

NEW CLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

Nak Eun Cho (釜山水産大学)

Shigeyoshi Owa (近畿大・理工 尾和重義)

Abstract. In this paper, we introduce new subclasses $C_{n,p}(\alpha)$ of meromorphically multivalent functions defined by the subordination relation. We also obtain the inclusion relations for the classes $C_{n,p}(\alpha)$ and investigate the integral preserving properties of functions in $C_{n,p}(\alpha)$.

1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$(1.1) \quad f(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (a_{-p} \neq 0, p \in N = \{1, 2, \dots\})$$

which are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$. Following Uralegaddi and Somanatha [4], we define

$$(1.2) \quad D^0 f(z) = f(z),$$

$$(1.3) \quad D^1 f(z) = \frac{a_{-p}}{z^p} + (p+1)a_0 + (p+2)a_1 z + (p+3)a_2 z^2 + \dots$$

$$= \frac{(z^{p+1} f(z))'}{z^p},$$

$$(1.4) \quad D^2 f(z) = D(D^1 f(z)),$$

* 1991 Mathematics Subject Classification: 30C45.

and for $n = 1, 2, \dots$,

$$\begin{aligned}
 (1.5) \quad D^n f(z) &= D(D^{n-1} f(z)) \\
 &= \frac{a_{-p}}{z^p} + \sum_{m=1}^{\infty} (p+m)^n a_{m-1} z^{m-1} \\
 &= \frac{(z^{p+1} D^{n-1} f(z))'}{z^p}.
 \end{aligned}$$

Using the operator D^n , Cho and Lee [2] introduced the subclasses $B_{n,p}(\alpha)$ of Σ_p whose members are characterized by the condition

$$(1.6) \quad \operatorname{Re} \left\{ z^{p+1} (D^n f(z))' \right\} < -p \frac{n+\alpha}{n+1} \quad (z \in U = \{z : |z| < 1\})$$

for some $\alpha (0 \leq \alpha < 1)$ and $n \in N_0 = N \cup \{0\}$. They proved that $B_{n+1,p}(\alpha) \subset B_{n,p}(\alpha)$, and since $B_{0,p}(\alpha)$ is the class of meromorphically p -valent functions [3]), all functions in $B_{n,p}(\alpha)$ are p -valent. Also they considered some properties in connection with certain integral transform.

In this paper, we introduce the new classes $C_{n,p}(\alpha)$ of meromorphically p -valent functions in U .

Let $C_{n,p}(\alpha)$ denote the class of functions $f \in \Sigma_p$ which satisfy the condition

$$(1.7) \quad -z^{p+1} (D^n f(z))' \prec p + \frac{p(1-\alpha)}{n+2} z \quad (0 \leq \alpha < 1, z \in U),$$

where $\prec$ denotes the subordination relation. From (1.7), we have that $C_{n,p}(\alpha) \subset B_{n,p}(\alpha)$ for $n \in N_0$. Hence the classes $C_{n,p}(\alpha)$ are subclasses of meromorphically p -valent functions. Also we shall prove that $C_{n+1,p}(\alpha) \subset C_{n,p}(\alpha)$. Furthermore we consider certain integral transform of functions in $C_{n,p}(\alpha)$.

2. Properties of the classes $C_{n,p}(\alpha)$

For the proofs of coming theorems, we need the following lemma due to Jack [1].

Lemma 1. *Let w be non-constant regular in $U = \{z : |z| < 1\}$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0 , we have $z_0 w'(z_0) = kw(z_0)$ where k is a real number, $k \geq 1$.*

Theorem 1. $C_{n+1,p}(\alpha) \subset C_{n,p}(\alpha)$ for each $n \in N_0$.

Proof. Let $f \in C_{n+1,p}(\alpha)$. Then

$$(2.1) \quad -z^{p+1}(D^{n+1}f(z))' \prec p + \frac{p(1-\alpha)}{n+3}z.$$

Define $w(z)$ in $U = \{z : |z| < 1\}$ by

$$(2.2) \quad -z^{p+1}(D^n f(z))' = p + \frac{p(1-\alpha)}{n+2}w(z).$$

Clearly $w(0) = 0$. Using the identity

$$(2.3) \quad z(D^n f(z))' = D^{n+1}f(z) - (p+1)D^n f(z),$$

the equation (2.2) may be written as

$$(2.4) \quad -z^p(D^{n+1}f(z) - (p+1)D^n f(z)) = p + \frac{p(1-\alpha)}{n+2}w(z).$$

Differentiating (2.4), we obtain

$$(2.5) \quad -z^{p+1}(D^{n+1}f(z))' = p + \frac{p(1-\alpha)}{n+2}\{w(z) + zw'(z)\}.$$

We claim that $|w(z)| < 1$ in U . Suppose that there exists a point $z_0 \in U$ such that $\max_{|z| < |z_0|} |w(z)| = |w(z_0)| = 1$ ($w(z_0) \neq 1$). Then, by Lemma 1, we have

$$(2.6) \quad z_0 w'(z_0) = kw(z_0),$$

where $k \geq 1$. The equation (2.5) in conjunction with (2.6) yields

$$(2.7) \quad \begin{aligned} |z_0^{p+1}(D^{n+1}f(z_0))' + p| &= \left| \frac{p(1-\alpha)}{n+2}(1+k) \right| \\ &> \frac{p(1-\alpha)}{n+3}, \end{aligned}$$

which is a contradiction to (2.1). Hence $|w(z)| < 1$ in U and from (2.2) it follows that $f \in C_{n,p}(\alpha)$.

Theorem 2. Let $f \in C_{n,p}(\alpha)$. Then

$$(2.8) \quad F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) dt \quad (c > 0)$$

belongs to $C_{n,p}(\alpha)$.

Proof. Let $f \in C_{n,p}(\alpha)$. Define $w(z)$ in U by

$$(2.9) \quad -z^{p+1}(D^n F(z))' = p + \frac{p(1-\alpha)}{n+2}w(z).$$

Clearly $w(0) = 0$. Using the equation

$$(2.10) \quad z(D^n F(z))' = cD^n f(z) - (c+p)D^n F(z),$$

the equation (2.9) may be written as

$$(2.11) \quad -z^p(cD^n f(z) - (c+p)D^n F(z)) = p + \frac{p(1-\alpha)}{n+2}w(z).$$

Differentiating (2.11), we have

$$(2.12) \quad -z^{p+1}(D^n f(z))' = p + \frac{p(1-\alpha)}{n+2}w(z) + \frac{p(1-\alpha)}{c(n+2)}zw'(z).$$

We claim that $|w(z)| < 1$ in U . For otherwise, by Lemma 1, there exists z_0 , $|z_0| < 1$ such that $z_0 w'(z_0) = kw(z_0)$, where $|w(z_0)| = 1$ and $k \geq 1$. Applying this result to (2.12), we obtain

$$(2.13) \quad \begin{aligned} |z_0^{p+1}(D^n f(z_0))' + p| &= \left| \frac{p(1-\alpha)}{n+2} + \frac{p(1-\alpha)k}{c(n+2)} \right| \\ &> \frac{p(1-\alpha)}{n+2}, \end{aligned}$$

which contradicts our assumption. Hence $|w(z)| < 1$ in U and from (2.12) we have that $F \in C_{n,p}(\alpha)$.

Taking $n = 0$ and $c = 1$ in Theorem 2, we have the following

Corollary 1. *Let $f \in C_{0,p}(\alpha)$. Then*

$$(2.14) \quad F(z) = \frac{1}{z^{1+p}} \int_0^z t^p f(t) dt$$

belongs to $C_{0,p}(\alpha)$.

References

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Nak Eun Cho
 Department of Applied Mathematics
 College of Natural Sciences
 National Fisheries University of Pusan
 Pusan 608-737
 Korea

Sigeyoshi Owa
 Department of Mathematics
 Kinki University
 Higashi-Osaka, Osaka 577
 Japan