positive constant and

$$\begin{aligned} \Delta(a) &= \prod_{\alpha \in \Phi_+} (e^{\alpha(\log a)} - e^{-\alpha(\log a)})^d \\ &= \left(\prod_{i=1}^r t_i \right)^{-\frac{d}{2}(r-1)} |\Delta(t_1, \dots, t_r)|^d, \end{aligned}$$

where $\Delta(t_1, \dots, t_r) = \prod_{i < j} (t_i - t_j)$ (cf. Helgason [4], Ch. X, § 1). Hence in view of (2.11) and (2.17) one has

$$(3.5) \quad \Gamma_{\mathcal{L}}(s) \zeta_{\mathcal{L}}(s; (v_j), (\alpha_j)) = c \int_0^\infty \dots \int_0^\infty \left(\prod_{i=1}^r t_i \right)^{\frac{n}{r}(s-1)} |\Delta(t)|^d F(t) \prod_{i=1}^r dt_i,$$

where

$$F(t_1, \dots, t_r) = \int_K \prod_{j=1}^l b(\langle v_j, k \sum t_i e_i \rangle, 1 - \alpha_j) dk.$$

It is clear that $F(t_1, \dots, t_r)$ is holomorphic for $\operatorname{Re} t_i > 0$ ($1 \leq i \leq r$).

Since K contains an element which induces any given permutation of $e_1, \dots, e_r$, the function F is symmetric. Hence, denoting by B_1 an open simplicial cone in R^r defined by $t_1 > \dots > t_r > 0$, one has

$$(3.5') \quad F_{\mathcal{Q}}(s) \zeta_{\mathcal{Q}}(s; (v_j), (\alpha_j)) = c r! \int_{B_1} \left(\prod t_i \right)^{\frac{n}{r}(s-1)} \Delta(t)^d F(t) \prod dt_i.$$

3.3. Still following Shintani [11], we make a change of variables $(t_i) \rightarrow (t_1, \tau_2, \dots, \tau_r)$ with $\tau_i = t_i/t_{i-1}$ ($2 \leq i \leq r$). Then B_1 can be expressed as

$$B_1 = \left\{ (t_i) \mid t_i = t_1 \prod_{j=2}^i \tau_j, 0 < t_1 < \infty, 0 < \tau_i < 1 \right\}.$$

Putting $\tau_1 = t_1$, one has

$$\begin{aligned} \frac{\varrho(t_1, \dots, t_r)}{\varrho(t_1, \tau_2, \dots, \tau_r)} &= \prod_{i=1}^r \tau_i^{r-i}, \\ \prod t_i &= \prod \tau_i^{r-i+1}, \\ \Delta(t) &= \prod \tau_i^{\frac{1}{2}(r-i+1)(r-i)} \prod_{2 \leq i < j \leq r} (1 - \tau_i \dots \tau_j). \end{aligned}$$

It follows that the exponent of τ_i in the integrand in (3.5') is equal to

$$\begin{aligned} (r-i+1) \frac{n}{r}(s-1) + \frac{d}{2}(r-i+1)(r-i) + r - i \\ = (r-i+1) \left\{ \frac{n}{r}s - \frac{d}{2}(i-1) \right\} - 1. \end{aligned}$$

Hence one has

$$(3.6) \quad \Gamma_{\mathcal{Q}}(s) \zeta_{\mathcal{Q}}(s; (v_j), (\alpha_j)) = c r! \int_0^\infty t^{ns-1} dt \int_0^1 \dots \int_0^1 \prod \tau_i^{(r-i+1) \left\{ \frac{n}{r}s - \frac{d}{2}(i-1) \right\} - 1} \tilde{F}(t_1, \tau) \prod_{i=2}^r d\tau_i,$$

where

$$(3.7) \quad \tilde{F}(t_1, \tau) = \prod_{2 \leq i < j \leq r} (1 - \tau_i \dots \tau_j)^d F(t_1, t_1 \tau_2, \dots, t_1 \tau_2 \dots \tau_r).$$

3.4. We now assume that all v_j 's are in $\mathcal{Q}$ (not on the boundary of $\mathcal{Q}$).

(In the situation explained in 3.1, this means that the $\mathbb{Q}$ -rank of G is equal to 1.) Then for any $v \in \bar{\mathcal{Q}} - \{0\}$, one has $\langle v_j, v \rangle > 0$; in particular,

$$(3.8) \quad \langle v_j, ke_i \rangle > 0 \quad \text{for all } k \in K, 1 \leq i \leq r.$$

Put

$$(3.9) \quad \begin{aligned} \xi_j &= \langle v_j, k \sum t_i e_i \rangle \\ &= t_1 \langle v_j, k(e_1 + \sum_{i=2}^r \tau_i e_i) \rangle \\ &= t_1 \langle v_j, ke_1 \rangle (1 + \sum_{i=2}^r \tau_i \frac{\langle v_j, ke_i \rangle}{\langle v_j, ke_1 \rangle}). \end{aligned}$$

For the fixed e_i, v_j , choose $\rho, \rho_i > 0$ in such a way that

$$(3.10) \quad \left\{ \begin{array}{ll} \sum_{i=2}^r \rho^{i-1} \frac{\langle v_j, ke_i \rangle}{\langle v_j, ke_1 \rangle} < 1 & \text{for all } k \in K, 1 \leq j \leq \ell, \\ \rho_i < \frac{\pi}{\langle v_j, ke_1 \rangle} & \text{for all } 1 \leq j \leq \ell. \end{array} \right.$$

Theⁿ/for

$$(3.11) \quad 0 < |t_i| < \rho_i, \quad |\tau_i| < \rho \quad (2 \leq i \leq r),$$

one has $0 < |\xi_j| < 2\pi$ and so $b(\xi_j, 1 - \alpha_j)$ is holomorphic. Hence the

function $F(t) = F(t_1, t_1 \tau_2, \dots, t_1 \tau_2 \dots \tau_r)$ has a Laurent expansion in

$t_1, \tau_2, \dots, \tau_r$ in the domain defined by (3.11). The coefficients in

this expansion is a $\mathbb{Q}$ -linear combination of the integrals of the form

$$(3.12) \quad I((v_{ij})) = \int_K \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq \ell}} \langle v_j, ke_i \rangle^{v_{ij}} dk$$

where $v_{ij} \geq 0$ for $2 \leq i \leq r$ and $v_{ij} \in \mathbb{Z}$ for all i, j .

3.5. Let $I(\varepsilon, 1)$ denote the contour consisting of the line segment $[\varepsilon, 1]$ taken twice in opposite directions and of a (small) circle of radius ε about the origin taken in the counterclockwise direction. When the τ_i ($2 \leq i \leq r$) are on $I(\varepsilon, 1)$, one has by (2.12)

$$|\langle v_j, k(e_1 + \sum_{i=2}^r \tau_i e_i) \rangle| \leq |v_j| \cdot \sum_{i=1}^r |e_i| = \sqrt{nr} |v_j|$$

and

$$\begin{aligned} \operatorname{Re} \langle v_j, k(e_1 + \sum_{i=2}^r \tau_i e_i) \rangle &= \langle v_j, ke_1 \rangle + \sum_{i=2}^r \operatorname{Re}(\tau_i) \langle v_j, ke_i \rangle \\ &\geq \langle v_j, ke_1 \rangle - \varepsilon |v_j| \sum_{i=2}^r |e_i| \\ &= \langle v_j, ke_1 \rangle - \varepsilon(r-1) \sqrt{\frac{n}{r}} |v_j|. \end{aligned}$$

We choose ε so that one has

$$(3.13) \quad \varepsilon \sqrt{nr} |v_j| < \min \{2\pi, \langle v_j, ke_1 \rangle \mid k \in K\} \quad \text{for all } 1 \leq j \leq l,$$

Theⁿ/the above inequalities show that $\langle v_j, k(e_1 + \sum_{i=2}^r \tau_i e_i) \rangle$ belongs to the domain

$$\left\{ z \in \mathbb{C} \mid |z| < \frac{2\pi}{\varepsilon}, \operatorname{Re} z > \varepsilon \sqrt{\frac{n}{r}} |v_j| \right\}.$$

It follows that, if t_1 is on the contour $I(\varepsilon, \infty)$, one has

$$|\xi_j| < 2\pi \quad \text{or} \quad \operatorname{Re} \xi_j > 0,$$

so that the function $b(\xi_j, 1 - \alpha_j)$ is holomorphic.

From this observation, it is clear that the integral on the r.h.s. of (3.6) is equal to the contour integral

$$(e^{2\pi i n s} - 1)^{-1} \int_{t_1 \in I(\varepsilon, \infty)} \cdot \prod_{i=2}^r (e^{2\pi i \frac{r-i+1}{r} n s} - 1)^{-1} \int_{\tau_i \in I(\varepsilon, 1)}$$

which is independent of the choice of ε satisfying (3.13). As is easily seen, the contour integral converges for all $s \in \mathbb{C}$. Hence the ^{above} integral

~~$\Gamma(v_1), \dots, \Gamma(v_l)$~~ , viewed as a function in s , can be continued to a meromorphic function on the whole plane; the possible poles are of the form

$$\frac{\Gamma v}{(r-i+1)n} \quad (v \in \mathbb{Z}).$$

§ 4. The special values of the zeta functions.

4.1. As a preliminary, we check the rationality of the constant c in (3.4).

For that purpose, we compute $\Gamma_{\mathcal{Q}}(s)$ by using the decomposition $G = KAK$.

$$\begin{aligned}
 (4.1) \quad \Gamma_{\mathcal{Q}}(s) &= \int_{\mathcal{Q}} N(u)^s e^{-\tau(u)} d_{\mathcal{Q}}(u) \\
 &= \int_G N(ge)^s e^{-\tau(ge)} dg \\
 &= c \int_A \det(a)^s e^{-\tau(ae)} \Delta(a) da \\
 &= c \int_0^\infty \int_0^\infty \left(\prod t_i \right)^{\frac{n}{r}(s-1)} |\Delta(t)|^d e^{-\frac{n}{r} \sum t_i} \prod dt_i.
 \end{aligned}$$

We make another change of variables:

$$t = \sum_{i=1}^r t_i, \quad t'_i = t_i/t.$$

Then

$$\frac{\partial(t_1, \dots, t_r)}{\partial(t, t'_1, \dots, t'_{r-1})} = (-1)^{r-1} t^{r-1},$$

and the exponent of t in the integrand in the last member of (4.1) is equal to

$$n(s-1) + \frac{d}{2} r(r-1) + r-1 = ns-1.$$

Hence one has

$$(4.2) \quad \Gamma_{\mathcal{Q}}(s) = c \cdot \gamma(s) \cdot \beta(s),$$

where

$$(4.3) \quad \begin{cases} \gamma(s) = \int_0^\infty t^{ns-1} e^{-\frac{n}{r}t} dt = \left(\frac{r}{n}\right)^{ns} \Gamma(ns), \\ \beta(s) = \int_{\substack{t'_i > 0 \\ \sum t'_i < 1}} \{t'_1 \dots t'_{r-1} (1 - \sum t'_i)\}^{\frac{n}{r}(s-1)} \times \\ \quad |\Delta(t'_1, \dots, t'_{r-1}, 1 - \sum t'_i)|^d \prod dt'_i. \end{cases}$$

For $s = 1$, one has

$$\Gamma_{\mathcal{Q}}(1) = c \cdot \gamma(1) \cdot \beta(1) = c \left(\frac{r}{n}\right)^n (n-1)! \beta(1),$$

$$\beta(1) = \int_{\substack{t'_i > 0 \\ \sum t'_i < 1}} |\Delta(t'_1, \dots, t'_{r-1}, 1 - \sum t'_i)|^d \prod dt'_i \in \mathbb{Q}.$$

By (2.23) one has

$$(4.4) \quad \Gamma_d(1) = (2\pi)^{\frac{n-r}{2}} \left(\frac{r}{n}\right)^{\frac{n}{2}} \prod_{j=1}^r \Gamma\left(1 + \frac{d}{2}(j-1)\right) \\ \sim_{\mathbb{Q}} \begin{cases} \pi^{\frac{n-r}{2}} & (d \text{ even}) \\ \pi^{\frac{1}{2}(n - [\frac{r+1}{2}])} & (d \text{ odd}), \end{cases}$$

where $a \sim_{\mathbb{Q}} b$ means that $a/b \in \mathbb{Q}$. Thus one has

$$(4.5) \quad c = \frac{(2\pi)^{\frac{n-r}{2}} \left(\frac{n}{r}\right)^{\frac{n}{2}} \prod_{j=1}^r \Gamma\left(1 + \frac{d}{2}(j-1)\right)}{(n-1)! \beta(1)} \sim_{\mathbb{Q}} \Gamma_d(1).$$

Since $\Gamma_d(1) \sim_{\mathbb{Q}} \Gamma_d(1 + \frac{r}{n}v)$ for $v \in \mathbb{Z}$, one obtains

$$(4.6) \quad c \Gamma_d(1 + \frac{r}{n}v) \sim_{\mathbb{Q}} \Gamma_d(1)^2 \sim_{\mathbb{Q}} \begin{cases} \pi^{n-r} & (d \text{ even}) \\ \pi^{n - [\frac{r+1}{2}]} & (d \text{ odd}). \end{cases}$$

4.2. We first consider the case where d is even. Then by (2.24) one has

$$\Gamma_d(s) \Gamma_d(1-s) = (2\pi i)^n e^{\pi i n s} (e^{2\pi i \frac{n}{r}s} - 1)^{-r}.$$

Hence

$$(4.7) \quad \zeta_d(s; (v_j), (\alpha_j)) = \frac{c \Gamma_d(1-s)}{(2\pi i)^{n-r} e^{\pi i n s}} \times R(s),$$

where

$$R(s) = \left(\frac{e^{2\pi i \frac{n}{r}s} - 1}{2\pi i} \right)^r \cdot r! \int_{B_1} \left(\prod t_i \right)^{\frac{n}{r}(s-1)} \Delta(t)^d F(t) \prod dt_i \\ = \prod_{j=1}^r \frac{e^{2\pi i \frac{n}{r}s} - 1}{e^{2\pi i \frac{r-j+1}{r}ns} - 1} \times \frac{1}{(2\pi i)^r} \int_{I(\varepsilon, \infty)} t_1^{ns-1} dt_1 \\ \left(\prod_{r=2}^r \int_{\tau_i \in I(\varepsilon, 1)} \right) \prod_{i=2}^r \tau_i^{(r-i+1)\{\frac{n}{r}s - \frac{d}{2}(i-1)\} - 1} r! \tilde{F}(t_1, \tau_i) \quad \left(\prod d\tau_i \right).$$

We are interested in the values of ζ_d at $s = -\frac{r}{n}v$ ($v = 0, 1, \dots$). The

first factor in the right hand side of (4.7) is holomorphic for $\operatorname{Re} s < \frac{r}{n}$ and by (4.6) the value at $s = -\frac{r}{n}v$ is rational:

$$(4.8) \quad \frac{c \Gamma_{\mathcal{Q}}(1 + \frac{r}{n}v)}{(2\pi i)^{n-r} e^{-rv\pi i}} = (-1)^{\frac{n-r}{2} + rv} \frac{c \Gamma_{\mathcal{Q}}(1 + \frac{r}{n}v)}{(2\pi i)^{n-r}} \in \mathbb{Q}.$$

On the other hand, it is clear that

$$\frac{e^{2\pi i \frac{n}{r} s} - 1}{e^{2\pi i \frac{r-i+1}{r} ns} - 1} \longrightarrow \frac{1}{r-i+1} \quad \text{when } s \rightarrow -\frac{r}{n}v.$$

Hence we see that $R(-\frac{r}{n}v)$ is equal to the coefficient of

$$t_1^{rv} \prod_{i=2}^r \tau_i^{(r-i+1)\{v + \frac{d}{2}(i-1)\}}$$

in the Laurent expansion of $\tilde{F}(t_1, \tau)$,

~~$$(4.9) \quad \tilde{F}(t_1, \tau) = \sum_{\alpha \in \mathbb{Z}^r} c_{\alpha} t_1^{\alpha_1} \tau_2^{\alpha_2} \cdots \tau_r^{\alpha_r} = \sum_{\alpha \in \mathbb{Z}^r} c_{\alpha} t_1^{\alpha_1} \tau_2^{\alpha_2} \cdots \tau_r^{\alpha_r},$$~~

which is a $\mathbb{Q}$ -linear combination of $I((v_i))$.

4.3. From now on we assume that d is odd. By the classification theory, it is known that this assumption implies that $r = 2$ ($n = d + 2$) or $d = 1$ ($n = \frac{1}{2}r(r+1)$). By (2.24) one has

$$\Gamma_{\mathcal{Q}}(s) \Gamma_{\mathcal{Q}}(1-s) = (2\pi i)^n e^{n\pi i s} (e^{2\pi i \frac{n}{r} s} - 1)^{-[\frac{r+1}{2}]} (e^{2\pi i \frac{n}{r} s} + 1)^{-[\frac{r}{2}]}.$$

Hence

$$(4.11) \quad \zeta_{\mathcal{Q}}(s; (v_j), (\alpha_j)) = \frac{c \Gamma_{\mathcal{Q}}(1-s)}{(2\pi i)^{n-[\frac{r+1}{2}]} e^{n\pi i s}} \times R^{(1)}(s) R^{(2)}(s),$$

where

$$R^{(1)}(s) = (2\pi i)^{[\frac{r}{2}]} r! \frac{(e^{2\pi i \frac{n}{r} s} - 1)^{[\frac{r+1}{2}]} (e^{2\pi i \frac{n}{r} s} + 1)^{[\frac{r}{2}]}}{\prod_{k=1}^r (e^{2\pi i (r-k+1)\{\frac{n}{r}s - \frac{d}{2}(k-1)\}} - 1)},$$

$$R^{(2)}(s) = (2\pi i)^{-r} \int_{I(\varepsilon, \omega)} t_1^{ns-1} dt_1 \int_{I(\varepsilon, 1)} \prod_{i=1}^{r-1} \tau_i^{(r-i+1)\{\frac{n}{r}s - \frac{d}{2}(i-1)\}} \tilde{F}(t_1, \tau) \prod d\tau_i.$$

The first factor in the right hand side of (4.11) is holomorphic for $\operatorname{Re} s < \frac{r}{n}$ and by (4.6) the value at $s = -\frac{r}{n}v$ ($v \geq 0$) is rational:

$$(4.12) \quad \frac{c \left[\frac{r}{n} v \right]}{(2\pi i)^{n - [\frac{r+1}{2}]} e^{-\pi i r v}} = (-1)^{\frac{1}{2}(n - [\frac{r+1}{2}]) + r v} \frac{c \left[\frac{r}{n} v \right]}{(2\pi)^{n - [\frac{r+1}{2}]} } \in \mathbb{Q}.$$

Note that one has

$$n \equiv \left[\frac{r+1}{2} \right] \pmod{2},$$

since

$$n = d+2 \equiv 1 = \left[\frac{3}{2} \right] \pmod{2} \quad \text{if } r = 2, \text{ and}$$

$$n = \frac{1}{2} r(r+1) \equiv \left[\frac{r+1}{2} \right] \pmod{2} \quad \text{if } d = 1.$$

4.4. To compute $R^{(1)}(s)$, we first note

$$e^{\pi i d(k-1)(r-k+1)} = \begin{cases} -1 & \text{if } k \equiv r \equiv 0 \pmod{2}, \\ 1 & \text{otherwise.} \end{cases}$$

We put

$$\left[\frac{r}{2} \right] = r_1, \quad \zeta = e^{2\pi i \frac{r}{r} s}.$$

The case r is odd. One has

$$\begin{aligned} R^{(1)}(s) &= (2\pi i)^{r_1} r! \frac{(\zeta - 1)^{r_1+1} (\zeta + 1)^{r_1}}{\prod_{k=1}^r (\zeta^k - 1)} \\ &= \frac{r!}{\prod_{k=1}^r (\zeta^{k-1} + \dots + \zeta + 1)} (2\pi i \cdot \frac{\zeta + 1}{\zeta - 1})^{r_1}. \end{aligned}$$

Hence, when $s \rightarrow -\frac{r}{n}v$, one has

$$(4.13) \quad (s + \frac{r}{n}v)^{r_1} R^{(1)}(s) \rightarrow (2\frac{r}{n})^{r_1}.$$

Thus $R^{(1)}(s)$ has a pole of order r_1 at $s = -\frac{r}{n}v$.

The case r is even. One has

$$R^{(1)}(s) = (2\pi i)^{r_1} r! \frac{(\zeta - 1)^{r_1} (\zeta + 1)^{r_1}}{\prod_{k=1}^r \{(-1)^k \zeta^k - 1\}}$$

$$= (-2\pi i)^{r_1} \frac{r!}{\prod_{\substack{1 \leq k \leq r \\ k \text{ even}}} (\zeta^{k-1} + \dots + \zeta + 1) \prod_{\substack{1 \leq k \leq r \\ k \text{ odd}}} (\zeta^{k-1} - \dots - \zeta + 1)}$$

Hence $R^{(1)}$ is holomorphic at $s = -\frac{r}{n}\nu$ and

$$(4.14) \quad R^{(1)}(-\frac{r}{n}\nu) = (-2\pi i)^{r_1} \frac{r!}{(2r_1)!} = (-\pi i)^{r_1} \frac{r!}{r_1!}.$$

4.5. When r is odd (hence $d = 1$, $n = \frac{1}{2}r(r+1)$), $R^{(2)}(s)$ for $s = -\frac{r}{n}\nu$ is given by the coefficient of

$$t_1^{r\nu} \prod_{i=2}^r \tau_i^{(r-i+1)(\nu + \frac{i-1}{2})}$$

in the Laurent expansion of $\tilde{F}(t_1, \tau)$. Hence $\zeta_{\mathcal{Q}}(s; (v_j), (\alpha_j))$ has at most a pole of order $r_1 = \frac{r-1}{2}$ at $s = -\frac{2\nu}{r+1}$ and one has

$$(4.15) \quad \lim_{s \rightarrow -\frac{2\nu}{r+1}} (s + \frac{2\nu}{r+1})^{r_1} \zeta_{\mathcal{Q}}(s; (v_j), (\alpha_j)) \sim_{\mathcal{Q}} R^{(2)}(-\frac{2\nu}{r+1}).$$

To treat the case r is even, we use the formula

$$\int_{I(\varepsilon, 1)} t^{\frac{m}{2}-1} dt = -\frac{4}{m} \quad (m \text{ odd}),$$

which can be verified easily. When r is even, the value of $R^{(2)}(s)$ for $s = -\frac{r}{n}\nu$ is given by

$$(4.16) \quad (-\pi i)^{-r_1} \sum_{\substack{m_1, \dots, m_{r_1} \in \mathbb{Z} \\ j=1}} \frac{a_{(m_j)}}{\prod_{j=1}^{r_1} (m_j - (r-2j+1) \{ \nu + \frac{d}{2}(2j-1) \})}$$

where $a_{(m_j)}$ is the coefficient of

$$t_1^{r\nu} \prod_{j=1}^{r_1} \tau_{2j-1}^{(r-2j+2)(\nu + d(j-1))} \prod_{j=1}^{r_1} \tau_{2j}^{m_j}$$

in $\tilde{F}(t_1, \tau)$. Hence for the value of $\zeta_{\mathcal{Q}}$, one has

$$(4.17) \quad \zeta_{\mathcal{Q}}(-\frac{r}{n}\nu; (v_j), (\alpha_j)) \sim_{\mathcal{Q}} (2\pi i)^{r_1} R^{(2)}(-\frac{r}{n}\nu).$$

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