

CONFORMALLY FLAT LORENTZ PARABOLIC MANIFOLD

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ABSTRACT. The purpose of this note is to introduce *Lorentz parabolic structure* on smooth manifolds. First we revisit (G, X) -structure on manifolds. Secondly we study *Lorentz similarity structure* and *Fefferman-Lorentz parabolic structure*.

1. INTRODUCTION

In the first part of this paper we review (G, X) -structure introduced by Thurston, Kulkarni et al. Many results are known when (G, X) is a homogeneous Riemannian geometry. In 1980-90s non-Riemannian homogeneous geometries have been studied intensively. Specifically conformally flat geometry, spherical CR -geometry and flat quaternionic CR -geometry. Those geometries are obtained on the projective limit of the isometric actions of hyperbolic spaces. Similarly, another kind of non-Riemannian homogeneous geometry is obtained as the boundary behavior of the isometric actions on *pseudo-hyperbolic spaces*. The typical example is *conformally flat Lorentz geometry*. In the second part of this paper, we introduce *conformally flat Lorentz parabolic geometry*. A Lorentz parabolic structure contains Lorentz similarity structure and Fefferman-Lorentz structure. It is explained that the fundamental group of a compact complete Lorentz similarity manifold M is virtually polycyclic. It turns out that a finite cover of M admits a Lorentz parabolic structure. We discuss *Fefferman-Lorentz parabolic geometry*. The conformally flat Lorentz geometry $(O(2n+2, 2), S^1 \times S^{2n+1})$ contains this as a subgeometry $(U(n+1, 1), S^1 \times S^{2n+1})$. Let Γ be a discrete subgroup of $U(n+1, 1)$ acting properly discontinuously on a domain $\#$ of $S^1 \times S^{2n+1}$. We present a classification of compact conformally flat Fefferman-Lorentz parabolic manifolds $\#/\Gamma$ admitting a 1-parameter group $H \leq \text{Conf}(\#/\Gamma)$. This

2000 *Mathematics Subject Classification.* 53C55, 57S25, 51M10.

Key words and phrases. (G, X) -structure, Lorentz structure, Fefferman-Lorentz structure. Parabolic structure, Similarity structure.

class contains $S^1 \times \mathcal{N}/\Delta$ where $\mathcal{N}$ is a 3-dimensional Heisenberg nil-manifold. Finally we discuss the deformation space of conformally flat Fefferman-Lorentz parabolic structures on the product $S^1 \times \mathcal{N}/\Delta$.

2. (G, X) -STRUCTURE

Our geometry is a pair (G, X) where G is a finite dimensional Lie group with finitely many components and X is an n dimensional homogeneous space of G . A geometric structure $((G, X)$ -structure) on a smooth n dimensional manifold M is a maximal collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ whose coordinate changes belong to G . More precisely, $M = \bigcup_{\alpha \in \Lambda} U_\alpha$, $\phi_\alpha : U_\alpha \rightarrow X$ is a diffeomorphism onto its image. If $U_\alpha \cap U_\beta \neq \emptyset$ then it satisfies that there exists a unique element $g_{\alpha\beta} \in G$ such that $g_{\alpha\beta} \cdot \phi_\alpha = \phi_\beta$ on $U_\alpha \cap U_\beta$. We say that M is uniformized over X with respect to G (or simply, M is locally modelled on (G, X)). An n -manifold M is called a (G, X) -manifold if M is uniformized over X with respect to G . Using a collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ we can construct a geometric invariant (ρ, dev) called a developing pair of M . (See [5].)

Lemma 2.1. *Given a (G, X) -structure on a smooth n -manifold M , there exists a pair $(\rho, \text{dev}) : (\pi_1(M), \tilde{M}) \rightarrow (G, X)$ unique up to conjugation of elements of G , where dev is a (G, X) -structure preserving immersion and ρ is a homomorphism such that the diagram is commutative for each element $\gamma \in \pi_1(M)$;*

$$(2.1) \quad \begin{array}{ccc} \tilde{M} & \xrightarrow{\text{dev}} & X \\ \gamma \downarrow & & \downarrow \rho(\gamma) \\ \tilde{M} & \xrightarrow{\text{dev}} & X. \end{array}$$

Proof. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ be a geometric structure on M . In the union $\bigcup_{\alpha \in \Lambda} (U_\alpha \times X)$, we define the following equivalence relation; for $(p, x) \in U_\alpha \times X$, $(q, y) \in U_\beta \times X$, then

$$(2.2) \quad \begin{aligned} (p, x) \sim (q, y) \text{ if and only if } & p = q \in U_\alpha \cap U_\beta, \ g_{\alpha\beta}x = y, \\ & (\exists g_{\alpha\beta} \in G). \end{aligned}$$

Put $E = \bigcup_{\alpha} (U_\alpha \times X) / \sim$. Let $\pi : E \rightarrow M$ be the map defined by $\pi([p, x]) = p$ if $p \in U_\alpha$. Then it is easy to see that $E \xrightarrow{\pi} M$ is a fiber bundle with fiber X . Recall that E is determined by the transitive functions $\{g_{\alpha\beta}\}$. Since $g_{\alpha\alpha} = 1$ and $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, $\{g_{\alpha\beta}\}$ is a 1-cocycle in the first cohomology $H^1(M; G)$. Here G is

viewed as the sheaf of germs of G -valued functions. Since $H^1(M; G) \approx \text{Hom}(\pi_1(M), G)$, $\{g_{\alpha\beta}\}$ determines a homomorphism $\rho : \pi_1(M) \rightarrow G$. More precisely it follows that $E \approx \tilde{M} \times X$ in which each element $\gamma \in \pi_1(M)$ acts on $\tilde{M} \times X$ by $(\gamma, (b, x)) = (\gamma b, \rho(\gamma)x)$.

We construct a developing map. Let $s : M \rightarrow E$ be a section defined by $s(p) = [p, \phi_\alpha(p)]$ if $p \in U_\alpha$. Consider the pull back of the bundle:

$$(2.3) \quad \begin{array}{ccccc} \pi_1(M) & \rightarrow & P^*E & \rightarrow & E \\ & & \downarrow & & \downarrow \\ \pi_1(M) & \rightarrow & \tilde{M} & \rightarrow & M. \end{array}$$

As before the bundle P^*E is determined by a lift $\{\tilde{g}_{\alpha\beta}\}$ of $\{g_{\alpha\beta}\}$. Since $H^1(\tilde{M}; G) = \{1\}$, the bundle P^*E is trivial. Choose a trivialization $\Psi : P^*E \rightarrow \tilde{M} \times X$. The section s extends to a section $\tilde{s} : \tilde{M} \rightarrow P^*E$. Put $\text{dev} = \text{Pr}_2 \cdot \Psi \cdot \tilde{s} : \tilde{M} \rightarrow X$. It is an immersion and preserves the (G, X) -structure. The map dev depends on the choice of sections and trivializations, however dev is unique up to elements of G .

On the other hand, we note that for $(\tilde{p}, x) \in \tilde{U}_\alpha \times X$, $(\tilde{q}, y) \in \tilde{U}_\beta \times X$ in $P^*E = \bigcup_\alpha (\tilde{U}_\alpha \times X)$, it follows that $(\tilde{p}, x) \sim (\tilde{q}, y)$ iff $\gamma\tilde{p} = \tilde{q}$, $\rho(\gamma)y = g_{\alpha\beta}x$ and $p = q \in U_\alpha \cap U_\beta$ for some $\gamma \in \pi_1(M)$ and $g_{\alpha\beta} \in G$. It is easy to see that

$$\text{dev} \cdot \gamma = \rho(\gamma) \cdot \text{dev} \text{ for every } \gamma \in \pi_1(M).$$

□

If $\text{Aut}(\tilde{M})$ is the group of all (G, X) -structure preserving diffeomorphisms on $\tilde{M}$. Then note that $\pi_1(M) \leq \text{Aut}(\tilde{M})$ and ρ extends naturally to a continuous homomorphism $\rho : \text{Aut}(\tilde{M}) \rightarrow G$.

Definition 2.2. The map dev is called a developing map for a (G, X) -manifold M and the map ρ is called a holonomy homomorphism of M .

Let $\hat{\#}(M)$ be the space consisting of all possible developing pairs (ρ, dev) . A topology on $\hat{\#}(M)$ is given by the following subbasis.

- $\mathcal{N}(U) = \{U\}$ where U is an open subset of $\text{Map}(\tilde{M}, X)$ in the compact open topology of $\text{Map}(\tilde{M}, X)$.
- $\mathcal{N}(K) = \{\text{dev} \in \hat{\#}(M) \mid \text{dev}|_K \text{ is embedding}\}$ for a compact subset $K \subset \tilde{M}$.

(Compare [1].) Recall that the deformation space $\mathcal{T}(M)$ is a space of (G, X) -structures on marked manifolds homeomorphic to M . $\mathcal{T}(M)$ consists of equivalence classes of diffeomorphisms $f : M \rightarrow M'$ from

M to a (G, X) -manifolds M' . Two such diffeomorphisms $f_i : M \rightarrow M_i$ ($i = 1, 2$) are equivalent if and only if there is an isomorphism (i.e. a (G, X) -structure preserving diffeomorphism) $h : M_1 \rightarrow M_2$ such that $h \circ f_1$ is *isotopic* to f_2 .

$$(2.4) \quad \begin{array}{ccc} M & \xrightarrow{f_1} & M_1 \\ f_2 \searrow & \simeq & \downarrow h \\ & & M_2 \end{array}$$

Denote by $\text{Diff}^0(M)$ the subgroup of $\text{Diff}(M)$ whose elements are *isotopic* to the identity map. Put $\pi = \pi_1(M)$. Consider the following exact sequences of the diffeomorphism groups, where $N_{\text{Diff}(\tilde{M})}(\pi)$ (resp. $C_{\text{Diff}(\tilde{M})}(\pi)$) is the normalizer (resp. centralizer) of π in $\text{Diff}(\tilde{M})$

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi & \longrightarrow & N_{\text{Diff}(\tilde{M})}(\pi) & \xrightarrow{\eta} & \text{Diff}(M) \longrightarrow 1 \\ & & & & \uparrow & & \uparrow \\ & & & & C_{\text{Diff}(\tilde{M})}(\pi) & \longrightarrow & \text{Diff}^0(M) \end{array}$$

Put $\widehat{\text{Diff}}(M) = \eta^{-1}(\text{Diff}(M))$ and let $\widehat{\text{Diff}}^0(M)$ be the identity component. Then $\eta(\widehat{\text{Diff}}(M)) = \text{Diff}^0(M)$ and $\widehat{\text{Diff}}^0(M) \leq C_{\text{Diff}(\tilde{M})}(\pi)$. The natural right action of $\widehat{\text{Diff}}(M)$ and the left action of G on $\hat{\#}(M)$ are given by

$$(2.5) \quad \begin{aligned} (\rho, \text{dev}) \circ \tilde{f} &= (\rho \circ \mu(\tilde{f}), \text{dev} \circ \tilde{f}), \\ g \circ (\rho, \text{dev}) &= (g \circ \rho \circ g^{-1}, g \circ \text{dev}), \end{aligned}$$

where $\mu(\tilde{f}) : \pi \rightarrow \pi$ is an isomorphism defined by $\mu(\tilde{f})(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$. Obviously both actions commute.

It is noted that two developing pairs (ρ_i, dev_i) ($i = 1, 2$) represent the same structure on M if and only if there exists an element $g \in G$ such that $g \circ \text{dev}_1 = \text{dev}_2$. Put

$$\#(M) = \hat{\#}(M) / \widehat{\text{Diff}}^0(M).$$

The action of G induces an action of $\#(M)$. Then it is easy to show that

Lemma 2.3. *The elements of $\mathcal{T}(M)$ are in one-to-one correspondence with the orbits of $G \setminus \#(M)$.*

If $f : M \rightarrow M'$ is a representative element of $\mathcal{T}(H, M)$ then there is a developing pair $(\rho, \text{dev}) : (\pi_1(M'), \tilde{M}') \rightarrow (G, X)$. We have the

holonomy representation $\rho \circ f_{\#} : \pi \rightarrow G$ up to conjugate by an element of G . We then obtain a map $hol : \mathcal{T}(M) \rightarrow \text{Hom}(\pi, G)/G$ which assigns to a marked structure its holonomy representation. By the definition hol lifts to a map $\widehat{hol} : \#(M) \rightarrow \text{Hom}(\pi, G)$ which makes the following diagram commute.

$$\begin{array}{ccc} \#(M) & \xrightarrow{\widehat{hol}} & \text{Hom}(\pi, G) \\ \downarrow & & \downarrow \\ \mathcal{T}(M) & \xrightarrow{hol} & \text{Hom}(\pi, G)/G. \end{array}$$

Thurston has shown the following. (See [Lo],[J-M],[Th] for the proof.)

Theorem 2.4 (Holonomy Theorem). $\widehat{hol} : \#(M) \rightarrow \text{Hom}(\pi, G)$ is a local homeomorphism.

3. EXAMPLES OF NON-RIEMANNIAN HOMOGENEOUS GEOMETRY

3.1. Homogeneous Riemannian geometry. Let $X = G_x \backslash G$ be the simply connected homogeneous space ($x \in X$). If G_x is compact, then (G, X) is called *homogeneous Riemannian geometry*. If M is a compact manifold which admits a (G, X) -structure, then it follows that $M = X/\rho(\pi)$ where $\rho : \pi = \pi_1(M) \rightarrow G$ is a discrete faithful representation. This is obtained by the following lemma.

Lemma 3.1. *If $f : M \rightarrow N$ is a Riemannian immersion and M is complete, then f is a covering map.*

A Riemannian manifold is complete if every Cauchy sequence converges relative to the Riemannian metric. Specifically a compact Riemannian manifold is complete.

Thus the deformation space $\mathcal{T}(M)$ is identified with the set of equivalence classes of discrete faithful representations $\text{R}(\pi, G)/G$. For example, when we take $G = \text{Isom}(\mathbb{H}_{\mathbb{K}}^n)$ the full isometry group of the $\mathbb{K}$ -hyperbolic space where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. The Mostow rigidity theorem says that $\text{R}(\pi, G)/G$ is a single point. By Margulis-Mostow rigidity, the same result holds for a noncompact semisimple Lie group G of $\mathbb{R}$ -rank ≥ 2 . If $X = K \backslash G$, then X/Γ is a compact nonpositively curved Riemannian manifold. On the other hand, if M is noncompact, there occurs a remarkably distinct feature, one is Thurston bending while the other is Margulis super rigidity. After Thurston's hyperbolization theory several non-Riemannian homogeneous geometry surrounding hyperbolic geometry came to our interest in 1980s~1990s. The $\mathbb{K}$ -hyperbolic space $\mathbb{H}_{\mathbb{K}}^{n+1}$ has the projective compactification $\partial \mathbb{H}_{\mathbb{K}}^{n+1}$ which

is diffeomorphic to the sphere $S^{|\mathbb{K}|(n+1)-1}$. It is well known that the isometric action $\text{Isom}(\mathbb{H}_{\mathbb{K}}^n)$ extends to a smooth action on $S^{|\mathbb{K}|(n+1)-1}$. This phenomenon occurs also for Hadamard manifolds (complete simply connected Riemannian manifold of nonpositive curvature). In general, an extended action on the boundary sphere is *topological*. But the above actions on $\partial\mathbb{H}_{\mathbb{K}}^{n+1}$ are known to be analytic. Denote $\text{Aut}(S^{|\mathbb{K}|(n+1)-1})$ the (extended) action of $\text{Isom}(\mathbb{H}_{\mathbb{K}}^n)$ on $S^{|\mathbb{K}|(n+1)-1}$. It is known that $\text{Aut}(S^{|\mathbb{K}|(n+1)-1})$ acts transitively on $S^{|\mathbb{K}|(n+1)-1}$ with *noncompact* stabilizer $\text{Aut}(S^{|\mathbb{K}|(n+1)-1})_{\infty}$ such that

$$S^{|\mathbb{K}|(n+1)-1} = \text{Aut}(S^{|\mathbb{K}|(n+1)-1})_{\infty} \backslash \text{Aut}(S^{|\mathbb{K}|(n+1)-1})$$

where $\infty \in S^{|\mathbb{K}|(n+1)-1}$. Hence we have a non-Riemannian homogeneous geometry $(\text{Aut}(S^{|\mathbb{K}|(n+1)-1}), S^{|\mathbb{K}|(n+1)-1})$. According to whether $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, it is said to be

$$(3.1) \quad \begin{array}{ll} \text{Conformally flat geometry} & (\text{PO}(n+1, 1), S^n) \\ \text{Spherical } CR\text{-geometry} & (\text{PU}(n+1, 1), S^{2n+1}) \\ \text{Quaternionic flat } CR\text{-geometry} & (\text{PSp}(n+1, 1), S^{4n+3}) \end{array}$$

It is an excellent result by Gromov-Lawson-Yau that a nontrivial S^1 -bundle M^3 over a closed surface Σ_g of genus $g > 1$ admits a conformally flat structure. It is trivial that the product $S^1 \times \Sigma_g$ is a conformally flat manifold. On the other hand, in spherical CR -geometry $(\text{PU}(2, 1), S^3)$, the complement of geometric circle $S^3 - S^1$ has an invariant subgroup $\text{U}(1, 1) = \text{P}(\text{U}(1, 1) \times \text{U}(1))$. Choosing a discrete cocompact subgroup $\Gamma \leq \text{U}(1, 1)$, we get a spherical CR -manifold $S^3 - S^1/\Gamma$ which is a nontrivial S^1 -bundle: $S^1 \rightarrow \text{U}(1) \backslash \text{U}(1, 1)/\Gamma \rightarrow \text{U}(1) \backslash \text{PU}(1, 1)/P(\Gamma)$. Here $\text{U}(1) \backslash \text{PU}(1, 1)/P(\Gamma) = \mathbb{H}_{\mathbb{C}}^1/P(\Gamma) = \Sigma_g$. However, to our knowledge, the following problem hasn't been yet proved rigorously.

Problem. Does the product $S^1 \times \Sigma_g$ admit a spherical CR -structure?

4. CONFORMALLY FLAT LORENTZ GEOMETRY

It is natural to consider how the isometry group of the pseudo-hyperbolic space acts on the compactification. Put $V_-^{m+2,2} = \{x \in \mathbb{R}^{m+4} \mid \mathcal{B}(x, x) = x_1^2 + \cdots + x_{m+2}^2 - x_{m+3}^2 - x_{m+4}^2 < 0\}$. If $P_{\mathbb{R}} : \mathbb{R}^{m+4} - \{0\} \rightarrow \mathbb{RP}^{m+3}$ is the canonical projection, then the real pseudo-hyperbolic space $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ is defined to be $P_{\mathbb{R}}(V_-^{m+2,2})$. For this reason, the $m+3$ -dimensional quadrics $V_{-1}^{m+2,2} = \{x \in \mathbb{R}^{m+4} \mid x_1^2 + \cdots + x_{m+2}^2 - x_{m+3}^2 - x_{m+4}^2 = -1\}$ with Lorentz metric g is the complete pseudo-Riemannian manifold of signature $(m+1, 1)$ and of constant curvature -1 such that $P_{\mathbb{R}}(V_{-1}^{m+2,2}) = P_{\mathbb{R}}(V_-^{m+2,2})$. Since $P_{\mathbb{R}} : V_{-1}^{m+2,2} \rightarrow \mathbb{H}_{\mathbb{R}}^{m+2,1}$

is a two-fold covering, so $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ is a complete pseudo-hyperbolic space form. The action $O(m+2, 2)$ on $V_-^{m+2,2}$ induces an action on $\mathbb{H}_{\mathbb{R}}^{m+2,1}$. The kernel of this action is the center $\mathbb{Z}/2 = \{\pm 1\}$ whose quotient is called *real pseudo-hyperbolic group* $PO(m+2, 2)$. The projective compactification of $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ is obtained by taking the closure $\overline{\mathbb{H}_{\mathbb{R}}^{m+2,1}}$ in $\mathbb{RP}^{m+3}$. Consider the commutative diagram:

$$\begin{array}{ccc} (GL(m+4, \mathbb{R}), \mathbb{R}^{m+4} - \{0\}) & \xrightarrow{P} & (PGL(m+4, \mathbb{R}), \mathbb{RP}^{m+3}) \\ \cup & & \cup \\ (O(m+2, 2), V_-^{m+2,2} \cup V_0) & \xrightarrow{P} & (PO(m+2, 2), \mathbb{H}_{\mathbb{R}}^{m+2,1} \cup S^{m+1,1}) \end{array}$$

Here $V_0 = V_0^{m+2,1} = \{x \in \mathbb{R}^{m+4} \mid x_1^2 + \dots + x_{m+2}^2 - x_{m+3}^2 - x_{m+4}^2 = 0\}$. It follows that

$$\overline{\mathbb{H}_{\mathbb{R}}^{m+2,1}} = \mathbb{H}_{\mathbb{R}}^{m+2,1} \cup S^{m+1,1}.$$

From this viewpoint, the pseudo-hyperbolic action of $PO(m+2, 2)$ on $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ extends to *conformal action* of $S^{m+1,1}$. We obtain *conformally flat Lorentz geometry* $(PO(m+2, 2), S^{m+1,1})$. This is of course non-Riemannian homogeneous geometry.

Let $(1, 0, \dots, 0, 1) \in V_0$ be a null vector. Put $\infty = P(1, 0, \dots, 0, 1) \in S^{m+1,1}$ which is called *the point at infinity*. The stabilizer $PO(m+2, 2)_{\infty}$ is $\mathbb{R}^{m+2} \rtimes (O(m+1, 1) \times \mathbb{R}^+)$ up to conjugacy. When $h \in PO(m+2, 2)_{\infty}$, the differential map $h_* : T_{\infty} S^{m+1,1} \rightarrow T_{\infty} S^{m+1,1}$ is an isomorphism, $h_* \in \text{Aut}(T_{\infty} S^{m+1,1}) = O(m+1, 1) \times \mathbb{R}^+$. Thus the structure group of $(PO(m+2, 2), S^{m+1,1})$ is $O(m+1, 1) \times \mathbb{R}^+$. Originally as a G -structure, conformal Lorentz structure is an $O(m+1, 1) \times \mathbb{R}^+$ -structure. In addition, an integrable $O(m+1, 1) \times \mathbb{R}^+$ -structure is *conformally flat Lorentz structure*. (Equivalently, the Weyl conformal curvature tensor vanishes.) When $\{\infty\}$ is the point at infinity of $S^m = \partial \mathbb{H}_{\mathbb{R}}^{m+1}$, we can consider the minimal parabolic group $O(m+1, 1)_{\infty}$ which is an amenable Lie subgroup of $O(m+1, 1)$. We remark that $O(m+1, 1)_{\infty}$ is isomorphic to the similarity group $\text{Sim}(\mathbb{R}^m)$.

Definition 4.1. If the structure group of a conformally flat Lorentz $(m+2)$ -manifold M belongs to $O(m+1, 1)_{\infty} \times \mathbb{R}^+$, then M is said to be a *conformally flat Lorentz parabolic manifold*.

We study a special class of conformally flat Lorentz parabolic manifolds called *Lorentz similarity manifold* of dimension $m+2$ and *Fefferman-Lorentz manifold* of dimension $2n+2$.

5. LORENTZIAN SIMILARITY GEOMETRY

Recall that $\mathbb{R}^{m+2}$ is the euclidean space with Lorentz inner product sitting in $S^{m+1,1} - \{\infty\}$. Then $\text{PO}(m+2, 2)_\infty = \mathbb{R}^{m+2} \rtimes (\text{O}(m+1, 1) \times \mathbb{R}^+)$. We define $\text{Sim}_L(\mathbb{R}^{m+2}) = \mathbb{R}^{m+2} \rtimes (\text{O}(m+1, 1) \times \mathbb{R}^+)$. The pair $(\text{Sim}_L(\mathbb{R}^{m+2}), \mathbb{R}^{m+2})$ is said to be *Lorentz similarity geometry*. In [3] we proved the following.

Theorem 5.1. *If M is a compact complete Lorentz similarity manifold of dimension $m+2$, then the fundamental group of M is virtually polycyclic. Furthermore, M is diffeomorphic to an infrasolmanifold.*

This theorem is originally proved by T. Aristide. Once $\pi_1(M)$ turns out to be virtually polycyclic, the holonomy group $L(\pi_1(M))$ belongs to either $\text{O}(m+1, 1)_\infty \times \mathbb{R}^+$ or $\text{O}(m+1) \times \text{O}(1) \times \mathbb{R}^+$. Here $\text{O}(m+1, 1)_\infty = \text{Sim}(\mathbb{R}^m) = \mathbb{R}^m \rtimes (\text{O}(m) \times \mathbb{R}^*)$. Since Γ acts freely as a'ne motions on $\mathbb{R}^{m+2}$, the matrix of holonomy group has no eigenvalue 1. The latter case shows that $L(\pi_1(M)) \leq \text{O}(m+1) \times \text{O}(1)$ so that M reduces to a compact euclidean space form. Then $\pi_1(M)$ is a Bieberbach group.

Corollary 5.2. *A finite cover of a compact complete Lorentz similarity manifold M is a conformally flat Lorentz parabolic manifold.*

We shall give a sketch of proof of Theorem 5.1. Put $M = \mathbb{R}^{m+2}/\Gamma$ where $\Gamma \leq \text{Sim}_L(\mathbb{R}^{m+2})$. There is the exact sequence: $1 \rightarrow \mathbb{R}^{m+2} \rightarrow \text{Sim}_L(\mathbb{R}^{m+2}) \xrightarrow{L} \text{O}(m+1, 1) \times \mathbb{R}^+ \rightarrow 1$. If $\mathbb{R}^{m+2} \cap \Gamma$ is nontrivial, say $\mathbb{Z}^k$, then a properly discontinuous action of Γ induces a properly discontinuous action of $L(\Gamma)$ on $\mathbb{R}^{m-k}$ as in the same argument of [3, (1) Proposition 2.2]. Then Γ is virtually polycyclic by induction. So we assume

$$(5.1) \quad \mathbb{R}^{m+2} \cap \Gamma = \{1\}.$$

Note also that $(\mathbb{R}^{m+2} \rtimes \mathbb{R}^+) \cap \Gamma = \{1\}$ because each element has the form $(a, \lambda \cdot I)$. As Γ acts freely on $\mathbb{R}^{m+2}$, $\lambda = 1$. It follows $(\mathbb{R}^{m+2} \rtimes \mathbb{R}^+) \cap \Gamma = \mathbb{R}^{m+2} \cap \Gamma$.

Consider the following exact sequence:

$$(5.2) \quad 1 \rightarrow \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \rightarrow \text{Sim}_L(\mathbb{R}^{m+2}) \xrightarrow{p} \text{O}(m+1, 1) \rightarrow 1.$$

If $p(\Gamma)$ is discrete in $\text{O}(m+1, 1)$, then the cohomological dimension $\text{cd } p(\Gamma) \leq m+1$. As $\mathbb{R}^{m+2}/\Gamma$ is compact, $\text{cd } \Gamma = m+2$. On the other hand, $\Gamma \cong p(\Gamma)$ by (5.1), $\text{cd } \Gamma = \text{cd } p(\Gamma)$ which yields a contradiction.

Suppose that $p(\Gamma)$ is indiscrete in $\text{O}(m+1, 1)$. Then the identity component of the closure $\overline{p(\Gamma)}^0$ is solvable in $\text{O}(m+1, 1)$.

Case I. If it is noncompact, then it belongs to the maximal amenable subgroup $\text{Sim}(\mathbb{R}^m)$ up to conjugate. The normalizer of $\overline{\mathfrak{p}(\Gamma)}^0$ is contained in $\text{Sim}(\mathbb{R}^m)$. In particular, $\mathfrak{p}(\Gamma) \leq \text{Sim}(\mathbb{R}^m)$. (5.2) induces an exact sequence:

$$1 \rightarrow \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \rightarrow \mathfrak{p}^{-1}(\text{Sim}(\mathbb{R}^m)) \xrightarrow{\mathfrak{p}} \text{Sim}(\mathbb{R}^m) \rightarrow 1$$

in which $\mathfrak{p}^{-1}(\text{Sim}(\mathbb{R}^m))$ is an amenable Lie subgroup. Any discrete subgroup of an amenable Lie group is virtually polycyclic so is Γ .

Case II. Suppose that $\overline{\mathfrak{p}(\Gamma)}^0$ is compact, say T^ℓ . We consider actions of subgroups of $O(m+1, 1)$ on $\mathbb{H}_{\mathbb{R}}^{m+1} \cup S^m$. If T^ℓ has no fixed point in S^m , then T^ℓ has a unique fixed point $0 \in \mathbb{H}_{\mathbb{R}}^{m+1}$ so that $\mathfrak{p}(\Gamma) \leq O(m+1) \times O(1)$. Thus $\Gamma \leq \text{Sim}(\mathbb{R}^{m+2})$. $\mathbb{R}^{m+2}/\Gamma$ turns out to be a compact complete similarity manifold and so Γ is virtually abelian (a Bieberbach group).

Suppose that T^ℓ has the fixed point set S^k in S^m for some $k < m$. As $\overline{\mathfrak{p}(\Gamma)}$ leaves invariant the complement $S^m - S^k = \mathbb{H}_{\mathbb{R}}^{k+1} \times S^{m-k-1}$. It follows $\overline{\mathfrak{p}(\Gamma)} \leq O(k+1, 1) \times O(m-k)$ for which $T^\ell = \overline{\mathfrak{p}(\Gamma)}^0 \leq O(m-k)$. If $\text{Pr} : O(k+1, 1) \times O(m-k) \rightarrow O(k+1, 1)$ is the canonical projection, then $\text{Pr}(\mathfrak{p}(\Gamma))$ is discrete. Note that $\text{Ker Pr} \circ \mathfrak{p} = \mathbb{R}^{m+2} \rtimes (O(m-k) \times \mathbb{R}^+)$. Put

$$(5.3) \quad \Delta = (\mathbb{R}^{m+2} \rtimes (O(m-k) \times \mathbb{R}^+)) \cap \Gamma.$$

It is nontrivial, because if trivial, $\Gamma \cong \text{Pr}(\mathfrak{p}(\Gamma))$ so $m+2 = \text{cd } \Gamma = \text{cd } \text{Pr}(\mathfrak{p}(\Gamma))$ but $\text{cd } \text{Pr}(\mathfrak{p}(\Gamma)) \leq k+1$ which is impossible by the inequality $k < m$.

Since T^ℓ is a maximal torus in $O(m-k)$, $N_{O(m-k)}(T^\ell)/T^\ell$ is finite for the normalizer $N_{O(m-k)}(T^\ell)$. As $\overline{\mathfrak{p}(\Gamma)}$ normalizes T^ℓ , there exists a finite index normal subgroup H of $\overline{\mathfrak{p}(\Gamma)}$ which centralizes T^ℓ with $H^0 = T^\ell$. Note that $H \cap \mathfrak{p}(\Gamma)$ is of finite index in $\mathfrak{p}(\Gamma)$. Put $\Gamma_1 = \mathfrak{p}^{-1}(H \cap \mathfrak{p}(\Gamma))$ which is a finite index subgroup of Γ .

Let $\mathfrak{p}_1 : \mathbb{R}^{m+2} \rtimes (O(m-k) \times \mathbb{R}^+) \rightarrow O(m-k)$ be the projection. Then $\mathfrak{p}_1(\Delta) \leq O(m-k)$ such that $\overline{\mathfrak{p}_1(\Delta)}^0$ is a torus in $O(m-k)$ from (5.3). Since $\overline{\mathfrak{p}_1(\Delta)}$ is a finite extension of $\overline{\mathfrak{p}_1(\Delta)}^0$, we choose Δ_1 such that $\mathfrak{p}_1(\Delta_1) = \overline{\mathfrak{p}_1(\Delta)}^0 \cap \mathfrak{p}_1(\Delta)$. As $\mathfrak{p}_1(\Delta_1) \leq \mathfrak{p}(\Gamma)$, $\overline{\mathfrak{p}_1(\Delta_1)}^0 \leq \overline{\mathfrak{p}(\Gamma)}^0$. Noting that $\mathfrak{p}(\Gamma_1) \leq H$ and $\mathfrak{p}(\Delta_1) \leq \overline{\mathfrak{p}(\Gamma)}^0$ for which H centralizes $T^\ell = \overline{\mathfrak{p}(\Gamma)}^0$ as above, it follows that $\mathfrak{p}(\Gamma_1)$ centralizes $\mathfrak{p}(\Delta_1)$.

Note that $\mathfrak{p}_1 : \Delta \rightarrow \mathfrak{p}_1(\Delta)$ is injective. In fact, if not, then $\Delta \cap \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \neq \{1\}$, so $\Gamma \cap \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \neq \{1\}$ which is impossible by the remark below (5.1). Since Γ normalizes Δ , it is easy to see that Γ_1 centralizes Δ_1 . Consider the exact sequences:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{R}^{m+2} & \longrightarrow & \mathbb{R}^{m+2} \rtimes O(m-k) & \xrightarrow{p} & O(m-k) \longrightarrow 1 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & 1 & \longrightarrow & \Delta_1 & \xrightarrow{p} & p(\Delta_1) \longrightarrow 1
\end{array}$$

where $\overline{p(\Delta_1)}^0 = T^s$ for some $s \leq m-k$. It is well known that the abelian discrete subgroup belongs to the following group (cf. [8]):

$$(5.4) \quad \Delta_1 \leq V \times T^s = \left\{ \left(\begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix} \right) \mid C \in T^s, a \in V \right\}$$

such that $V \times T^s / \Delta_1$ is compact. Here $V \cong \mathbb{R}^{k+2}$.

Let $\Gamma_1 \leq \mathbb{R}^{m+2} \rtimes (O(k+1, 1) \times O(m-k) \times \mathbb{R}^+)$ be as before and choose an arbitrary element $\gamma = \left(\begin{bmatrix} x \\ y \end{bmatrix}, \lambda \cdot \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right)$ and take an element $\alpha = \left(\begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix} \right)$ from Δ_1 . As Γ_1 centralizes Δ_1 , the equation $\gamma\alpha\gamma^{-1} = \alpha$ implies that

$$\lambda \cdot Aa = a, BCB^{-1} = C \text{ and } y - BCB^{-1}y = 0.$$

The projection $P : \mathbb{R}^{k+2} - \{0\} \rightarrow \mathbb{RP}^{k+1}$ maps the cone V_0 onto S^k . We observe that if $\langle a, a \rangle = 0$ with respect to the Lorentz inner product, then $P(a) = [a] \in S^k$. Put $[a] = \infty \in S^k$ up to conjugacy. The equality $\lambda \cdot Aa = a$ implies $A\infty = \infty$ so $A \in O(k+1, 1)_\infty$. This holds for arbitrary elements of Γ_1 . It follows

$$(5.5) \quad \Gamma \leq \mathbb{R}^{m+2} \rtimes (O(k+1, 1)_\infty \times O(m-k) \times \mathbb{R}^+)$$

which is an amenable Lie subgroup. Thus Γ is virtually polycyclic. When $\langle a, a \rangle \neq 0$, as $\langle a, a \rangle = \langle \lambda Aa, \lambda Aa \rangle = \lambda^2 \langle a, a \rangle$, it follows $\lambda = 1$. Thus

$$(5.6) \quad \Gamma \leq \mathbb{R}^{m+2} \rtimes (O(k+1, 1) \times O(m-k)) \leq E(m+1, 1).$$

$\mathbb{R}^{m+2}/\Gamma$ becomes a compact complete Lorentz flat space form. It is well known that Γ is virtually polycyclic. This proves the theorem 5.1.

6. CONFORMALLY FLAT FEFERMAN-LORENTZ GEOMETRY

Let $(O(2n+2, 2), S^1 \times S^{2n+1})$ be the conformally flat Lorentz geometry (which is a 2-fold cover.) There is the natural embedding $U(n+1, 1) \rightarrow O(2n+2, 2)$. $U(n+1, 1)$ acts transitively on $S^1 \times S^{2n+1}$ so we have a subgeometry $(U(n+1, 1), S^1 \times S^{2n+1})$.

Proposition 6.1. *A manifold locally modelled on $(U(n+1, 1), S^1 \times S^{2n+1})$ admits a Lorentz parabolic structure.*

Proof. We see that

$$\hat{U}(n+1, 1) \cap O(2n+2, 2)_\infty = \mathbb{R}^{2n+2} \rtimes (O(2n+1, 1) \times \mathbb{R}^*).$$

Then the intersection $\hat{U}(n+1, 1)_\infty = \mathcal{N} \rtimes (U(n) \times \mathbb{R}^+)$ which is amenable. Here $\mathcal{N}$ is the Heisenberg Lie group. So $\hat{U}(n+1, 1)_\infty$ belongs to the maximal amenable group $\mathbb{R}^{2n+2} \rtimes (O(2n+1, 1)_\infty \times \mathbb{R}^*)$. Thus the structure group of $(U(n+1, 1), S^1 \times S^{2n+1})$ belongs to the parabolic group $O(2n+1, 1)_\infty \times \mathbb{R}^*$. So does any manifold modelled on $(U(n+1, 1), S^1 \times S^{2n+1})$. $\square$

Definition 6.2. A manifold locally modelled on $(U(n+1, 1), S^1 \times S^{2n+1})$ is said to be a conformally flat Fefferman-Lorentz parabolic manifold.

To the rest of this section we shall give our recent results concerning compact conformally flat Fefferman-Lorentz parabolic manifolds. The details will be given elsewhere.

Recall that the center S^1 acts freely on the 2-fold covering $S^1 \times S^{2n+1}$ of $S^{2n+1,1}$, there is the equivariant principal bundle:

$$(6.1) \quad (S^1, S^1) \rightarrow (U(n+1, 1), S^1 \times S^{2n+1}) \xrightarrow{(P, p)} (PU(n+1, 1), S^{2n+1}).$$

Let X be a domain of $S^1 \times S^{2n+1}$. If h is an element of the group of conformal Lorentz transformations $\text{Conf}(X)$, then $h : X \rightarrow X$ extends uniquely to a conformal diffeomorphism of $S^1 \times S^{2n+1}$ by Liouville's theorem. We assume that

$$(6.2) \quad \text{Conf}(X) \leq U(n+1, 1).$$

Suppose that a discrete subgroup Γ of $U(n+1, 1)$ acts properly discontinuously on X such that the quotient X/Γ is compact. Note that there is a covering group extension:

$$(6.3) \quad 1 \longrightarrow \Gamma \longrightarrow N_{\text{Conf}(X)}(\Gamma) \xrightarrow{\nu} \text{Conf}(X/\Gamma) \longrightarrow 1.$$

We shall determine X/Γ when X/Γ admits a 1-parameter subgroup H whose lift H to $U(n+1, 1)$ is not the center $ZU(n+1, 1)$.

Theorem 6.3. *Let X/Γ be a $2n+2$ -dimensional compact conformally flat Fefferman-Lorentz parabolic manifold. If X/Γ admits a 1-parameter subgroup H whose lift H to $U(n+1, 1)$ is not the center $ZU(n+1, 1)$, then X/Γ is a Seifert fiber space over a spherical CR-orbifold. Moreover X/Γ is either one of (i), ..., (v). As a consequence, a finite covering of such X/Γ is a Fefferman-Lorentz manifold.*

- (i) $X/\Gamma = S^1 \times_{\mathbb{Z}_\ell} S^{2n+1}$ where $\mathbb{Z}_\ell \leq T^{n+1}$.
- (ii) $S^1 \rightarrow X/\Gamma \rightarrow \mathcal{N}/Q$ where $Q \leq \mathcal{N} \rtimes U(n)$.

- (iii) $S^1 \rightarrow X/\Gamma \rightarrow S^{2n} \times_F S^1$ where $F \leq U(n)$.
- (iv) $S^1 \rightarrow X/\Gamma \rightarrow S^{2n+1}/F$ where $F \leq T^{n+1}$.
- (v) $S^1 \rightarrow X/\Gamma \rightarrow (S^{2n+1} - L(Q))/Q$ where
 $Q \leq P(U(k, 1) \times U(n - k + 1))$ ($k = 1, \dots, n$).

The idea of proof is as follows. Let $S^1 = \mathcal{Z}U(n + 1, 1)$ be the center of $U(n + 1, 1)$. Then $S^1 \cdot \tilde{H} \leq U(n + 1, 1)$. There is an equivariant fibration:

$$(6.4) \quad (S^1, S^1) \longrightarrow (S^1 \cdot \tilde{H}, \Gamma, X) \xrightarrow{(P, p)} (G, Q, W)$$

where we put $G = S^1 \cdot \tilde{H}/S^1$, $Q = \Gamma/S^1 \cap \Gamma$ and $W = X/S^1$. As $Q, G \leq PU(n + 1, 1)$, the quotient W/Q is a spherical CR -orbifold with CR -action G . To determine X/Γ reduces to the classification of CR -manifolds (Q, W) with the 1-parameter group G of CR -transformations. The classification is accomplished by the result in [4].

When $\dim X/\Gamma = 4$, then $Q \leq U(1, 1)$ so that $L(Q) \subset S^1$ ($k = 1$). According to whether $L(Q)$ is a Cantor set in S^1 or $L(Q) = S^1$, it is well known that $S^3 - L(Q)/Q = S^1 \times S^2 \# \dots \# S^1 \times S^2$ or some finite cover of $S^3 - L(Q)/Q = V_{-1}^3/Q$ is a principal S^1 -bundle with nonzero euler class over a closed surface of genus $g \geq 2$.

6.1. Non Fefferman-Lorentz manifold. It is conceivable whether some finite cover of any compact conformally flat Fefferman-Lorentz parabolic manifold is a Fefferman-Lorentz manifold. It is not true in general. It will be shown

Proposition 6.4. *There exists a compact conformally flat Fefferman-Lorentz parabolic manifold P of dimension $2n + 2$ ($n \geq 1$) but no finite covering is a Fefferman-Lorentz manifold.*

This manifold P supports a principal fiber space: $T^2 \rightarrow P \rightarrow \mathbb{H}_{\mathbb{C}}^n/Q_0$ where $\mathbb{H}_{\mathbb{C}}^n/Q_0$ is a compact complex hyperbolic manifold.

7. REPRESENTATION SPACE

Let X/Γ be a compact conformally flat Lorentz manifold with S^1 -action so that $X \subset \mathbb{R} \times S^{2n+1}$, $\Gamma, \tilde{S}^1 \leq \widetilde{O(m+2, 2)}$. If $p: \widetilde{O(m+2, 2)} \rightarrow O(m+2, 2)$ is the covering homomorphism, put $G = p(\tilde{S}^1) \leq O(m+2, 2)$. We will prove that

- If G is compact, then $m = 2n$ and $G = S^1$, $C_{O(m+2, 2)}(S^1) = U(n + 1, 1)$. $(S^1, X/\Gamma)$ is locally modelled on $(U(n + 1, 1), S^1 \times S^{2n+1})$ where $S^1 = \mathcal{Z}U(n + 1, 1)$, i.e. X/Γ is a conformally flat Fefferman-Lorentz parabolic manifold.

- If G is noncompact, then either $\Gamma \leq \mathbb{R}^{m+2} \rtimes O(m+1, 1)$ and $G = \mathbb{R}$ or $\Gamma \leq O(m+1, 1) \times \mathbb{R}^+$ and $G = \mathbb{R}^+$.

Proposition 7.1. *Let $X/\Gamma = S^1 \times \mathcal{N}^3/\Delta$ which is a conformally flat Lorentz parabolic manifold and $\Gamma = \mathbb{Z} \times \Delta \leq \widetilde{O(4, 2)}$, $\# \subset \mathbb{R} \times S^3$. There are exactly two distinct faithful representations up to conjugate in $O(4, 2)$:*

$$(7.1) \quad \begin{aligned} \rho_1 : \Gamma &\rightarrow \mathbb{R} \times (\mathcal{N} \rtimes U(1)), \quad S^1 \text{ is lightlike.} \\ \rho_2 : \Gamma &\rightarrow \mathbb{R}^3 \times (\mathbb{R}^2 \rtimes O(2)) \leq \mathbb{R}^4 \rtimes O(3, 1), \quad S^1 \text{ is spacelike.} \end{aligned}$$

Then the space of discrete faithful representations $R(\Gamma, \widetilde{O(4, 2)})$ consists of two components $R(\Gamma, \mathbb{R} \times (\mathcal{N} \rtimes U(1)))$, $R(\Gamma, \mathbb{R}^3 \rtimes (\mathbb{R}^2 \rtimes O(2)))$.

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