

Pair correlation of zeros of quadratic L -functions near the real axis

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Abstract

In this article, we investigate the non-trivial zeros of quadratic L -functions near the real axis. Assuming the Generalized Riemann Hypothesis, we give an asymptotic formula for the weighted pair correlation function of quadratic L -functions. From this formula, we prove that there exists a number of “close low lying zeros”.

1 Introduction

About forty years ago, H. L. Montgomery [9] published his famous paper titled “The pair correlation of zeros of the zeta function”. Under the assumption of the Riemann Hypothesis (RH), he investigated the function

$$F(\alpha, T) = \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma'),$$

where $w(u) = 4/(4 + u^2)$ and γ, γ' run over the set of the imaginary parts of the non-trivial zeros of the Riemann zeta-function $\zeta(s)$ in $0 < \text{Im}(s) \leq T$. (Note that the number of non-trivial zeros of $\zeta(s)$ in this domain is asymptotic to $(1/2\pi)T \log T$.) He obtained some asymptotic formula for $F(\alpha, T)$ ($0 < \alpha \leq 1 - \epsilon$), and using this formula, he obtained several results on the distances of the non-trivial zeros. For example, under the assumption of the RH, he proved that at least 67% of the non-trivial zeros are simple, and that

$$\liminf_{n \rightarrow \infty} \frac{(\gamma_{n+1} - \gamma_n) \log \gamma_n}{2\pi} \leq \lambda < 1$$

holds for specific λ , where γ_n denotes the imaginary part of the n -th non-trivial zero of $\zeta(s)$ in the upper half plane.

Later, Montgomery's idea was extended to many types of L -functions or other situations. For example, Özlük [10] investigated the non-trivial zeros of the Dirichlet L -functions near the real axis. Assuming the Generalized Riemann Hypothesis (GRH), he proved that at least 86% of such zeros are simple in some sense. One of the other interesting generalizations is the work of Hejhal [5].

From his explicit formula of the Riemann zeta-function, he constructed certain asymptotic formula for the function involving the pairs of three distinct zeros of $\zeta(s)$. Further, the result of Hejhal was generalized by Rudnick and Sarnak [14], and the n -level correlation of the zeros of principal L -functions was obtained. In particular, their results agree with the prediction for the Gaussian unitary ensemble of random matrix theory. Today there are several papers considering such kind of problem (n -level density). For example, see [1], [2], [3], [6], [7], [8], [12], [13].

Our aim in this paper is to investigate the pair correlation of the zeros of the quadratic L -functions near the real axis. As a prior research, Özlük and Snyder [11] investigated such zeros. Under the assumption of GRH, they studied the asymptotic behavior of the function

$$G_K(\alpha, D) = \left(\frac{1}{2} K \left(\frac{1}{2} \right) D \right)^{-1} \sum_{d \neq 0} e^{-\frac{\pi d^2}{D^2}} \sum_{\rho \in Z_d} K(\rho) D^{i\alpha\gamma}$$

as $D \rightarrow \infty$ for $|\alpha| < 2$, where $\rho = 1/2 + i\gamma$ runs over the set of all non-trivial zeros of $L(s, \chi_d)$, the quadratic L -function associated to the Kronecker symbol $\chi_d = (d/\cdot)$, and $K(s)$ is some weight function. From their asymptotic formula, they proved that assuming the GRH, not more than 6.25% of all integers d have the property that $L(s, \chi_d)$ vanishes at the central point $s = 1/2$. Subsequently Soundararajan [16] unconditionally proved that $L(1/2, \chi_d) \neq 0$ for at least 87.5% of all fundamental discriminants d .

In this paper, assuming the GRH (including RH), we investigate the function $F_K(\alpha, D)$ defined as follows. Let $K(s)$ be analytic in $-1 < \operatorname{Re}(s) < 2$ and satisfy $K(1/2 - it) = \overline{K(1/2 + it)}$ for any $t \in \mathbf{R}$. Moreover, we assume that its Mellin inverse transform

$$a(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) x^{-s} ds \quad (1.1)$$

converges absolutely for any $-1 < c < 2$, $x > 0$, and that $a(x)$ is real, non-negative, belongs to C^1 class, and has a support in $[A, B]$ for some $0 < A < B < \infty$. Then, $K(s)$ is given by the Mellin transform of $a(x)$:

$$K(s) = \int_0^\infty a(x) x^s \frac{dx}{x}. \quad (1.2)$$

For $d \in \mathbf{Z}$, let $\chi_d = (d/\cdot)$ be the Kronecker symbol and $L(s, \chi_d)$ be the L -function associated to χ_d . We denote the set of non-trivial zeros of $L(s, \chi_d)$ by Z_d . For $x > 0$, $D > 0$, we put

$$f_K(x, D) = \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\rho_1, \rho_2 \in Z_d} K(\rho_1) \overline{K(\rho_2)} x^{\rho_1 + \overline{\rho_2}},$$

and for $\alpha \in \mathbf{R}$, we define the correlation function $F_K(\alpha, D)$ by

$$\begin{aligned} F_K(\alpha, D) &= \left[\frac{1}{xD \log D} f_K(x, D) \right]_{x=D^\alpha} \\ &= \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\rho_1, \rho_2 \in Z_d} K(\rho_1) \overline{K(\rho_2)} D^{i\alpha(\gamma_1 - \gamma_2)}, \end{aligned} \quad (1.3)$$

where $\rho_j = 1/2 + i\gamma_j$ for $j = 1, 2$. Then, the main theorem is stated as follows:

Theorem 1.1. *Assuming the Generalized Riemann Hypothesis, for any small $\delta > 0$, we have*

$$\begin{aligned} F_K(\alpha, D) &= L(1)\alpha + a(D^{-\alpha})^2 D^{-\alpha} \log D + a(D^{-\alpha}) \cdot O(\alpha D^{-\frac{\alpha}{2}} \log D) \\ &\quad + a(D^{-\alpha})^2 \cdot O(D^{-\alpha}) + O(\min\{1, \alpha D^{-\alpha} \log^2 D\}) \\ &\quad + O(\min\{D^\alpha (\log D)^{-1}, \alpha^2 D^{-\alpha} \log^3 D\}) + o(1) \end{aligned} \quad (1.4)$$

uniformly for $0 < \alpha < 1 - \delta$ as $D \rightarrow \infty$, where

$$L(1) = \int_0^\infty a(x)^2 dx.$$

The implied constants depend only on $K(s)$ and $\delta > 0$.

In the next section, we introduce the outline of the proof. The author recommends the reader to see the preprint [15] to check the detailed computations. Several results on the average gaps of the non-trivial zeros can be obtained. Among others, in Section 3, we prove that there are quite a few pairs of zeros $(1/2 + i\gamma_1, 1/2 + i\gamma_2)$ of $L(s, \chi_d)$ ($d \in \mathbf{Z} \setminus \{0\}$) near the real axis satisfying $0 < |\gamma_1 - \gamma_2| \leq (2\pi\lambda)/\log D$, if λ is large to a certain extent.

2 The proof of Theorem 1.1 (outline)

We start from Özlük's explicit formula

$$\begin{aligned} \sum_{\rho \in Z_d} K(\rho) x^\rho &= K(1) E(\chi_d) x - \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \left(\frac{d}{n}\right) \\ &\quad + a\left(\frac{1}{x}\right) \log\left(\frac{|d|}{\pi}\right) + O(\min\{x, \log |d| \log x\}) \quad (x \geq 1), \end{aligned}$$

introduced in [11]. Here, $E(\chi) = 1$ if χ is a principal character, and otherwise $E(\chi) = 0$. The error term is interpreted as $O(1)$ if $x = 1$. Since the main terms of the right hand side are real, we have

$$\begin{aligned}
& \sum_{\rho_1, \rho_2 \in Z_d} K(\rho_1) \overline{K(\rho_2)} x^{\rho_1 + \overline{\rho_2}} \\
&= \left\{ K(1) E(\chi_d) x - \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \left(\frac{d}{n}\right) \right. \\
&\quad \left. + a\left(\frac{1}{x}\right) \log\left(\frac{|d|}{\pi}\right) + O(\min\{x, \log |d| \log x\}) \right\}^2 \\
&= K(1)^2 E(\chi_d)^2 x^2 + \sum_{n, m=1}^{\infty} a\left(\frac{n}{x}\right) a\left(\frac{m}{x}\right) \Lambda(n) \Lambda(m) \left(\frac{d}{n}\right) \left(\frac{d}{m}\right) \\
&\quad + a\left(\frac{1}{x}\right)^2 \log^2\left(\frac{|d|}{\pi}\right) - 2K(1) E(\chi_d) x \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \left(\frac{d}{n}\right) \\
&\quad + 2K(1) E(\chi_d) x a\left(\frac{1}{x}\right) \log\left(\frac{|d|}{\pi}\right) - 2a\left(\frac{1}{x}\right) \log\left(\frac{|d|}{\pi}\right) \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \left(\frac{d}{n}\right) \\
&\quad + O\left(\max\left\{K(1) E(\chi_d) x, \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \left(\frac{d}{n}\right), a\left(\frac{1}{x}\right) \log\left(\frac{|d|}{\pi}\right)\right\}\right) \\
&\quad \times O(\min\{x, \log |d| \log x\}) \\
&\quad + O(\min\{x^2, \log^2 |d| \log^2 x\}).
\end{aligned}$$

By multiplying both sides by $e^{-\frac{\pi d^2}{D^2}}$ and taking the sum over d , we have

$$f_K(x, D) = \sum_{i=1}^6 M_i + \sum_{i=1}^4 O_i, \quad (2.1)$$

where

$$\begin{aligned}
M_1 &= K(1)^2 x^2 \sum_d E(\chi_d)^2 e^{-\frac{\pi d^2}{D^2}}, \\
M_2 &= \sum_{n, m=1}^{\infty} a\left(\frac{n}{x}\right) a\left(\frac{m}{x}\right) \Lambda(n) \Lambda(m) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{n}\right) \left(\frac{d}{m}\right), \\
M_3 &= a\left(\frac{1}{x}\right)^2 \sum_d e^{-\frac{\pi d^2}{D^2}} \log^2\left(\frac{|d|}{\pi}\right), \\
M_4 &= -2K(1) x \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \sum_d e^{-\frac{\pi d^2}{D^2}} E(\chi_d) \left(\frac{d}{n}\right), \\
M_5 &= 2K(1) x a\left(\frac{1}{x}\right) \sum_d e^{-\frac{\pi d^2}{D^2}} E(\chi_d) \log\left(\frac{|d|}{\pi}\right), \\
M_6 &= -2a\left(\frac{1}{x}\right) \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{n}\right) \log\left(\frac{|d|}{\pi}\right),
\end{aligned}$$

$$O_1 = O(\min\{O_{11}, O_{12}\}), \quad O_2 = O(\min\{O_{21}, O_{22}\}),$$

$$O_3 = O(\min\{O_{31}, O_{32}\}), \quad O_4 = O(\min\{O_{41}, O_{42}\}),$$

with

$$O_{11} = K(1)x^2 \sum_d e^{-\frac{\pi d^2}{D^2}} E(\chi_d), \quad O_{12} = K(1)x \log x \sum_d e^{-\frac{\pi d^2}{D^2}} E(\chi_d) \log |d|,$$

$$O_{21} = x \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{n}\right),$$

$$O_{22} = \log x \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{n}\right) \log |d|,$$

$$O_{31} = xa \left(\frac{1}{x}\right) \sum_d e^{-\frac{\pi d^2}{D^2}} \log \left(\frac{|d|}{\pi}\right),$$

$$O_{32} = a \left(\frac{1}{x}\right) \log x \sum_d e^{-\frac{\pi d^2}{D^2}} \log |d| \log \left(\frac{|d|}{\pi}\right),$$

$$O_{41} = x^2 \sum_d e^{-\frac{\pi d^2}{D^2}}, \quad O_{42} = \log^2 x \sum_d e^{-\frac{\pi d^2}{D^2}} \log^2 |d|.$$

First, by a standard technique, we find that the error terms O_1, O_2, O_3, O_4 are evaluated by

$$O_1 \ll \min\{x^2 D^{1/2}, x D^{1/2} \log x \log D\},$$

$$O_2 \ll \min\{x^2 D, x D \log x \log D, x^{3/2} \log^2 x \log D\},$$

$$O_3 \ll \min\{x D \log D, D \log x \log^2 D\},$$

$$O_4 \ll \min\{x^2 D, D \log^2 x \log^2 D\}.$$

Next, by using partial summation and prime number theorem, the main terms except for M_2 are obtained as follows:

$$M_1 = IK(1)^2 x^2 D^{1/2} - \frac{1}{2} K(1)^2 x^2 + O(x^2 D^{-1/2}),$$

$$M_3 = a \left(\frac{1}{x}\right) \{D \log^2 D + O(D \log D)\},$$

$$M_4 = -2IK(1)^2 D^{1/2} x^2 + O(D^{1/2} x^{3/2} \log^2 x) + O(\min\{x^2, x^3 D^{-1/2}\}),$$

$$M_5 \ll xa \left(\frac{1}{x}\right) D^{1/2} \log D,$$

$$M_6 \ll a \left(\frac{1}{x}\right) \{x^{3/2} \log x \log D + D x^{1/2} \log x \log D\},$$

where $I = 4^{-1}\pi^{-1/4}\Gamma(1/4)$. These are obtained by some computations similar to those in the paper of Özlük and Snyder [11], hence we omit the detail. Finally, we compute

$$M_2 = \sum_{k,l=1}^{\infty} \sum_{p,q \in P} a\left(\frac{p^k}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{p^k}\right) \left(\frac{d}{q^l}\right),$$

where P denotes the set of all prime numbers. It will be convenient to keep in mind that only k, l satisfying $k, l \ll \log x$ contribute to the sum above, since $a(x)$ has a support in such range. First, we evaluate the contribution of the part $p = 2$ to M_2 . The contribution of the part $p = q = 2$ is

$$\ll \sum_{k,l=1}^{\infty} a\left(\frac{2^k}{x}\right) a\left(\frac{2^l}{x}\right) \sum_d e^{-\frac{\pi d^2}{D^2}} \ll D \log^2 x. \quad (2.2)$$

The contribution of the part $p = 2, q \geq 3, l \geq 2$ is

$$\begin{aligned} &\ll \sum_k \sum_{l \geq 2} \sum_{q \in P} a\left(\frac{2^k}{x}\right) a\left(\frac{q^l}{x}\right) (\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \\ &\ll Dx^{\frac{1}{2}} \log^2 x. \end{aligned} \quad (2.3)$$

Since $(\cdot/2^k q)$ is a non-principal character whose conductor is at most $2q$, by combining partial summation and Pólya-Vinogradov inequality for the sum involving quadratic characters, we find that

$$\sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{2^k}\right) \left(\frac{d}{q}\right) \ll q^{\frac{1}{2}} \log q$$

for primes $q \geq 3$. Therefore, the contribution of the part $p = 2, q \geq 3, l = 1$ is

$$\begin{aligned} &\sum_k \sum_{q \in P_{\geq 3}} a\left(\frac{2^k}{x}\right) a\left(\frac{q}{x}\right) (\log 2)(\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{2^k}\right) \left(\frac{d}{q}\right) \\ &\ll (\log x) \sum_{q \in P} a\left(\frac{q}{x}\right) (\log q) \cdot q^{\frac{1}{2}} \log q \\ &\ll x^{\frac{3}{2}} \log^2 x, \end{aligned} \quad (2.4)$$

where $P_{\geq 3}$ denotes the set of all prime numbers greater than 2. By (2.2), (2.3) and (2.4), the contribution of the part $p = 2$ to M_2 is at most $O(Dx^{1/2} \log^2 x + x^{3/2} \log^2 x)$. The contribution of the part $q = 2$ is the same. Hence

$$\begin{aligned} M_2 &= \sum_{k,l=1}^{\infty} \sum_{p,q \in P_{\geq 3}} a\left(\frac{p^k}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{p^k}\right) \left(\frac{d}{q^l}\right) \\ &\quad + O(Dx^{\frac{1}{2}} \log^2 x + x^{\frac{3}{2}} \log^2 x) \\ &=: \sum_{k,l=1}^{\infty} M_2^{(k,l)} + O(Dx^{\frac{1}{2}} \log^2 x + x^{\frac{3}{2}} \log^2 x), \end{aligned} \quad (2.5)$$

say. Moreover, by prime number theorem and partial summation, we have

$$\begin{aligned} M_2^{(k,l)} &\ll \sum_{p,q} a\left(\frac{p^k}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \\ &\ll klDx^{\frac{1}{k}+\frac{1}{l}} \end{aligned}$$

uniformly for each k, l . Hence the contribution of the part $k \geq 3, l \geq 2$ or $k \geq 2, l \geq 3$ is at most $O(Dx^{5/6} \log^4 x)$. Therefore,

$$M_2 = M_2^{(1,1)} + M_2^{(2,2)} + 2 \sum_{l \geq 2} M_2^{(1,l)} + O(Dx^{\frac{5}{6}} \log^4 x + x^{\frac{3}{2}} \log^2 x). \quad (2.6)$$

By the computation above, $M_2^{(2,2)}$ is evaluated by

$$M_2^{(2,2)} \ll Dx. \quad (2.7)$$

Next, we compute $M_2^{(1,l)}$ for $l \geq 1$.

A) First, we consider the case that l is odd. We decompose

$$\begin{aligned} M_2^{(1,l)} &= \left(\sum_{\substack{p,q \in P_{\geq 3} \\ p=q}} + \sum_{\substack{p,q \in P_{\geq 3} \\ p \neq q}} \right) a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{p}\right) \left(\frac{d}{q}\right) \\ &=: M_{2,1}^{(1,l)} + M_{2,2}^{(1,l)}. \end{aligned} \quad (2.8)$$

Easily, we find that

$$M_{2,1}^{(1,1)} = L(1)Dx \log x + O(Dx + x \log x),$$

$$M_{2,1}^{(1,l)} \ll lDx^{1/l} \log x \quad (l \geq 2).$$

Next, we compute $M_{2,2}^{(1,l)}$.

a) If $x = o(D^{1/2})$, by prime number theorem and Pólya-Vinogradov inequality, we obtain

$$M_{2,2}^{(1,l)} \ll lx^{3/2(1+1/l)} \log^2 x.$$

b) If $D^{1/2-\delta} \ll x \ll D^{1-\delta}$ ($\delta > 0$), by the translation formula of twisted theta function, we find that

$$M_{2,2}^{(1,l)} = D \sum_{p \geq 3} \sum_{\substack{q \geq 3 \\ q \neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \frac{1}{\sqrt{pq}} \sum_m \left(\frac{m}{pq}\right) e^{-\frac{\pi m^2 D^2}{p^2 q^2}}.$$

We decompose this by

$$M_{2,2}^{(1,l)} = M_s^{(1,l)} - M_p^{(1,l)} - M_q^{(1,l)} + M_{pq}^{(1,l)} + E, \quad (2.9)$$

where

$$\begin{aligned}
M_s^{(1,l)} &= D \sum_{p \geq 3} \sum_{\substack{q \geq 3 \\ q \neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \frac{1}{\sqrt{pq}} \sum_{m=\square} e^{-\frac{\pi m^2 D^2}{p^2 q^2}}, \\
M_p^{(1,l)} &= D \sum_{p \geq 3} \sum_{\substack{q \geq 3 \\ q \neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \frac{1}{\sqrt{pq}} \sum_{\substack{m=\square \\ p|m}} e^{-\frac{\pi m^2 D^2}{p^2 q^2}}, \\
M_q^{(1,l)} &= D \sum_{p \geq 3} \sum_{\substack{q \geq 3 \\ q \neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \frac{1}{\sqrt{pq}} \sum_{\substack{m=\square \\ q|m}} e^{-\frac{\pi m^2 D^2}{p^2 q^2}}, \\
M_{pq}^{(1,l)} &= D \sum_{p \geq 3} \sum_{\substack{q \geq 3 \\ q \neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \frac{1}{\sqrt{pq}} \sum_{\substack{m=\square \\ pq|m}} e^{-\frac{\pi m^2 D^2}{p^2 q^2}},
\end{aligned}$$

and

$$E = D \sum_{p \geq 3} \sum_{\substack{q \geq 3 \\ q \neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \frac{1}{\sqrt{pq}} \sum_{m \neq \square} \left(\frac{m}{pq}\right) e^{-\frac{\pi m^2 D^2}{p^2 q^2}}.$$

Here, $\square$ denotes the square of natural numbers. Using partial summation and prime number theorem, we obtain

$$\begin{aligned}
M_s^{(1,l)} &= ID^{1/2} l^{-1} K(1) K(1/l) x^{1+1/l} + O(l^{-1} D^{1/2} x^{1+1/2l} \log^2 x) + O(D x^{1/2+1/2l}), \\
M_p^{(1,l)} &\ll l D^{1/2} x^{1/l} + D x^{1/2+1/2l}, \\
M_q^{(1,l)} &\ll l^{-1} D^{1/2} x \log x + D x^{1/2+1/2l}, \\
M_{pq}^{(1,l)} &\ll D x^{1/2+1/2l},
\end{aligned}$$

and after slightly complicated computations (we use the assumption of the GRH to evaluate the sum involving quadratic symbols), the error term E is evaluated by

$$E \ll D x^{1/l} + x^{1+1/l} \log^4 x.$$

Hence we obtain

$$\begin{aligned}
M_{2,2}^{(1,l)} &= ID^{\frac{1}{2}} l^{-1} K(1) K\left(\frac{1}{l}\right) x^{1+\frac{1}{l}} + O(l^{-1} D^{\frac{1}{2}} x^{1+\frac{1}{2l}} \log^2 x) + O(D x^{\frac{1}{2}+\frac{1}{2l}}) \\
&\quad + O(x^{1+\frac{1}{l}} \log^4 x)
\end{aligned} \tag{2.10}$$

for odd l and $D^{1/2-\delta} \ll x \ll D^{1-\delta}$. Therefore we have

$$\begin{aligned}
M_2^{(1,l)} &= ID^{\frac{1}{2}} l^{-1} K(1) K\left(\frac{1}{l}\right) x^{1+\frac{1}{l}} + O(l^{-1} D^{\frac{1}{2}} x^{1+\frac{1}{2l}} \log^2 x) + O(l D x^{\frac{1}{l}} \log x) \\
&\quad + O(D x^{\frac{1}{2}+\frac{1}{2l}}) + O(x^{1+\frac{1}{l}} \log^4 x)
\end{aligned} \tag{2.11}$$

for odd l and $D^{1/2-\delta} \ll x \ll D^{1-\delta}$. If $l = 1$, $D^{1/2-\delta} \ll x \ll D^{1-\delta}$, we have

$$M_2^{(1,1)} = L(1)Dx \log x + IK(1)^2 D^{\frac{1}{2}} x^2 + O(Dx + D^{\frac{1}{2}} x^{\frac{3}{2}} \log^2 x + x^2 \log^4 x). \quad (2.12)$$

On the other hand, for odd $l \geq 3$ and $x = o(D^{1/2})$, we have

$$M_2^{(1,l)} \ll lDx^{\frac{1}{l}} \log x + lx^{\frac{3}{2}(1+\frac{1}{l})} \log^2 x, \quad (2.13)$$

and for $l = 1$, $x = o(D^{1/2})$, we have

$$M_2^{(1,1)} = L(1)Dx \log x + O(Dx + x^3 \log^2 x). \quad (2.14)$$

B) Next, we consider the case that l is even. In this case, we have

$$\begin{aligned} M_2^{(1,l)} &= \sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}} \\ &\quad - \sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_{q|d} \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}}. \end{aligned} \quad (2.15)$$

Since

$$\sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}} \ll \sqrt{p} \log p,$$

the first term of the right hand side of (2.15) is

$$\begin{aligned} &\sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}} \\ &\ll \sum_p a\left(\frac{p}{x}\right) \sqrt{p} \log^2 p \sum_q a\left(\frac{q^l}{x}\right) \log q \\ &\ll x^{\frac{3}{2}} \log x \cdot lx^{\frac{1}{l}} \\ &\ll lx^{\frac{3}{2}+\frac{1}{l}} \log x. \end{aligned} \quad (2.16)$$

The second term of the right hand side of (2.15) is

$$\begin{aligned} &\sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_{q|d} \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}} \\ &= \sum_p a\left(\frac{p}{x}\right) (\log p) \sum_q \left(\frac{q}{p}\right) a\left(\frac{q^l}{x}\right) (\log q) \sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi q^2 d^2}{D^2}} \\ &\ll \sum_p a\left(\frac{p}{x}\right) (\log p) \sum_q a\left(\frac{q^l}{x}\right) (\log q) \left| \sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi q^2 d^2}{D^2}} \right|. \end{aligned} \quad (2.17)$$

Now, since q satisfies $q \leq (Bx)^{1/l} \ll D^{1-\delta} \ll D$, we have $D/q \gg 1$. Therefore, by using Pólya-Vinogradov inequality, we have

$$\sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi q^2 d^2}{D^2}} \ll \sqrt{p} \log p. \quad (2.18)$$

Therefore,

$$\begin{aligned}
& \sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_{q|d} \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}} \\
& \ll \sum_p a\left(\frac{p}{x}\right) \sqrt{p} \log^2 p \sum_q a\left(\frac{q^l}{x}\right) \log q \\
& \ll lx^{\frac{3}{2}+\frac{1}{l}} \log x.
\end{aligned} \tag{2.19}$$

By combining (2.16) and (2.19), we obtain

$$M_2^{(1,l)} \ll lx^{\frac{3}{2}+\frac{1}{l}} \log x \tag{2.20}$$

for even l . Now, we have computed or evaluated $M_2^{(1,l)}$ for all l . If $x = o(D^{1/2})$, by (2.13) and (2.20), we have

$$\sum_{l \geq 2} M_2^{(1,l)} \ll Dx^{\frac{1}{3}} \log x + x^2 \log^2 x. \tag{2.21}$$

By inserting (2.7), (2.14), (2.21) into (2.6), we obtain

$$M_2 = L(1)Dx \log x + O(Dx + x^3 \log^2 x).$$

If $D^{1/2-\delta} \ll x \ll D^{1-\delta}$, by (2.11),

$$\sum_{l \geq 3, \text{ odd}} M_2^{(1,l)} \ll D^{\frac{1}{2}} x^{\frac{4}{3}} + Dx^{\frac{2}{3}} + x^{\frac{4}{3}} \log^4 x. \tag{2.22}$$

(Notice that $K(1/l) \ll l$.) On the other hand, by (2.20),

$$\sum_{l \geq 2, \text{ even}} M_2^{(1,l)} \ll x^2 \log x. \tag{2.23}$$

By inserting (2.7), (2.12), (2.22), (2.23) into (2.6), we obtain

$$\begin{aligned}
M_2 &= L(1)Dx \log x + IK(1)^2 D^{\frac{1}{2}} x^2 \\
&\quad + O(Dx + D^{\frac{1}{2}} x^{\frac{3}{2}} \log^2 x + x^2 \log^4 x)
\end{aligned}$$

for $D^{1/2-\delta} \ll x \ll D^{1-\delta}$.

Now, we have computed or evaluated all the terms appearing in (2.1), hence we obtain the asymptotic formula for $f_K(x, D)$. By dividing this by $x D \log D$ and putting $x = D^\alpha$ ($\alpha > 0$), we obtain the asymptotic formula in Theorem 1.1.

3 The pairs of close zeros near the real axis

The asymptotic formula (1.4) of Theorem 1.1 is useful to investigate the distribution of low lying zeros of quadratic L -functions. As a corollary, we prove that

assuming the GRH and simple zero conjecture for each relevant L -functions, there exists a number of “close zeros” near the real axis. First, we mention to the L -functions associated to Kronecker symbols. If $d \not\equiv 3 \pmod{4}$, $\chi_d = (d/\cdot)$ becomes a Dirichlet character modulo $4|d|$ or $|d|$. In this case, we denote the conductor of χ_d by d^* . If $d \equiv 3 \pmod{4}$, the L -function associated to χ_d is expressed by

$$L(s, \chi_d) = \frac{1}{1 - \left(\frac{2}{d}\right) 2^{-s}} \prod_{p \geq 3} \frac{1}{1 - \eta_4(p) \left(\frac{p}{d}\right) p^{-s}},$$

where η_4 is the non-principal character of modulo 4. In this case, we denote the conductor of $\eta_4(\cdot)(\cdot/d)$ by d^* . We define the constants A_+^* , A_-^* by

$$A_-^* = \liminf_{D \rightarrow \infty} \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \log d^*, \quad A_+^* = \limsup_{D \rightarrow \infty} \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \log d^*. \quad (3.1)$$

Notice that since $\log d^* \leq \log d + O(1)$, we have $A_+^* \leq 1$.

Corollary 3.1. *Assume the Generalized Riemann Hypothesis and that all non-trivial zeros of $L(s, \chi_d)$ are simple. Then, for $0 < \lambda < 1$, we have*

$$\begin{aligned} & \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\substack{\rho_1, \rho_2 \in Z_d \\ 0 < |\gamma_1 - \gamma_2| \leq \frac{2\pi\lambda}{\log D}}} K(\rho_1) K(\rho_2) \\ & \geq \frac{2}{3}\lambda - \frac{2}{9}\lambda^2 - \frac{\cos 2\pi\lambda}{6\pi^2} + \frac{\sin 2\pi\lambda}{12\pi^3\lambda} - B_+^* + o(1) \end{aligned} \quad (3.2)$$

as $D \rightarrow \infty$, where $B_+^* = A_+^*/3$ and

$$K(s) = \left(\frac{e^{s-\frac{1}{2}} - e^{-s+\frac{1}{2}}}{2s-1} \right)^2.$$

In particular, if $\lambda > \lambda_0 = 0.6073$, we have

$$\sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\substack{\rho_1, \rho_2 \in Z_d \\ 0 < |\gamma_1 - \gamma_2| \leq \frac{2\pi\lambda}{\log D}}} K(\rho_1) K(\rho_2) \gg D \log D \quad (3.3)$$

as $D \rightarrow \infty$.

Proof. We use Selberg’s minorant function

$$h(u) = \left(\frac{\sin \pi u}{\pi u} \right)^2 \frac{1}{1 - u^2}.$$

This function is bounded and satisfies $h(u) \leq 1$, $h(u) < 0$ if $|u| > 1$. The Fourier transform of $h(u)$ is given by

$$\hat{h}(\alpha) = \begin{cases} 1 - |\alpha| + \frac{\sin 2\pi|\alpha|}{2\pi} & (|\alpha| \leq 1) \\ 0 & (|\alpha| > 1) \end{cases}$$

(for example, see [4]). For $0 < \lambda < 1$, we give lower and upper bounds for the integral

$$\int_{-\infty}^{\infty} F_K(\alpha, D) \cdot \lambda \hat{h}(\lambda \alpha) d\alpha.$$

First, since the integrant is non-negative and $1/\lambda > 1$, by (1.4), we have

$$\begin{aligned} & \int_{-\infty}^{\infty} F_K(\alpha, D) \cdot \lambda \hat{h}(\lambda \alpha) d\alpha \\ & \geq \int_{-1}^1 F_K(\alpha, D) \cdot \lambda \hat{h}(\lambda \alpha) d\alpha \\ & = \lambda L(1) \int_{-1}^1 |\alpha| \left\{ 1 - |\lambda \alpha| + \frac{\sin 2\pi \lambda |\alpha|}{2\pi} \right\} d\alpha \\ & \quad + \lambda \log D \int_{-1}^1 a(D^{-|\alpha|})^2 D^{-|\alpha|} \left\{ 1 - |\lambda \alpha| + \frac{\sin 2\pi \lambda |\alpha|}{2\pi} \right\} d\alpha + o(1) \\ & = \frac{2}{3} \lambda - \frac{2}{9} \lambda^2 - \frac{\cos 2\pi \lambda}{6\pi^2} + \frac{\sin 2\pi \lambda}{12\pi^3 \lambda} + o(1). \end{aligned} \tag{3.4}$$

In the computation above, we used

$$\begin{aligned} L(1) &= \int_0^{\infty} a(v)^2 dv = \frac{1}{3}, \\ \int_{-\infty}^{\infty} a(D^{-\alpha})^2 D^{-\alpha} d\alpha &= \frac{1}{\log D} \int_0^1 a(v)^2 dv = \frac{1}{6 \log D}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_{-\infty}^{\infty} F_K(\alpha, D) \cdot \lambda \hat{h}(\lambda \alpha) d\alpha \\ & = \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\rho_1, \rho_2 \in Z_d} K(\rho_1) K(\rho_2) h\left(\frac{(\gamma_1 - \gamma_2) \log D}{2\pi \lambda}\right). \end{aligned} \tag{3.5}$$

Now, since $h((\gamma_1 - \gamma_2) \log D / (2\pi \lambda))$ is negative if $|\gamma_1 - \gamma_2| > (2\pi \lambda) / \log D$, by (3.5), we have

$$\begin{aligned} & \int_{-\infty}^{\infty} F_K(\alpha, D) \cdot \lambda \hat{h}(\lambda \alpha) d\alpha \\ & \leq \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\substack{\rho_1, \rho_2 \in Z_d \\ |\gamma_1 - \gamma_2| \leq \frac{2\pi \lambda}{\log D}}} K(\rho_1) K(\rho_2) h\left(\frac{(\gamma_1 - \gamma_2) \log D}{2\pi \lambda}\right) \\ & = \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\rho \in Z_d} m_{\rho} K(\rho)^2 \\ & \quad + \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\substack{\rho_1, \rho_2 \in Z_d \\ 0 < |\gamma_1 - \gamma_2| \leq \frac{2\pi \lambda}{\log D}}} K(\rho_1) K(\rho_2) h\left(\frac{(\gamma_1 - \gamma_2) \log D}{2\pi \lambda}\right) \end{aligned}$$

$$\leq B_+^* + \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\substack{\rho_1, \rho_2 \in Z_d \\ 0 < |\gamma_1 - \gamma_2| \leq \frac{2\pi\lambda}{\log D}}} K(\rho_1)K(\rho_2) + o(1),$$

where m_ρ is the multiplicity of the zero of $L(s, \chi_d)$ at $s = \rho$, and by our assumption, this equals 1. By combining (3.4) and above, we obtain (3.2). Since $B_+^* = A_+^*/3 \leq 1/3$, the right hand side of (3.2) becomes positive if $\lambda > \lambda_0 = 0.6073$. Therefore, we obtain (3.3). $\square$

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