

On ϵ -Optimality Theorems and ϵ -Duality Theorems for Convex Semidefinite Optimization Problems with Conic Constraints

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Abstract

In this paper, we review results in Lee [13] and Lee et al. [14] without proofs. We consider ϵ -approximate solutions for a convex semidefinite optimization problem (SDP) with conic constraints and present ϵ -optimality theorems and ϵ -saddle point theorems for such solutions which hold under a weakened constraint qualification or which hold without any constraint qualification. We give a Wolfe type dual problem for (SDP), and then we present ϵ -duality results, which hold under a weakened constraint qualification.

1 Introduction and Preliminaries

Convex semidefinite optimization problem is to optimize an objective convex function over a linear matrix inequality. When the objective function is linear and the corresponding matrices are diagonal, this problem becomes a linear optimization problem. So, this problem is an extension of a linear optimization problem. For convex semidefinite optimization problem, strong duality without constraint qualification [17], complete dual characterization conditions of solutions ([7, 11]), saddle point theorems [1] and characterizations of optimal solution sets [9] have been investigated.

To get the ϵ -approximate solution, many authors have established ϵ -optimality conditions, ϵ -saddle point theorems and ϵ -duality theorems for several kinds of op-

timization problems ([2, 3, 4, 15, 16, 18, 19]). Recently, Jeyakumar et al. [7] established sequential optimality conditions for exact solutions of convex optimization problem which holds without any constraint qualification. Jeyakumar et al. [6] gave ϵ -optimality conditions for convex optimization problems, which hold without any constraint qualification. Yokoyama et al. [19] gave a special case of convex optimization problem which ϵ -optimality conditions. Kim et al. [12] proved sequential ϵ -saddle point theorems and ϵ -saddle point theorems for convex semidefinite optimization problems which have not conic constraints. Recently, Lee [13] and Lee et al. [14] extended results in Kim et al. [12] to convex semidefinite optimization problems with conic constraints.

In this paper, we review the results in Lee [13] and Lee et al. [14] without proofs. We consider ϵ -approximate solutions for a convex semidefinite optimization problem with conic constraints and present ϵ -optimality theorems and ϵ -saddle point theorems for such solutions which hold under a weakened constraint qualification or which hold without any constraint qualification. We give a Wolfe type dual problem for the convex semidefinite optimization problem with conic constraint and then we present ϵ -duality results, which hold under a weakened constraint qualification.

Consider the following convex semidefinite optimization problem:

$$\begin{aligned} \text{(SDP)} \quad & \text{Minimize} \quad f(x) \\ & \text{subject to} \quad F_0 + \sum_{i=1}^m x_i F_i \succeq 0, \quad (x_1, x_2, \dots, x_m) \in C, \end{aligned}$$

where $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is a convex function, the space of $n \times n$ real symmetric matrices, C is a closed convex cone of $\mathbb{R}^m$, and for $i = 0, 1, \dots, m$, $F_i \in S_n$. The space S_n is partially ordered by the Löwner order, that is, for $M, N \in S_n$, $M \succeq N$ if and only if $M - N$ is positive semidefinite. The inner product in S_n is defined by $(M, N) = \text{Tr}[MN]$, where $\text{Tr}[\cdot]$ is the trace operation.

Let $S := \{M \in S_n \mid M \succeq 0\}$. Then S is self-dual, that is,

$$S^+ := \{\theta \in S_n \mid (\theta, Z) \geq 0, \text{ for any } Z \in S\} = S.$$

Let $F(x) := F_0 + \sum_{i=1}^m x_i F_i$, $\hat{F}(x) := \sum_{i=1}^m x_i F_i$, $x = (x_1, \dots, x_m) \in \mathbb{R}^m$. Then $\hat{F}$ is a linear operator from $\mathbb{R}^m$ to S_n and its dual is defined by

$$\hat{F}^*(Z) = (\text{Tr}[F_1 Z], \dots, \text{Tr}[F_m Z])$$

for any $Z \in S_n$. Clearly, $A := \{x \in C \mid F(x) \in S\}$ is the feasible set of (SDP).

We define the ϵ -approximate solution of (SDP) as follows:

Definition 1.1 Let $\epsilon \geq 0$. Then $\bar{x}$ is called an ϵ -approximate solution of (SDP) if for any $x \in A$,

$$f(x) \geq f(\bar{x}) - \epsilon.$$

Now we give the definitions of subdifferential and ϵ -subdifferential of convex function in [5].

Definition 1.2 Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function.

(1) The subdifferential of g at a is given by

$$\partial g(a) = \{v \in \mathbb{R}^n \mid g(x) \geq g(a) + \langle v, x - a \rangle, \text{ for all } x \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$

(2) The ϵ -subdifferential of g at a is given by

$$\partial_\epsilon g(a) = \{v \in \mathbb{R}^n \mid g(x) \geq g(a) + \langle v, x - a \rangle - \epsilon, \text{ for all } x \in \mathbb{R}^n\}.$$

Definition 1.3 Let C be a closed convex set in $\mathbb{R}^n$ and $x \in C$.

(1) Let $N_C(x) = \{v \in \mathbb{R}^n \mid \langle v, y - x \rangle \leq 0, \text{ for all } y \in C\}$.

Then $N_C(x)$ is called the normal cone to C at x .

(2) Let $\epsilon \geq 0$. Let $N_C^\epsilon(x) = \{v \in \mathbb{R}^n \mid \langle v, y - x \rangle \leq \epsilon, \text{ for all } y \in C\}$.

Then $N_C^\epsilon(x)$ is called the ϵ -normal set to C at x .

(3) When C is a closed convex cone in $\mathbb{R}^n$, $N_C(0)$ is denoted by C^* and called the negative dual cone of C .

Following the proof of Lemma 2.2 in [8], we can obtain the following lemma.

Lemma 1.1 [13, 14] Let $F_i \in S_n$, $i = 0, 1, \dots, m$. Suppose that $A \neq \emptyset$. Let $u \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. Then the following are equivalent:

$$(i) \quad \{x \in C \mid F_0 + \sum_{i=1}^m F_i x_i \succeq 0\} \subset \{x \in \mathbb{R}^m \mid \langle u, x \rangle \geq \alpha\}.$$

$$(ii) \quad \begin{pmatrix} u \\ \alpha \end{pmatrix} \in cl \left(\bigcup_{(Z, \delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -Tr[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+ \right).$$

Using the above Lemma 1.1, we can obtain the following lemma:

Lemma 1.2 [13, 14] *Suppose that $A \neq \emptyset$. Let $\bar{x} \in A$ and $\epsilon \geq 0$. Then $\bar{x}$ is an ϵ -approximate solution of (SDP) if and only if there exist $\epsilon_0, \epsilon_1 \geq 0$, $v \in \partial_{\epsilon_0} f(\bar{x})$ such that $\epsilon_0 + \epsilon_1 = \epsilon$, and*

$$\begin{pmatrix} v \\ \langle v, \bar{x} \rangle - \epsilon_1 \end{pmatrix} \in cl \left(\bigcup_{(Z, \delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -Tr[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+ \right).$$

2 ϵ -Optimality Conditions

Now, using the above Lemma 1.2, we can give the following two ϵ -optimality conditions for (SDP).

Theorem 2.1 [13] *Let $\bar{x} \in A$ and $\bigcup_{(Z, \delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -Tr[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+$ is closed in $\mathbb{R}^m \times \mathbb{R}$. Then $\bar{x} \in A$ is an ϵ -approximate solution of (SDP) if and only if there exist $\epsilon_0, \epsilon_1 \geq 0$, $v \in \partial_{\epsilon_0} f(\bar{x})$, $Z \in S$, $c^* \in C^*$ such that $\epsilon_0 + \epsilon_1 = \epsilon$,*

$$v = \hat{F}^*(Z) - c^*$$

and

$$0 \leq Tr[ZF(\bar{x})] \leq \epsilon_1.$$

Theorem 2.2 [13] *Let $\bar{x} \in A$. Then $\bar{x}$ is an ϵ -approximate solution of (SDP) if and only if there exist $\epsilon_0, \epsilon_1 \geq 0$, $v \in \partial_{\epsilon_0} f(\bar{x})$, $c_n^* \in C^*$, $Z_n \in S$, $\delta_n \geq 0$ such that $\epsilon_0 + \epsilon_1 = \epsilon$,*

$$v = \lim_{n \rightarrow \infty} (\hat{F}^*(Z_n) - c_n^*)$$

and

$$\langle v, \bar{x} \rangle - \epsilon_1 = \lim_{n \rightarrow \infty} (-Tr[Z_n F_0] - \delta_n).$$

3 ϵ -Saddle Point Theorems and ϵ -Duality Theorem

Now we give ϵ -saddle point theorems and ϵ -duality theorems for **(SDP)**. Using Lemma 1.1, we can obtain the following two lemmas which are useful in proving our ϵ -saddle point theorems for **(SDP)**.

Lemma 3.1 [13] *Let $\bar{x} \in A$. Then $\bar{x} \in A$ is an ϵ -approximate solution of **(SDP)** if and only if there exists a sequence $\{Z_n\}$ in S such that*

$$f(x) - \liminf_{n \rightarrow \infty} \text{Tr}[Z_n F(x)] \geq f(\bar{x}) - \epsilon, \quad \text{for any } x \in C.$$

Lemma 3.2 [13, 14] *Let $\bar{x} \in A$. Suppose that $\bigcup_{(Z, \delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -\text{Tr}[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+$ is closed. Then $\bar{x}$ is an ϵ -approximate solution of **(SDP)** if and only if there exists $Z \in S$ such that for any $x \in C$,*

$$f(x) - \text{Tr}[ZF(x)] \geq f(\bar{x}) - \epsilon.$$

Let $\epsilon \geq 0$. Consider the following sequential ϵ -saddle point problem and ϵ -saddle point problem:

(SSP) $_{\epsilon}$ Find $\bar{x} \in C$ and a sequence $\{\bar{Z}_n\} \subset S$ such that

$$\begin{aligned} f(\bar{x}) - \liminf_{n \rightarrow \infty} \text{Tr}[Z_n F(\bar{x})] - \epsilon &\leq f(\bar{x}) - \liminf_{n \rightarrow \infty} \text{Tr}[\bar{Z}_n F(\bar{x})] \\ &\leq f(x) - \liminf_{n \rightarrow \infty} \text{Tr}[\bar{Z}_n F(x)] + \epsilon \end{aligned}$$

for any $x \in C$ and any sequence $\{Z_n\} \subset S$.

(SP) $_{\epsilon}$ Find $\bar{x} \in C$ and $\bar{Z} \in S$ such that

$$\begin{aligned} f(\bar{x}) - \text{Tr}[ZF(\bar{x})] - \epsilon &\leq f(\bar{x}) - \text{Tr}[\bar{Z}F(\bar{x})] \\ &\leq f(x) - \text{Tr}[\bar{Z}F(x)] + \epsilon \end{aligned}$$

for any $x \in C$ and any $Z \in S$.

Now we give a useful characterization of solution of **(SSP) $_{\epsilon}$** .

Lemma 3.3 [13] Suppose that $A \neq \emptyset$. Let $(\bar{x}, \{\bar{Z}_n\}) \in C \times S$, $n = 1, 2, \dots$. Then $(\bar{x}, \{\bar{Z}_n\})$ is a solution of $(\text{SSP})_\epsilon$ if and only if

$$f(\bar{x}) - \liminf_{n \rightarrow \infty} \text{Tr}[\bar{Z}_n F(\bar{x})] \leq f(x) - \liminf_{n \rightarrow \infty} \text{Tr}[\bar{Z}_n F(x)] + \epsilon$$

for any $x \in C$,

$$0 \leq \liminf_{n \rightarrow \infty} \text{Tr}[\bar{Z}_n F(\bar{x})] \leq \epsilon$$

and $F(\bar{x}) \in S$.

Using Lemma 3.1 and 3.3, we can give a sequential ϵ -saddle point theorem which holds between (SDP) and $(\text{SSP})_\epsilon$.

Theorem 3.1 (Sequential ϵ -Saddle Point Theorem) [13]

(1) If $\bar{x} \in A$ is an ϵ -approximate solution of (SDP) , then there exists a sequence $\{\bar{Z}_n\}$ such that $(\bar{x}, \{\bar{Z}_n\})$ is a solution of $(\text{SSP})_\epsilon$

(2) If $A \neq \emptyset$ and $(\bar{x}, \{\bar{Z}_n\})$ is a solution of $(\text{SSP})_\epsilon$, then $\bar{x}$ is an 2ϵ -approximate solution of (SDP) .

Using Lemma 3.2, we can give an ϵ -saddle point theorem which holds between (SDP) and $(\text{SP})_\epsilon$.

Theorem 3.2 [13] (ϵ - Saddle Point Theorem) Suppose that

$$\bigcup_{(Z, \delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -\text{Tr}[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+$$

is closed. If $\bar{x} \in A$ is an ϵ -approximate solution of (SDP) , then there exists $\bar{Z} \in S$ such that $(\bar{x}, \bar{Z})$ is a solution of $(\text{SP})_\epsilon$.

Theorem 3.3 [13] If $(\bar{x}, \bar{Z})$ is a solution of $(\text{SP})_\epsilon$, then $\bar{x}$ is an 2ϵ -approximate solution of (SDP) .

Now we can formulate dual problem (SDD) of (SDP) as follows:

$$\begin{aligned} (\text{SDD}) \quad & \text{Maximize} && f(x) - \text{Tr}[ZF(x)] \\ & \text{subject to} && 0 \in \partial_{\epsilon_0} f(x) - \hat{F}^*(Z) + N_C^{\epsilon_1}(x), \\ & && Z \succeq 0, \\ & && \epsilon_0 + \epsilon_1 \in [0, \epsilon]. \end{aligned}$$

We can prove ϵ -weak and ϵ -strong duality theorems which hold between (SDP) and (SDD).

Theorem 3.4 (ϵ -Weak Duality) [13, 14] *For any feasible x of (SDP) and any feasible (y, Z) of (SDD),*

$$f(x) \geq f(y) - \text{Tr}[ZF(y)] - \epsilon.$$

Theorem 3.5 (ϵ -Strong Duality) [13, 14] *Suppose that*

$$\bigcup_{(Z, \delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -\text{Tr}[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+ \text{ is closed.}$$

If $\bar{x}$ is an ϵ -approximate solution of (SDP), then there exists $\bar{Z} \in S$ such that $(\bar{x}, \bar{Z})$ is an 2ϵ -approximate solution of (SDD).

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