

Principal series Whittaker functions on $SL(3, \mathbf{R})$

HIROYUKI MANABE, TAKU ISHII, AND TAKAYUKI ODA

眞鍋廣幸、石井卓、織田孝幸

Graduate School of Mathematical Sciences, The University of Tokyo
東京大学数理科学研究科

This is an extract from a preprint with the same title. The full proofs are contained in that. Here we write only the major results. *The numbering of the statement are the same as the original full paper. Some statements in the original are skipped.*

Introduction

The study of Whittaker models of algebraic groups over local fields has already some history. The Jacquet integral is named after the investigation of H.Jacquet [7]. Multiplicity free theorem by J.Shalika for quasi-split groups, was later enhanced for the case of the real field by N.Wallach. For reductive groups over the real field, this theme was investigated by M.Hashizume [5], B.Kostant, D. Vogan, H.Matsumoto, and the joint work of R.Goodman and N.Wallach [4].

More specifically $GL(n, \mathbf{R})$, explicit expressions for class 1 Whittaker functions are obtained, firstly for $n = 3$ by D.Bump [2]. The main contributor for the case of general n seems to be E.Stade. Other related results will be find in the references of the papers of him ([9],[10]).

Let us explain the outline of this paper. The purpose of the master thesis [1] refered above is to investigate the Whittaker functions belonging to the non-spherical principal series representations of $SL(3, \mathbf{R})$. The minimal K -type of such representations is 3-dimensional. So we have to consider vector-valued functions. The main results are, firstly, to obtain the holonomic system of the A -radial part of such Whittaker functions with minimal K -type explicitly (§4), and secondly to have 6 formal solutions (§5, Theorem (5.5)), which are considered as examples of confluent hypergeometric series of two variables. We also have integral expressions of these 6 solutions (§5, Theorem (5.6)). In the subsequent section, the Jacquet integral (so to say, the primary Whittaker function) is written as a sum of these 6 *secondary* Whittaker functions (§6-8).

1 Preliminaries. Basic terminology

1.1 Whittaker model

Given an irreducible admissible representation (π, H) of $G = SL(3, \mathbf{R})$, we consider its model or realization in the space of Whittaker functions. This means, for a non-

degenerate unitary character ψ of a maximal unipotent subgroup $N = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G \right\}$ of G defined by

$$\psi\left(\begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}\right) = \exp\{2\pi\sqrt{-1}(c_1x_{12} + c_2x_{23})\}$$

with $c_1, c_2 \in \mathbf{R}$ being non-zero, we consider a smooth induction $C^\infty\text{-Ind}_N^G(\psi)$ to G , and the space of intertwining operators of smooth G -modules

$$\text{Hom}_G(H_\infty, C^\infty\text{-Ind}_N^G(\psi))$$

with H_∞ the subspace consisting of C^∞ -vectors in H . Or more algebraically speaking, we might consider the corresponding space in the context of $(\mathfrak{g}, K)$ -modules (with $\mathfrak{g} = \text{Lie}(G)$, $K = SO(3)$):

$$\text{Hom}_{(\mathfrak{g}, K)}(H_\infty, C^\infty\text{-Ind}_N^G(\psi)).$$

1.2 Principal series representations

Let P_0 be a minimal parabolic subgroup of G given by the upper triangular matrices in G , and $P_0 = MAN$ be a Langlands decomposition of P_0 with $M = K \cap \{\text{diagonals in } G\}$, $A = \exp \mathfrak{a}$, with

$$\mathfrak{a} = \{\text{diag}(t_1, t_2, t_3) | t_i \in \mathbf{R}, t_1 + t_2 + t_3 = 0\}.$$

In order to define a principal series representation with respect to the minimal parabolic subgroup P_0 of G , we firstly fix a character σ of the finite abelian group M of type $(2, 2)$ and a linear form $\nu \in \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C} = \text{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$. For such data, we can define a representation $\sigma \otimes e^\nu$ of MA , and extend this to P_0 by the identification $P_0/N \cong MA$. Then we set

$$\pi_{\sigma, \nu} = L^2\text{-Ind}_{P_0}^G(\sigma \otimes e^{\nu+\rho} \otimes 1_N).$$

Here $\nu(\text{diag}(t_1, t_2, t_3)) = \sum_{i=1}^3 \nu_i t_i$ with $\nu_i \in \mathbf{C}$ and ρ is the half-sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$ for P_0 , given as follows. For $i < j$ ($1 \leq i, j \leq 3$), we put $\eta_{ij}(a) = a_i/a_j$ for $a = \text{diag}(a_1, a_2, a_3)$ ($a_1 a_2 a_3 = 1$). Then we have $a^{2\rho} = \prod_{i < j} a_i/a_j = a_1^2/a_3^2 = a_1^4 a_2^2$ by definition. Hence $a^\rho = a_1^2 a_2$.

Here the characters σ_j of M are identified as follows. The group M consisting of 4 elements is a finite abelian group of $(2, 2)$ type, and its elements except for the unity is given by the matrices

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since M is commutative, all the irreducible unitary representations of it is 1-dimensional. For any $\sigma \in \widehat{M}$, we have $\sigma^2 = 1$. Therefore the set $\widehat{M}$ consisting of

4 characters $\{\sigma_j : j = 0, 1, 2, 3\}$, where each σ_j , except for the trivial character σ_0 , is specified by the following table of values at the elements m_i .

	m_1	m_2	m_3
σ_1	1	-1	-1
σ_2	-1	1	-1
σ_3	-1	-1	1

Proposition (1.1) (i) *If σ is the trivial character of M , the representation $\pi_{\sigma,\nu}$ is spherical or class 1, i.e., it has a (unique) K -invariant vector in the representation space $H_{\sigma,\nu}$.*

(ii) *If σ is not trivial, then the minimal K -type of the restriction $\pi_{\sigma,\nu}|_K$ to K is a 3-dimensional representation of $K = SO(3)$, which is isomorphic to the unique standard one (τ_2, V_2) . The multiplicity of this minimal K -type is one:*

$$\dim_{\mathbb{C}} \operatorname{Hom}_K(\tau_2, H_{\sigma,\nu}) = 1,$$

namely there is a unique non-zero K -homomorphism

$$\iota : (\tau_2, V_2) \rightarrow (\pi_{\sigma,\nu}|_K, H_{\sigma,\nu})$$

up to constant multiple.

2 Representations of $K = SO(3)$

2.1 The spinor covering

To describe the finite dimensional irreducible representations of $SO(3)$, the simplest way seems to utilize the double covering $s : SU(2) = Spin(3) \rightarrow SO(3)$, which is realized as follows.

The Hamilton quaternion algebra $\mathbf{H}$ is realized in $M_2(\mathbb{C})$ by

$$\mathbf{H} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbb{C}) \mid a, b \in \mathbb{C} \right\}.$$

Then $SU(2)$ is the subgroup of the multiplicative group consisting of quaternions with reduced norm 1, i.e.,

$$SU(2) = \{x \in \mathbf{H} \mid \det x = 1\}.$$

Let $\mathbf{P} = \{x \in \mathbf{H} \mid \operatorname{tr} x = 0\}$ be the 3-dimensional real Euclidean space consisting of pure quaternions. Then for each $x \in SU(2)$, the map

$$p \in \mathbf{P} \mapsto x \cdot p \cdot x^{-1} \in \mathbf{P}$$

preserve the Euclid norm $p \mapsto \det p$ and the orientation, hence we have a homomorphism

$$s : SU(2) \rightarrow SO(\mathbf{P}, \det) = SO(3),$$

which is surjective, since the range is a connected group. The kernel of this homomorphism is given by $\{\pm 1_2\}$.

By the derivation of $s : \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)$, the standard generators:

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

are mapped to $2K_1, 2K_2, 2K_3$ with

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{k},$$

respectively. Here $\mathfrak{k}$ is the Lie algebra of K .

2.2 Representations of $SU(2)$

The set of equivalence classes of the finite dimensional continuous representations of $SU(2)$ is exhausted by the symmetric tensor products τ_l ($l = 0, 1, \dots$) of the standard representation. These are realized as follows.

Let V_l be the subspace consisting of homogeneous polynomials of two variables x, y in the polynomial ring $\mathbb{C}[x, y]$. For $g \in SU(2)$ with $g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, and $f(x, y) \in V_l$ we set

$$\tau_l(g)f(x, y) := f(ax + by, -\bar{b}x + \bar{a}y).$$

Passing to the Lie algebra $\text{Lie}(SU(2)) = \mathfrak{su}(2)$, the derivation of τ_l , denoted by the same symbol, is described as follows by using the standard basis $\{v_k = x^k y^{l-k} \mid 0 \leq k \leq l\}$ and the standard generators

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Namely we have

$$\tau_l(u_1)v_k = \sqrt{-1}(l - 2k)v_k, \quad \tau_l(X_+)v_k = (l - k)v_{k+1}, \quad \tau_l(X_-)v_k = -k \cdot v_{k-1}.$$

Here we put $X_+ = \frac{1}{2}(u_2 + \sqrt{-1}u_3)$, $X_- = \frac{1}{2}(u_2 - \sqrt{-1}u_3)$.

The condition that τ_l defines a representation of $SO(3)$ by passing to the quotient with respect $s : SU(2) \rightarrow SO(3)$ is that $\tau_l(-1_2) = (-1)^l = +1$, i.e., l is even. Therefore the dimension of V_l , $l + 1$ is odd in this case.

The representation τ_2 of $SU(2)$ is equivalent to the spinor homomorphism. Hence passing to the quotient, τ_2 is equivalent to the tautological representation $SO(3) \rightarrow GL(3, \mathbb{C})$.

2.3 Irreducible components of $\tau_2 \otimes \tau_4$ and $\tau_2 \otimes Ad_{\mathfrak{p}}$

For our later use, we want to specify the standard basis of the unique irreducible constituent τ_2 in the tensor product $\tau_2 \otimes \tau_4$.

Lemma (2.1) Let $\{v_i \ (i = 0, 1, 2)\}$ and $\{w_j \ (0 \leq j \leq 4)\}$ be the standard basis of (τ_2, V_2) and (τ_4, V_4) , respectively. Then the elements

$$\begin{aligned} v'_0 &= v_0 \otimes w_2 - 2v_1 \otimes w_1 + v_2 \otimes w_0, \\ v'_1 &= v_0 \otimes w_3 - 2v_1 \otimes w_2 + v_2 \otimes w_1, \\ v'_2 &= v_0 \otimes w_4 - 2v_1 \otimes w_3 + v_2 \otimes w_2 \end{aligned}$$

define a set of standard basis in $\tau_2 \subset \tau_2 \otimes \tau_4$, which is unique up to a common scalar multiple.

2.4 The K -module isomorphism between $\mathfrak{p}_\mathbb{C}$ and V_4

We denote by $\mathfrak{p}_\mathbb{C}$ the complexification of the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ with respect to the Killing form, on which the group K acts via the adjoint action $Ad_\mathfrak{p}$. We denote by E_{ij} the matrix unit with 1 at (i, j) -th entry and 0 at other entries. Then E_{ii} and $E_{ij} + E_{ji}$ are considered as elements in $\mathfrak{p}$. We set $H_{ij} = E_{ii} - E_{jj}$ for $i \neq j$.

Lemma (2.2) Via the unique isomorphism V_4 and $\mathfrak{p}_\mathbb{C}$ as K -modules we have the identification

$$\begin{aligned} w_0 &= -2\{H_{23} - \sqrt{-1}(E_{23} + E_{32})\}, \\ w_1 &= \sqrt{-1}\{(E_{12} + E_{21}) - \sqrt{-1}(E_{13} + E_{31})\}, \\ w_2 &= \frac{2}{3}(H_{12} + H_{13}), \\ w_3 &= \sqrt{-1}\{(E_{12} + E_{21}) + \sqrt{-1}(E_{13} + E_{31})\}, \\ w_4 &= -2\{H_{23} + \sqrt{-1}(E_{23} + E_{32})\}. \end{aligned}$$

3 Principal series $(\mathfrak{g}, K)$ -modules

3.1 The case of the class one principal series

3.1.1 The Capelli elements

A set of generators for the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{sl}_3$ is obtained as Capelli elements, because $\mathfrak{sl}_3$ is of type A_2 .

Let

$$E'_{ii} = E_{ii} - \frac{1}{3}\left(\sum_{a=1}^3 E_{aa}\right), \quad E'_{ij} = E_{ij} \ (i \neq j).$$

Then $E'_{ij} \in \mathfrak{g}$. Define a matrix C of size 3 with entries in $\mathfrak{g}$ by

$$C = \begin{pmatrix} E'_{11} & E'_{12} & E'_{13} \\ E'_{21} & E'_{22} & E'_{23} \\ E'_{31} & E'_{32} & E'_{33} \end{pmatrix} - \text{diag}(-1, 0, 1).$$

Then for

$$\mathcal{A} = (A_{ij})_{1 \leq i, j \leq 3} = x \cdot 1_3 - C \in M_3(\mathfrak{g}[x]) \subset M_3(U(\mathfrak{g})[x]),$$

we define its *vertical* determinant by

$$\det \downarrow (\mathcal{A}) = \sum_{\sigma \in \mathfrak{S}_3} \text{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} A_{3\sigma(3)}.$$

Then it is written in the form $x^3 + Cp_2x - Cp_3 \in U(\mathfrak{g})[x]$ with some elements Cp_2 and Cp_3 in $Z(\mathfrak{g})$.

Proposition (3.1) *The set $\{Cp_2, Cp_3\}$ is a system of independent generators of $Z(\mathfrak{g})$. Here are explicit formulae of Cp_2 and Cp_3 :*

$$\begin{aligned} Cp_2 &= (E'_{11} - 1)E'_{22} + E'_{22}(E'_{33} + 1) + (E'_{11} - 1)(E'_{33} + 1) \\ &\quad - E_{23}E_{32} - E_{13}E_{31} - E_{12}E_{21}, \\ Cp_3 &= (E'_{11} - 1)E'_{22}(E'_{33} + 1) + E_{12}E_{23}E_{31} + E_{13}E_{21}E_{32} \\ &\quad - (E'_{11} - 1)E_{23}E_{32} - E_{13}E'_{22}E_{31} - E_{12}E_{21}(E'_{33} + 1). \end{aligned}$$

Eigenvalues of Cp_2, Cp_3

We compute the value $Cp_2f_0(e)$ and $Cp_3f_0(e)$. Let $S_2(a, b, c) = ab + bc + ca$ and $S_3(a, b, c) = abc$ be the elementary symmetric functions of three variables of degree 2 and 3, respectively. Then we have the following.

Proposition (3.2) *The infinitesimal character of $\pi_{\sigma_0, \nu}$ is given by*

$$Cp_2f_0 = S_2\left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)\right)f_0$$

and

$$Cp_3f_0 = S_3\left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)\right)f_0.$$

3.2 $(\mathfrak{g}, K)$ -module structure of non-spherical principal series at the minimal K -type

3.2.1 Construction of K -equivariant differential operators

Lemma (3.3) *Let $\{f_i \ (i = 0, 1, 2)\}$ be the set of the standard basis of the minimal K -type $\tau \subset \pi_{\sigma, \nu}$ of a non-spherical principal series representation $\pi_{\sigma, \nu} = \pi$. Define another three C^∞ -elements $\{\varphi_i \ (i = 0, 1, 2)\}$ by the formulae:*

$$\begin{aligned} \varphi_0 &= \frac{2}{3}\pi(2E_{11} - E_{22} - E_{33})f_0 \\ &\quad - 2\sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31}))f_1 \\ &\quad - 2\pi(E_{12} + E_{21} - \sqrt{-1}(E_{23} + E_{32}))f_2, \\ \varphi_1 &= \sqrt{-1}\pi(E_{12} + E_{21} + \sqrt{-1}(E_{13} + E_{31}))f_0 \\ &\quad - \frac{4}{3}\pi(2E_{11} - E_{22} - E_{33})f_1 \\ &\quad + \sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31}))f_2, \\ \varphi_2 &= -2\pi(E_{22} - E_{33} + \sqrt{-1}(E_{23} + E_{32}))f_0 \\ &\quad - 2\sqrt{-1}\pi(E_{12} + E_{21}) + \sqrt{-1}(E_{13} + E_{31}))f_1 \\ &\quad + \frac{2}{3}\pi(2E_{11} - E_{22} - E_{33})f_2. \end{aligned}$$

Then $(\varphi_0, \varphi_1, \varphi_2)$ is a constant multiple of (f_0, f_1, f_2) .

3.2.2 Computation of eigenvalues

The previous lemma tells that there exist a scalar $\lambda(\sigma, \nu)$ depending on σ and ν such that $\varphi_i = \lambda(\sigma, \nu)f_i$ ($i = 0, 1, 2$). We determine this eigenvalue $\lambda(\sigma, \nu)$ by using explicit models of the principal series $\pi_{\sigma, \nu}$.

To do this, we have to find functions in

$$L^2\text{-Ind}_M^K(\sigma_i) = L_{M, \sigma_i}^2(K) = \{f \in L^2(K) | f(mk) = \sigma(m)f(k) \text{ for all } m \in M, k \in K\}$$

corresponding to the standard basis in the minimal K -type for each i .

In the larger space $L^2(K)$, the τ_2 -isotypic component is generated by the 9 matrix elements $s_{ij}(k)$ ($1 \leq i, j \leq 3$) of the tautological representation

$$k \in K \mapsto S(k) = (s_{ab}(k))_{1 \leq a, b \leq 3} \in SO(3).$$

It is directly confirmed that $s_{ib}(k)$ ($b = 0, 1, 2$) belong to the subspace $L_{M, \sigma_i}^2(K)$ for each i .

Diagonalizing the action of u_1 , we find that s_{i1} corresponds to v_1 for each i . And finally we find that the standard basis is given by

$$v_0 = \sqrt{-1}(s_{i2} - \sqrt{-1}s_{i3}), \quad v_1 = s_{i1}, \quad \text{and} \quad v_2 = \sqrt{-1}(s_{i2} + \sqrt{-1}s_{i3}).$$

We need the values of these standard functions $f_a(k) = v_a$ ($a = 0, 1, 2$) at the identity $e \in K$.

Lemma (3.4) *The values of the standard functions at $e \in K$ is given as follows.*

1. If $\sigma = \sigma_1$, $(f_0(e), f_1(e), f_2(e)) = (0, 1, 0)$.
2. If $\sigma = \sigma_2$, $(f_0(e), f_1(e), f_2(e)) = (\sqrt{-1}, 0, \sqrt{-1})$.
3. If $\sigma = \sigma_3$, $(f_0(e), f_1(e), f_2(e)) = (1, 0, -1)$.

Now we can proceed to the computation of the value $\lambda(\sigma_i, \nu)$.

Lemma (3.5)

$$\lambda(\sigma_1, \nu) = -\frac{4}{3}(2\nu_1 - \nu_2), \quad \lambda(\sigma_2, \nu) = \frac{4}{3}(\nu_1 - 2\nu_2), \quad \lambda(\sigma_3, \nu) = \frac{4}{3}(\nu_1 + \nu_2).$$

Summing up the lemmata in this section, we have the following.

Proposition (3.6) *Let $\{f_i$ ($i = 0, 1, 2$)* be the set of the standard basis of the minimal K -type $\tau \subset \pi_{\sigma, \nu}$ of a non-spherical principal series representation $\pi_{\sigma, \nu} = \pi$. Define another three C^∞ -elements $\{\varphi_i$ ($i = 0, 1, 2$) by the formulae:

$$\begin{aligned} \varphi_0 &= \frac{2}{3}\pi(H_{12} + H_{13})f_0 \\ &\quad - 2\sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31}))f_1 \\ &\quad - 2\pi(H_{23} - \sqrt{-1}(2E_{23} + \frac{1}{2}u_1))f_2, \\ \varphi_1 &= \sqrt{-1}\pi(E_{12} + E_{21} + \sqrt{-1}(E_{13} + E_{31}))f_0 \\ &\quad - \frac{4}{3}\pi(H_{12} + H_{13})f_1 \\ &\quad + \sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31}))f_2, \\ \varphi_2 &= -2\pi(H_{23} + \sqrt{-1}(2E_{23} + \frac{1}{2}u_1))f_0 \\ &\quad - 2\sqrt{-1}\pi(E_{12} + E_{21}) + \sqrt{-1}(E_{13} + E_{31}))f_1 \\ &\quad + \frac{2}{3}\pi(H_{12} + H_{13})f_2. \end{aligned}$$

Then we have

$$(\varphi_0, \varphi_1, \varphi_2) = \lambda(\sigma_i, \nu)(f_0, f_1, f_2)$$

with eigenvalue $\lambda(\sigma_i, \nu)$ given by

$$\lambda(\sigma_1, \nu) = -\frac{4}{3}(2\nu_1 - \nu_2), \quad \lambda(\sigma_2, \nu) = \frac{4}{3}(\nu_1 - 2\nu_2), \quad \lambda(\sigma_3, \nu) = \frac{4}{3}(\nu_1 + \nu_2).$$

In the next section, we consider the Whittaker realization of the equation of the above proposition. Then we need the following Iwasawa decomposition of standard elements of $\mathfrak{g}$.

Lemma (3.7) *We have the following decomposition of standard generators of $\mathfrak{g}$ with respect to the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$. For $H_{ij} \in \mathfrak{a}$ we have*

$$H_{ij} = 0 + H_{ij} + 0.$$

Since $E_{ij} + E_{ji} = 2E_{ij} - (E_{ij} - E_{ji})$, we have

$$E_{12} + E_{21} = 2E_{12} + 0 + K_3, \quad E_{13} + E_{31} = 2E_{13} + 0 + (-K_2), \quad E_{23} + E_{32} = 2E_{23} + 0 + K_1.$$

4 The holonomic system for the A -radial part of the principal series Whittaker functions

4.1 The case of the class one principal series

Let I be a non-zero Whittaker functional from the class one principal series $\pi_{\sigma_0, \nu}$ to $C^\infty\text{-Ind}_N^G(\psi)$. Let F be the restriction of the image $I(f_0)$ of the K -fixed vector f_0 to A . We write here the holonomic system for F with respect to the variables $y_1 = \eta_{12}(a) = a_1/a_2$, $y_2 = \eta_{23}(a) = a_2/a_3 = a_1/a_2^2$.

Proposition (4.1) *Put $F(y_1, y_2) = y_1 y_2 G(y_1, y_2)$ (note $a^\rho = y_1 y_2$). Then $G(y_1, y_2)$ satisfies the partial differential equations:*

$$\Delta_2 G = \frac{1}{3}(\nu_1^2 + \nu_2^2 - \nu_1 \nu_2)G$$

and

$$\{\partial_1(\partial_1 - \partial_2)\partial_2 + 4\pi^2 c_2^2 y_2^2 \partial_1 - 4\pi^2 c_1^2 y_1^2 \partial_2\}G = -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2)G.$$

Here ∂_i is the Euler operator $y_i \frac{\partial}{\partial y_i}$ for $i = 1, 2$. and we write

$$\Delta_2 = (\partial_1^2 + \partial_2^2 - \partial_1 \partial_2) - 4\pi^2(c_1^2 y_1^2 + c_2^2 y_2^2).$$

Remark From these equations for the monodromy exponents α_1, α_2 at the origin $y_1 = 0$, $y_2 = 0$, we have an equality of sets of complex numbers:

$$\{\alpha_1, -\alpha_1 + \alpha_2, -\alpha_2\} = \left\{\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), -\frac{1}{3}(\nu_1 + \nu_2)\right\}.$$

4.2 The holonomic system for the A -radial part of non-spherical Whittaker functions

Let I be a non-zero Whittaker functional from the principal series $\pi_{\sigma, \nu}$. For the set $\{f_i | (i = 0, 1, 2)\}$ of standard functions, we put $F_i = I(f_i)$.

Theorem (4.4) *Let $F(a) = {}^t(F_0(a), F_1(a), F_2(a)) = (y_1 y_2)^t(G_0(y), G_1(y), G_2(y))$ be the vector of the A -radial part of the standard Whittaker functions with minimal K -type of the principal series representation $\pi_{\sigma, \nu}$ with non-trivial $\sigma = \sigma_i$. Then it satisfies the following partial differential equations:*

(i):

$$\begin{pmatrix} \partial_1 & 4\pi c_1 y_1 & \partial_1 - 2\partial_2 - 4\pi c_2 y_2 \\ -2\pi c_1 y_1 & -2\partial_1 & -2\pi c_1 y_1 \\ \partial_1 - 2\partial_2 + 4\pi c_2 y_2 & 4\pi c_1 y_1 & \partial_1 \end{pmatrix} \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix} = \frac{1}{2} \lambda_i \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix},$$

(ii):

$$\Delta_2 \cdot \mathbf{1}_3 \cdot \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix} - 2\pi c_2 y_2 \begin{pmatrix} G_0(y) \\ 0 \\ -G_2(y) \end{pmatrix} + 2\pi c_1 y_1 \begin{pmatrix} G_1(y) \\ \frac{1}{2}(G_0(y) + G_2(y)) \\ G_1(y) \end{pmatrix} = \frac{1}{3} \mu \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix}.$$

Moreover the eigenvalues λ_i and μ depending on the representation $\pi_{\sigma, \nu}$ are given by

$$\begin{cases} \lambda_1 = -\frac{4}{3}(2\nu_1 - \nu_2) & (\sigma = \sigma_1) \\ \lambda_2 = \frac{4}{3}(\nu_1 - 2\nu_2) & (\sigma = \sigma_2) \\ \lambda_3 = \frac{4}{3}(\nu_1 + \nu_2) & (\sigma = \sigma_3) \end{cases} \quad \text{and} \quad \mu = \nu_1^2 + \nu_2^2 - \nu_1 \nu_2.$$

Remark We can write the differential equations (i) and (ii) of the above Theorem as

$$(i): \quad \mathcal{D}_1 G = \lambda_i G \quad (ii): \quad \mathcal{D}_2 G = \mu G,$$

with $\mathcal{D}_i$ ($i = 1, 2$) 3 by 3 matrix-valued differential operators. Then we have

$$\mathcal{D}_1 \cdot \mathcal{D}_2 - \mathcal{D}_2 \cdot \mathcal{D}_1 = 0.$$

4.3 The equations via the tautological basis

Let $k \in K \mapsto S(k) = (s_{ij}(k))_{1 \leq i, j \leq 3}$ be the tautological representation of $K = SO(3)$. Let $I \in \text{Hom}_{\mathfrak{g}, K}(\pi_{\sigma, \nu}, \text{Ind}_N^G(\psi))$ be a Whittaker functional and define function T_{ij} on A by

$$I(s_{ij})|_A = y_1 y_2 T_{ij}(y) \quad (1 \leq i, j, \leq 3).$$

Then

$$\begin{pmatrix} G_0 \\ G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1} & 1 \\ 1 & 0 & 0 \\ 0 & \sqrt{-1} & -1 \end{pmatrix} \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix}.$$

Then for each i , the equation (i) of the above theorem is transformed to

$$\begin{pmatrix} -\partial_1 & -2\pi\sqrt{-1}c_1y_1 & 0 \\ -2\pi\sqrt{-1}c_1y_1 & \partial_1 - \partial_2 & -2\pi\sqrt{-1}c_2y_2 \\ 0 & -2\pi\sqrt{-1}c_2y_2 & \partial_2 \end{pmatrix} \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix} = \frac{1}{2}\lambda_i \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix},$$

and the equation (ii) to

$$\left[\Delta_2 \cdot 1_3 + \begin{pmatrix} 0 & 2\pi\sqrt{-1}c_1y_1 & 0 \\ -2\pi\sqrt{-1}c_1y_1 & 0 & 2\pi\sqrt{-1}c_2y_2 \\ 0 & -2\pi\sqrt{-1}c_2y_2 & 0 \end{pmatrix} \right] \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix} = \frac{1}{3}\mu \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix}.$$

5 Power series solutions at the origin

We determine 6 linearly independent formal power series at the origin $(y_1, y_2) = (0, 0)$ for generic parameter ν in this section. These formal solutions converges because the singularity at the origin is a regular singularity. These solutions do not have exponential decay at infinity, different from the unique ‘good’ solution given by Jacquet integral. We refer to these solutions as *secondary* Whittaker functions sometimes.

5.1 The case of the class one principal series

This case is more or less discussed in the paper of Bump [2], up to some difference of notations. We omit its explicit formula.

An integral expression of this power series solution was found by Stade ([9, Lemma 3.10], [11, Theorem 2]) as an analogue of an integral formula for Jacquet integral by Vinogradov and Takhadzhyan [12]. The same as non-spherical case discussed later, we let $\{e_1, e_2, e_3\}$ be a permutation of the three complex numbers $\{-\frac{1}{3}(2\nu_1 - \nu_2), -\frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)\} = \{\frac{1}{4}\lambda_1, \frac{1}{4}\lambda_2, \frac{1}{4}\lambda_3\}$

Theorem (5.2) For $\text{Re}(e_2 - e_1) > 2$,

$$\begin{aligned} \Phi(y_1, y_2) &= \Gamma\left(\frac{e_2 - e_1}{2} + 1\right) \Gamma\left(\frac{e_3 - e_1}{2} + 1\right) \Gamma\left(\frac{e_2 - e_3}{2} + 1\right) (\pi c_1 y_1)^{\frac{e_3}{2}} (\pi c_2 y_2)^{-\frac{e_3}{2}} (\pi c_1)^{e_1} (\pi c_2)^{-e_2} \\ &\quad \cdot \frac{1}{2\pi\sqrt{-1}} \int_{|u|=1} I_{\frac{e_2 - e_1}{2}}(2\pi c_1 y_1 \sqrt{1 + 1/u}) I_{\frac{e_2 - e_1}{2}}(2\pi c_2 y_2 \sqrt{1 + u}) u^{-\frac{3}{4}e_3} \frac{du}{u}. \end{aligned}$$

5.2 The case of the non-spherical principal series

In this case also, the holonomic system obtained in Theorem (4.4) has regular singularities at the origin $(y_1, y_2) = (0, 0)$ with rank 6, i.e., the order of the Weyl group of $SL(3, \mathbf{R})$, for generic values of parameter ν . We determine the characteristic indices and the convergent formal power series solutions at $y = 0$. Here to abridge the notation, we write the set of variables (y_1, y_2) as y collectively.

By inspection we find that it is convenient to introduce scalar functions $\Phi_i(y_1, y_2)$ ($i = 0, 1, 2$) by

$$F(y) = y_1 y_2 G(y) = y_1 y_2 \left\{ \Phi_0(y) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \Phi_1(y) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \Phi_2(y) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

5.3 The holonomic system for $\Phi_i(y)$

Now we can rewrite the holonomic system for G_i to that for Φ_i .

Proposition (5.3) *The holonomic system in Theorem (4.4) is equivalent to the following system for $\Phi_i = \Phi_i(y_1, y_2)$ ($i = 0, 1, 2$).*

- (1) (i) $[\partial_1 + \frac{1}{4}\lambda_i]\Phi_0 + (2\pi c_1 y_1)\Phi_1 = 0,$
(ii) $[\partial_1 - \partial_2 - \frac{1}{4}\lambda_i]\Phi_1 + (2\pi c_1 y_1)\Phi_0 + (2\pi c_2 y_2)\Phi_2 = 0,$
(iii) $[\partial_2 - \frac{1}{4}\lambda_i]\Phi_2 - (2\pi c_2 y_2)\Phi_1 = 0,$
- (2) (i) $[\Delta_2 - \frac{1}{3}\mu]\Phi_0 + (2\pi c_1 y_1)\Phi_1 = 0,$
(ii) $[\Delta_2 - \frac{1}{3}\mu]\Phi_1 + (2\pi c_1 y_1)\Phi_0 - (2\pi c_2 y_2)\Phi_2 = 0,$
(iii) $[\Delta_2 - \frac{1}{3}\mu]\Phi_2 - (2\pi c_2 y_2)\Phi_1 = 0.$

5.4 The characteristic indices at the origin $(y_1, y_2) = (0, 0)$ and the recurrence formulae.

Let

$$\Phi_k(y) = y_1^{-e_1} y_2^{e_2} \sum_{n_1, n_2 \geq 0} c_{k; n_1, n_2} (\pi c_1 y_1)^{n_1} (\pi c_2 y_2)^{n_2}, \quad (k = 0, 1, 2)$$

be a system of formal power series solutions at the origin $y = 0$.

Now we can determine the 6 pairs $(-e_1, e_2)$ of characteristic indices at the origin, and the corresponding initial values conditions for F or Φ_i . the system at the origin and to determine the first coefficients Moreover we have the recurrence relations between the coefficients.

Lemma (5.4) *When $\sigma = \sigma_i$ for $i = 1, 2$ or 3 , we have the following:*

- (1) *The characteristic indices take the six values:*

$$(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_l) \quad (1 \leq k \neq l \leq 3).$$

- (2) *For each case, the set of first coefficients, or the initial values at the origin are given as follows:*

- (i) *If $(-e_1, e_2) = (-\frac{1}{4}\lambda_i, \frac{1}{4}\lambda_k) \quad (k \neq i),$*

$$(y_1^{e_1} y_2^{-e_2} G)(0, 0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ i.e., } (y_1^{e_1} y_2^{-e_2} \Phi_0)(0, 0) = 1, \text{ and } (y_1^{e_1} y_2^{-e_2} \Phi_j)(0, 0) =$$

0 for other j .

- (ii) *If $(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_l) \quad (k \neq i, l \neq i, k \neq l),$*

$$(y_1^{e_1} y_2^{-e_2} G)(0, 0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \text{ i.e., } (y_1^{e_1} y_2^{-e_2} \Phi_1)(0, 0) = 1, \text{ and } (y_1^{e_1} y_2^{-e_2} \Phi_j)(0, 0) =$$

0 for other j .

(iii) If $(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_i)$ ($k \neq i$),

$$(y_1^{e_1} y_2^{-e_2} G)(0, 0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \text{ i.e., } (y_1^{e_1} y_2^{-e_2} \Phi_2)(0, 0) = 1, \text{ and } (y_1^{e_1} y_2^{-e_2} \Phi_j)(0, 0) = 0 \text{ for other } j.$$

(3) We have the following recurrence relations for the coefficients:

- (i) $(n_1 - e_1 + \frac{1}{4}\lambda_i)c_{0;n_1,n_2} + 2c_{1;n_1-1,n_2} = 0;$
- (ii) $(n_1 - n_2 - e_1 - e_2 - \frac{1}{4}\lambda_i)c_{1;n_1,n_2} + 2c_{0;n_1-1,n_2} + 2c_{2;n_1,n_2-1} = 0;$
- (iii) $(n_2 + e_2 - \frac{1}{4}\lambda_i)c_{2;n_1,n_2} - 2c_{1;n_1,n_2-1} = 0.$

5.5 Power series solutions at the origin

Now we can show the following formulae for the power series solutions.

Theorem (5.5) Assume that $\frac{1}{4}(\lambda_k - \lambda_l) \notin \mathbb{Z}$. Then we have the following.

(I) When $\sigma = \sigma_1$ we have the following six independent solutions.

$$\begin{aligned} {}^t(\Phi_0^{1,I}, \Phi_1^{1,I}, \Phi_2^{1,I}) &= y_1^{-\frac{\lambda_1}{4}} y_2^{\frac{\lambda_2}{4}} \\ &\left(\sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1+m_2}}{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1} (\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_2 - \lambda_3}{8} + 1)_{m_2}} \right. \\ &\quad - \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1+1} (\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1} (\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_1+1} (\frac{\lambda_2 - \lambda_3}{8} + 1)_{m_2}} \\ &\quad \left. - \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1+1} (\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_2+1}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1} (\pi c_2 y_2)^{2m_2+1}}{m_1! m_2! (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_1+1} (\frac{\lambda_2 - \lambda_3}{8} + 1)_{m_2}} \right) \\ {}^t(\Phi_0^{1,III}, \Phi_1^{1,III}, \Phi_2^{1,III}) &= y_1^{-\frac{\lambda_2}{4}} y_2^{\frac{\lambda_3}{4}} \\ &\left(\sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1+m_2}}{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1} (\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1+1} (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_2}} \right. \\ &\quad - \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1+m_2}}{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1} (\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_2}} \\ &\quad \left. - \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1+m_2}}{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1} (\pi c_2 y_2)^{2m_2+1}}{m_1! m_2! (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_2+1}} \right) \\ {}^t(\Phi_0^{1,V}, \Phi_1^{1,V}, \Phi_2^{1,V}) &= y_1^{-\frac{\lambda_2}{4}} y_2^{\frac{\lambda_1}{4}} \\ &\left(- \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1+1} (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_2+1}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1} (\pi c_2 y_2)^{2m_2+1}}{m_1! m_2! (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_1 - \lambda_3}{8} + \frac{1}{2})_{m_2+1}} \right. \\ &\quad - \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_2+1}} \cdot \frac{(\pi c_1 y_1)^{2m_1} (\pi c_2 y_2)^{2m_2+1}}{m_1! m_2! (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_1 - \lambda_3}{8} + \frac{1}{2})_{m_2+1}} \\ &\quad \left. - \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1+m_2}}{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1} (\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_1 - \lambda_3}{8} + \frac{1}{2})_{m_2}} \right) \end{aligned}$$

and other three solutions $\Phi_i^{1,II}$, $\Phi_i^{1,IV}$ and $\Phi_i^{1,VI}$ are given by exchanging the role of λ_2 and λ_3 in the expression for $\Phi_i^{1,I}$, $\Phi_i^{1,III}$ and $\Phi_i^{1,V}$, respectively.

(II) When $\sigma = \sigma_2$, exchange λ_1 and λ_2 in the part (I).

(III) When $\sigma = \sigma_3$, exchange λ_1 and λ_3 in the part (I).

5.6 Integral representations of the secondary Whittaker functions

In this subsection, we rewrite the power series solutions of the previous subsection by integral expressions.

Theorem (5.6) (I) When $\sigma = \sigma_1$ we have

$$\begin{aligned} {}^t(\Phi_0^{1,I}, \Phi_1^{1,I}, \Phi_2^{1,I}) &= (\pi c_1 y_1)^{\frac{\lambda_3}{8} + \frac{1}{2}} (\pi c_2 y_2)^{-\frac{\lambda_3}{8} + \frac{1}{2}} \\ &\cdot (2\pi\sqrt{-1})^{-1} \Gamma\left(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_2 - \lambda_3}{8} + 1\right) (\pi c_1)^{\frac{\lambda_1}{4}} (\pi c_2)^{-\frac{\lambda_2}{4}} \\ &\cdot \left(\begin{aligned} &\int_{|u|=1} I_{\frac{\lambda_2 - \lambda_1}{8} - \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_2 - \lambda_1}{8} - \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 + \frac{1}{4}} \frac{du}{u} \\ &(-1) \int_{|u|=1} I_{\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_2 - \lambda_1}{8} - \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{4}} (1+u)^{\frac{1}{2}} \frac{du}{u} \\ &(-1) \int_{|u|=1} I_{\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{4}} \frac{du}{u} \end{aligned} \right) \end{aligned}$$

for $\operatorname{Re}(\frac{\lambda_2 - \lambda_1}{8}) > \frac{3}{2}$,

$$\begin{aligned} {}^t(\Phi_0^{1,III}, \Phi_1^{1,III}, \Phi_2^{1,III}) &= (\pi c_1 y_1)^{\frac{\lambda_1}{8}} (\pi c_2 y_2)^{-\frac{\lambda_1}{8}} \\ &\cdot (2\pi\sqrt{-1})^{-1} \Gamma\left(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_3 - \lambda_2}{8} + 1\right) (\pi c_1)^{\frac{\lambda_2}{4}} (\pi c_2)^{-\frac{\lambda_3}{4}} \\ &\cdot \left(\begin{aligned} &(\pi c_1 y_1) \int_{|u|=1} I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{2}} \frac{du}{u} \\ &(-1) \int_{|u|=1} \left[\pi c_1 y_1 \sqrt{1+1/u} I_{\frac{\lambda_3 - \lambda_2}{8} - 1}(2\pi c_1 y_1 \sqrt{1+1/u}) + \left(\frac{\lambda_1 - \lambda_3}{8} + \frac{1}{2}\right) \right. \\ &\quad \left. \cdot I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_1 y_1 \sqrt{1+1/u}) \right] I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{2}} \frac{du}{u} \\ &(-1)(\pi c_2 y_2) \int_{|u|=1} I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 + \frac{1}{2}} \frac{du}{u} \end{aligned} \right) \end{aligned}$$

for $\operatorname{Re}(\frac{\lambda_3 - \lambda_2}{8}) > 1$,

$$\begin{aligned} {}^t(\Phi_0^{1,V}, \Phi_1^{1,V}, \Phi_2^{1,V}) &= (\pi c_1 y_1)^{\frac{\lambda_3}{8} + \frac{1}{2}} (\pi c_2 y_2)^{-\frac{\lambda_3}{8} + \frac{1}{2}} \\ &\cdot (2\pi\sqrt{-1})^{-1} \Gamma\left(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_1 - \lambda_3}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_3 - \lambda_2}{8} + 1\right) (\pi c_1)^{\frac{\lambda_2}{4}} (\pi c_2)^{-\frac{\lambda_1}{4}} \\ &\cdot \left(\begin{aligned} &(-1) \int_{|u|=1} I_{\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 + \frac{1}{4}} \frac{du}{u} \\ &\int_{|u|=1} I_{\frac{\lambda_1 - \lambda_2}{8} - \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{4}} (1+u)^{\frac{1}{2}} \frac{du}{u} \\ &\int_{|u|=1} I_{\frac{\lambda_1 - \lambda_2}{8} - \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_1 - \lambda_2}{8} - \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{4}} \frac{du}{u} \end{aligned} \right) \end{aligned}$$

for $\operatorname{Re}(\frac{\lambda_1 - \lambda_2}{8}) > \frac{3}{2}$.

To have the integral expression for $\Phi_i^{1,II}, \Phi_i^{1,IV}$ and $\Phi_i^{1,VI}$, we have to exchange the role of λ_2 and λ_3 in the expression for $\Phi_i^{1,I}, \Phi_i^{1,III}$ and $\Phi_i^{1,V}$, respectively.

(II) When $\sigma = \sigma_2$, exchange λ_1 and λ_2 in (I).

(III) When $\sigma = \sigma_3$, exchange λ_1 and λ_3 in (I).

6 Evaluation of Jacquet integrals

We give explicit descriptions of Jacquet integrals for non-spherical principal series Whittaker functions here. These are similar to the class one case ([12]).

6.1 Jacquet integrals

Let us denote by $g = n(g)a(g)k(g)$ the Iwasawa decomposition of $g \in G$. We define Jacquet integral J_{ij} for $\sigma_i \in \widehat{M}$ ($1 \leq i, j \leq 3$) as

$$J_{ij}(g) = \int_N \psi(n)^{-1} a(s_0^{-1}ng) s_{ij}(k(s_0^{-1}ng)) dn$$

for $1 \leq j \leq 3$. Here

$$s_0 = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$$

the longest element in the Weyl group of $SL(3, \mathbf{R})$ and $s_{ij}(k)$ is the element of the tautological representation of K (cf. [4, (7.1)]).

Since

$$v_0 = \sqrt{-1}(s_{i2} - \sqrt{-1}s_{i3}), v_1 = s_{i1}, v_2 = \sqrt{-1}(s_{i2} + \sqrt{-1}s_{i3})$$

(§3.2.2) and

$$\Phi_0 = G_1, 2\Phi_1 = G_0 + G_2, 2\Phi_2 = G_0 - G_2,$$

(§5.2) the vector of integrals ${}^t(J_{i1}, \sqrt{-1}J_{i2}, J_{i3})$ has the same K -type as ${}^t(\Phi_0, \Phi_1, \Phi_2)$.

For an element $a \in A$, we use the coordinates $(y_1, y_2) = (a_1/a_2, a_1a_2^2)$. In the Iwasawa decomposition of the element $s_0^{-1}na$ its A -part $a(s_0^{-1}na)$ is given by

$$a(s_0^{-1}na) = \left(\frac{y_1^{\frac{1}{3}} y_2^{\frac{2}{3}}}{\sqrt{\Delta_1}}, \left(\frac{y_2}{y_1} \right)^{\frac{1}{3}} \sqrt{\frac{\Delta_1}{\Delta_2}} \right)$$

with

$$\Delta_1 = y_1^2 y_2^2 + y_1^2 n_2^2 + (n_1 n_2 - n_3)^2, \quad \Delta_2 = y_1^2 y_2^2 + y_2^2 n_1^2 + n_3^2.$$

Under the symbol above

$$J_{ij}(y) = y_1^{(2\nu_1 - \nu_2)/3 + 1} y_2^{(\nu_1 + \nu_2)/3 + 1} \cdot \int_{\mathbf{R}^3} \Delta_1^{(\nu_2 - \nu_1 - 1)/2} \Delta_2^{(-\nu_2 - 1)/2} k_{ij} \exp(-2\pi\sqrt{-1}(c_1 n_1 + c_2 n_2)) dn_1 dn_2 dn_3.$$

Here $(k_{ij})_{1 \leq i, j \leq 3} = k(s_0^{-1}na)$.

6.2 Integral representations of Jacquet integrals

To write down our results, we use the following notation.

Notation.

$$K(\alpha, \beta, \gamma, \delta; y) := 4\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}(y_1 y_2)(\pi|c_1|y_1)^{\frac{\lambda_2}{8}}(\pi|c_2|y_2)^{-\frac{\lambda_2}{8}} \\ \cdot \int_0^\infty K_{\frac{\lambda_3-\lambda_1}{8}+\alpha}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\lambda_3-\lambda_1}{8}+\beta}(2\pi|c_2|y_2\sqrt{1+v})v^{-\frac{3}{16}\lambda_2+\gamma}(1+v)^\delta \frac{dv}{v}$$

with $K_\nu(z)$ the K -Bessel function.

6.2.1 The case of the class one principal series

In the case of class one, the Jacquet integral $J_0(y)$ is

Theorem (6.2) ([12]) For $\operatorname{Re}(\lambda_2 - \lambda_1) > 0$, $\operatorname{Re}(\lambda_3 - \lambda_2) > 0$,

$$J_0(y) = \frac{1}{\Gamma(\frac{\lambda_2-\lambda_1}{8} + \frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8} + \frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8} + \frac{1}{2})} K(0, 0, 0, 0; y).$$

6.2.2 The case of the non-spherical principal series

Theorem (6.3) For $\operatorname{Re}(\lambda_2 - \lambda_1) > 0$, $\operatorname{Re}(\lambda_3 - \lambda_2) > 0$, the Jacquet integrals J_{ij} can be written as follows.

$$\begin{pmatrix} J_{11}(y) \\ J_{12}(y) \\ J_{13}(y) \end{pmatrix} = \frac{(\pi|c_1|)^{\frac{1}{2}}(\pi|c_2|)^{\frac{1}{2}}(\pi|c_1|y_1)^{\frac{1}{2}}(\pi|c_2|y_2)^{\frac{1}{2}}}{\Gamma(\frac{\lambda_2-\lambda_1}{8} + 1)\Gamma(\frac{\lambda_3-\lambda_2}{8} + \frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8} + 1)} \cdot \begin{pmatrix} \varepsilon_1 \varepsilon_2 K(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, 0; y) \\ -\sqrt{-1} \varepsilon_2 K(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{2}; y) \\ -K(\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, 0; y) \end{pmatrix} \\ \begin{pmatrix} J_{21}(y) \\ J_{22}(y) \\ J_{23}(y) \end{pmatrix} = \frac{1}{\Gamma(\frac{\lambda_2-\lambda_1}{8} + 1)\Gamma(\frac{\lambda_3-\lambda_2}{8} + 1)\Gamma(\frac{\lambda_3-\lambda_1}{8} + \frac{1}{2})} \cdot \begin{pmatrix} -\sqrt{-1} \varepsilon_1 K(0, 0, -\frac{1}{2}, 0; y) \\ -K(0, 0, \frac{1}{2}, -1; y) \\ \sqrt{-1} \varepsilon_2 K(0, 0, \frac{1}{2}, 0; y) \end{pmatrix} \\ \begin{pmatrix} J_{31}(y) \\ J_{32}(y) \\ J_{33}(y) \end{pmatrix} = \frac{(\pi|c_1|)^{\frac{1}{2}}(\pi|c_2|)^{\frac{1}{2}}(\pi|c_1|y_1)^{\frac{1}{2}}(\pi|c_2|y_2)^{\frac{1}{2}}}{\Gamma(\frac{\lambda_2-\lambda_1}{8} + \frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8} + 1)\Gamma(\frac{\lambda_3-\lambda_1}{8} + 1)} \cdot \begin{pmatrix} -K(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, 0; y) \\ \sqrt{-1} \varepsilon_1 K(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2}; y) \\ \varepsilon_1 \varepsilon_2 K(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{4}, 0; y) \end{pmatrix}.$$

Here ε_i ($i = 1, 2$) means 1 if $c_i > 0$ and -1 if $c_i < 0$.

7 Integral expression of Mellin-Barnes type

As in [9], we consider the Mellin-Barnes integral expression for $J_{ij}(y)$ to find linear relations between Jacquet integrals J_{ij} and power series solutions $\Phi_k^{i,*}$. We discuss only the non-spherical case.

Lemma (7.1) For $p, q \in \mathbb{C}$,

$$\begin{aligned} & (\pi|c_1|y_1)^p(\pi|c_2|y_2)^q \int_0^\infty K_\alpha(2\pi|c_1|y_1\sqrt{1+1/v})K_\beta(2\pi|c_2|y_2\sqrt{1+v})v^\gamma(1+v)^\delta \frac{dv}{v} \\ &= \frac{1}{2^4(2\pi\sqrt{-1})^2} \int_{\rho_1-\sqrt{-1}\infty}^{\rho_1+\sqrt{-1}\infty} \int_{\rho_2-\sqrt{-1}\infty}^{\rho_2+\sqrt{-1}\infty} V_0(s_1, s_2)(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2} ds_1 ds_2, \end{aligned}$$

$$V_0(s_1, s_2) = \frac{\Gamma(\frac{s_1+p+\alpha}{2})\Gamma(\frac{s_1+p-\alpha}{2})\Gamma(\frac{s_1+p+2\gamma}{2})\Gamma(\frac{s_2+q+\beta}{2})\Gamma(\frac{s_2+q-\beta}{2})\Gamma(\frac{s_2+q-2\gamma-2\delta}{2})}{\Gamma(\frac{s_1+s_2+p+q}{2}-\delta)}.$$

Here the lines of integration are taken as to the right of all poles of the integrand.

Proposition (7.2) Let

$$M(a_1, a_2, a_3; b_1, b_2, b_3; c; y) = \frac{1}{(2\pi\sqrt{-1})^2} \int_{\rho_1-\sqrt{-1}\infty}^{\rho_1+\sqrt{-1}\infty} \int_{\rho_2-\sqrt{-1}\infty}^{\rho_2+\sqrt{-1}\infty} V(s_1, s_2) (\pi|c_1|y_1)^{-s_1} (\pi|c_2|y_2)^{-s_2} ds_1 ds_2,$$

with

$$V(s_1, s_2) = \frac{\Gamma(\frac{s_1+a_1-\lambda_1}{2})\Gamma(\frac{s_1+a_2-\lambda_2}{2})\Gamma(\frac{s_1+a_3-\lambda_3}{2})\Gamma(\frac{s_1+b_1+\lambda_1}{2})\Gamma(\frac{s_1+b_2+\lambda_2}{2})\Gamma(\frac{s_1+b_3+\lambda_3}{2})}{\Gamma(\frac{s_1+s_2+c}{2})}.$$

Here the lines of integration are taken as to the right of all poles of the integrand. Then

$$\begin{pmatrix} J_{11}(y) \\ J_{12}(y) \\ J_{13}(y) \end{pmatrix} = \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \cdot \begin{pmatrix} \varepsilon_1\varepsilon_2M(0,1,1;1,0,0;1;y) \\ -\sqrt{-1}\varepsilon_2M(1,0,0;1,0,0;0;y) \\ -M(1,0,0;0,1,1;1;y) \end{pmatrix}$$

$$\begin{pmatrix} J_{21}(y) \\ J_{22}(y) \\ J_{23}(y) \end{pmatrix} = \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})} \cdot \begin{pmatrix} -\sqrt{-1}\varepsilon_1M(1,0,1;0,1,0;1;y) \\ -M(0,1,0;0,1,0;0;y) \\ \sqrt{-1}\varepsilon_2M(0,1,0;1,0,1;1;y) \end{pmatrix}$$

$$\begin{pmatrix} J_{31}(y) \\ J_{32}(y) \\ J_{33}(y) \end{pmatrix} = \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \cdot \begin{pmatrix} -M(1,1,0;0,0,1;1;y) \\ \sqrt{-1}\varepsilon_1M(0,0,1;0,0,1;0;y) \\ \varepsilon_1\varepsilon_2M(0,0,1;1,1,0;1;y) \end{pmatrix}.$$

Proof. It is obvious from Lemma (7.1). $\square$

Remark. In view of this proposition, we can see the following symmetry for J_{ij} with respect to the parameter $(\lambda_1, \lambda_2, \lambda_3)$. This is natural but is not immediately seen from the formulae for J_{ij} (Theorem 6.3). We denote

$$\tilde{J}_i(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+p_i)\Gamma(\frac{\lambda_3-\lambda_2}{8}+q_i)\Gamma(\frac{\lambda_3-\lambda_1}{8}+r_i)} \right)^{-1} {}^t(J_{i1}(y), J_{i2}(y), J_{i3}(y))$$

with $(p_i, q_i, r_i) = (1, \frac{1}{2}, 1)$ ($i = 1$), $(1, 1, \frac{1}{2})$ ($i = 2$), $(\frac{1}{2}, 1, 1)$ ($i = 3$). Then

$$\tilde{J}_2(\lambda_1, \lambda_2, \lambda_3) = (-\sqrt{-1})\varepsilon_2\tilde{J}_1(\lambda_2, \lambda_1, \lambda_3), \quad \tilde{J}_3(\lambda_1, \lambda_2, \lambda_3) = -\varepsilon_1\varepsilon_2\tilde{J}_1(\lambda_3, \lambda_2, \lambda_1).$$

8 Relation between Jacquet integrals and power series solutions.

We omit the case of the class one principal series here, which is discussed by other people. In the same way of [9] for class one case, we move the lines of Mellin-Barnes

integral expression in Proposition (7.2) to the left and sum up the residues at the poles. Then we obtain the following.

Theorem (8.2)

$$\begin{aligned}
 {}^t(J_{11}(y), J_{12}(y), J_{13}(y)) &= \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \\
 &\cdot \left[\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}) {}^t(\Phi_0^{1,I}, \Phi_1^{1,I}, \Phi_2^{1,I}) \right. \\
 &+ \varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8}) {}^t(\Phi_0^{1,II}, \Phi_1^{1,II}, \Phi_2^{1,II}) \\
 &- \varepsilon_2(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8}) {}^t(\Phi_0^{1,III}, \Phi_1^{1,III}, \Phi_2^{1,III}) \\
 &- \varepsilon_2(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}) {}^t(\Phi_0^{1,IV}, \Phi_1^{1,IV}, \Phi_2^{1,IV}) \\
 &- (\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8}) {}^t(\Phi_0^{1,V}, \Phi_1^{1,V}, \Phi_2^{1,V}) \\
 &\left. - (\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}) {}^t(\Phi_0^{1,VI}, \Phi_1^{1,VI}, \Phi_2^{1,VI}) \right], \\
 {}^t(J_{21}(y), J_{22}(y), J_{23}(y)) &= \frac{-\sqrt{-1}\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})} \\
 &\cdot \left[\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}) {}^t(\Phi_0^{2,I}, \Phi_1^{2,I}, \Phi_2^{2,I}) \right. \\
 &+ \varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}) {}^t(\Phi_0^{2,II}, \Phi_1^{2,II}, \Phi_2^{2,II}) \\
 &- (\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}) {}^t(\Phi_0^{2,III}, \Phi_1^{2,III}, \Phi_2^{2,III}) \\
 &- (\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}) {}^t(\Phi_0^{2,IV}, \Phi_1^{2,IV}, \Phi_2^{2,IV}) \\
 &- \varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}) {}^t(\Phi_0^{2,V}, \Phi_1^{2,V}, \Phi_2^{2,V}) \\
 &\left. - \varepsilon_2(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}) {}^t(\Phi_0^{2,VI}, \Phi_1^{2,VI}, \Phi_2^{2,VI}) \right], \\
 {}^t(J_{31}(y), J_{32}(y), J_{33}(y)) &= \frac{-\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \\
 &\cdot \left[(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8}) {}^t(\Phi_0^{3,I}, \Phi_1^{3,I}, \Phi_2^{3,I}) \right. \\
 &+ (\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8}) {}^t(\Phi_0^{3,II}, \Phi_1^{3,II}, \Phi_2^{3,II}) \\
 &- \varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8}) {}^t(\Phi_0^{3,III}, \Phi_1^{3,III}, \Phi_2^{3,III}) \\
 &- \varepsilon_1(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8}) {}^t(\Phi_0^{3,IV}, \Phi_1^{3,IV}, \Phi_2^{3,IV}) \\
 &- \varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8}) {}^t(\Phi_0^{3,V}, \Phi_1^{3,V}, \Phi_2^{3,V}) \\
 &\left. - \varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8}) {}^t(\Phi_0^{3,VI}, \Phi_1^{3,VI}, \Phi_2^{3,VI}) \right].
 \end{aligned}$$

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