

Poly-Cauchy numbers

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1 Introduction.

In 1997 M. Kaneko ([7]) introduced the poly-Bernoulli numbers $B_n^{(k)}$ for an integer k and a non-negative integer n by

$$\frac{\text{Li}_k(1 - e^{-x})}{1 - e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!},$$

where

$$\text{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

denotes the k -th polylogarithm function ($k \geq 1$) It becomes a rational function if $k \leq 0$. When $k = 1$, $B_n^{(1)} = B_n$ are the Bernoulli numbers (with $B_1 = 1/2$), defined by the generating function

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n^{(1)} \frac{x^n}{n!}.$$

$B_n^{(k)}$ is explicitly expressed as

$$B_n^{(k)} = (-1)^n \sum_{m=0}^n \frac{(-1)^m m!}{(m+1)^k} \left\{ \begin{matrix} n \\ m \end{matrix} \right\},$$

where

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \frac{(-1)^m}{m!} \sum_{l=0}^m (-1)^l \binom{m}{l} l^m$$

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are the Stirling numbers of the second kind ([7, Theorem 1]).

For a positive integer k and a non-negative integer n , *poly-Cauchy numbers* (of the first kind) $c_n^{(k)}$ are given by

$$\text{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}, \quad (1)$$

where

$$\text{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}$$

are called the k -th *polylogarithm factorial* function. When $k = 1$, $c_n^{(1)} = c_n$ are the Cauchy numbers ([4]) defined by the generating function defined by the generating function

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}.$$

$c_n^{(k)}$ is explicitly expressed as

$$c_n^{(k)} = (-1)^n \sum_{m=0}^n \frac{(-1)^m}{(m+1)^k} \begin{bmatrix} n \\ m \end{bmatrix},$$

where $\begin{bmatrix} n \\ m \end{bmatrix}$ are the (unsigned) Stirling numbers of the first kind appeared as the coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} x^m.$$

We record the values of $c_n^{(k)}$ for $n = 0, 1, \dots, 5$.

$$c_0^{(k)} = 1,$$

$$c_1^{(k)} = \frac{1}{2^k},$$

$$c_2^{(k)} = -\frac{1}{2^k} + \frac{1}{3^k},$$

$$c_3^{(k)} = \frac{2}{2^k} - \frac{3}{3^k} + \frac{1}{4^k},$$

$$c_4^{(k)} = -\frac{6}{2^k} + \frac{11}{3^k} - \frac{6}{4^k} + \frac{1}{5^k},$$

$$c_5^{(k)} = \frac{24}{2^k} - \frac{50}{3^k} + \frac{35}{4^k} - \frac{10}{5^k} + \frac{1}{6^k}.$$

Poly-Cauchy numbers (of the first kind) may be defined by using integrals.

Theorem 1. For $n \geq 0$ and $k \geq 1$, let $C_n^{(k)}$ be

$$C_n^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_k (x_1 x_2 \dots x_k) (x_1 x_2 \dots x_k - 1) \dots (x_1 x_2 \dots x_k - n + 1) dx_1 dx_2 \dots dx_k.$$

Then $C_n^{(k)} = c_n^{(k)}$.

As the Stirling numbers of the second kind is related to $e^t - 1$ and the Stirling numbers of the second is to $1/\ln(1 - t)$ via Riordan arrays (see e.g. [9, 11, 12]), it may be natural to consider if some properties which hold on Bernoulli numbers (polynomials) would also hold on Cauchy numbers (polynomials).

2 Polylogarithm factorial function

Note that for $k \geq 2$

$$\frac{d}{dz} \text{Li}_k(z) = \frac{1}{z} \text{Li}_{k-1}(z),$$

so

$$\text{Li}_k(z) = \int_0^z \frac{\text{Li}_{k-1}(t)}{t} dt;$$

on the other hand,

$$\frac{d}{dz} (z \text{Lif}_k(z)) = \text{Lif}_{k-1}(z),$$

so

$$\text{Lif}_k(z) = \frac{1}{z} \int_0^z \text{Lif}_{k-1}(t) dt. \quad (2)$$

A Riordan array is a pair $(d(t), h(t))$ where d and h are analytic functions and $d(0) \neq 0$ ([11, 12]). This pair then defines an infinite lower triangular array $\{d_{n,k}\}$, where

$$\sum_{n=0}^{\infty} d_{n,m} t^n = d(t) (t \cdot h(t))^m.$$

From this definition, $d(t)(t \cdot h(t))^m$ is the generating function of column k in the array. It is known that Pascal triangle $\{P_{n,m}\}_{n,k \geq 0}$ is represented by a Riordan array:

$$\frac{1}{1-t} \left(\frac{t}{1-t} \right)^m = \sum_{n=0}^{\infty} \binom{n}{m} t^n = \sum_{n=0}^{\infty} P_{n,m} t^n$$

(Unsigned) Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$, which arise as coefficients of the rising factorial

$$x(x+1) \dots (x+n-1) = \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] x^m,$$

is represented by

$$1 \cdot \left(\ln \frac{1}{1-t} \right)^m = \sum_{n=0}^{\infty} \frac{m!}{n!} \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] t^n$$

Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$, which are determined by

$$\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} = \frac{1}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j)^n,$$

is represented by

$$1 \cdot (e^t - 1)^m = \sum_{n=0}^{\infty} \frac{m!}{n!} \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} t^n$$

Notice that $\text{Li}_1(z) = -\ln(1-z)$ and $\text{Lif}_1(z) = (e^z - 1)/z$.

3 Poly-Bernoulli numbers and poly-Cauchy numbers

The generating function of the poly-Bernoulli numbers can be written in terms of iterated integrals:

$$e^x \cdot \underbrace{\frac{1}{e^x - 1} \int_0^x \frac{1}{e^x - 1} \int_0^x \dots \frac{1}{e^x - 1} \int_0^x}_{k-1} \frac{x}{e^x - 1} \underbrace{dx dx \dots dx}_{k-1} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}.$$

The generating function of the poly-Cauchy numbers of the first kind $c_n^{(k)}$ ($k \geq 2$) are also written in terms of iterated integrals:

$$\underbrace{\frac{1}{\ln(1+x)} \int_0^x \frac{1}{(1+x) \ln(1+x)} \int_0^x \cdots \frac{1}{(1+x) \ln(1+x)} \int_0^x \frac{x}{(1+x) \ln(1+x)} dx dx \cdots dx}_{k-1} \\ = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}.$$

It is known that the identity

$$\sum_{m=0}^n (-1)^m \begin{bmatrix} n+1 \\ m+1 \end{bmatrix} B_m^{(k)} = \frac{n!}{(n+1)^k}$$

holds. On the other hand,

$$\sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} c_m^{(k)} = \frac{1}{(n+1)^k}.$$

It is known that the duality theorem holds for poly-Bernoulli numbers. Namely,

$$B_n^{(-k)} = B_k^{(-n)} \quad \text{for } n, k \geq 0.$$

It is due to the symmetric formula:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$

However, the duality theorem does not hold for poly-Cauchy numbers. In fact, we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = e^y (1+x)^{e^y}.$$

4 Poly-Cauchy polynomials

We introduce the *poly-Cauchy polynomials* (of the first kind) $c_n^{(k)}(z)$ for a positive integer k and a non-negative integer n , given by the generating function

$$\frac{\text{Lif}_k(\ln(1+x))}{(1+x)^z} = \sum_{n=0}^{\infty} c_n^{(k)}(z) \frac{x^n}{n!}, \quad (3)$$

where

$$\text{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}.$$

When $z = 0$, $c_n^{(k)}(0) = c_n^{(k)}$ are the poly-Cauchy numbers. We may also define the poly-Cauchy polynomials of the first kind $c_n^{(k)}(z)$ by

$$c_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_k \binom{x_1 x_2 \dots x_k - z}{n} dx_1 dx_2 \dots dx_k.$$

The first several polynomials are

$$\begin{aligned} c_0^{(k)}(z) &= 1, \\ c_1^{(k)}(z) &= \frac{1}{2^k} - z, \\ c_2^{(k)}(z) &= -\frac{1}{2^k} + \frac{1}{3^k} + \left(1 - \frac{2}{2^k}\right)z + z^2, \\ c_3^{(k)}(z) &= \frac{2}{2^k} - \frac{3}{3^k} + \frac{1}{4^k} + \left(-2 + \frac{6}{2^k} - \frac{3}{3^k}\right)z \\ &\quad + \left(-3 + \frac{3}{2^k}\right)z^2 - z^3, \\ c_4^{(k)}(z) &= -\frac{6}{2^k} + \frac{11}{3^k} - \frac{6}{4^k} + \frac{1}{5^k} + \left(6 - \frac{22}{2^k} + \frac{18}{3^k} - \frac{4}{4^k}\right)z \\ &\quad + \left(11 - \frac{18}{2^k} + \frac{6}{3^k}\right)z^2 + \left(6 - \frac{4}{2^k}\right)z^3 + z^4. \end{aligned}$$

Poly-Bernoulli polynomials $B_n^{(k)}(z)$ were defined as

$$\frac{\text{Li}_k(1 - e^{-x})}{1 - e^{-x}} e^{xz} = \sum_{n=0}^{\infty} B_n^{(k)}(z) \frac{x^n}{n!}.$$

([1]). Note that $B_n^{(k)}(z)$ are defined in [5] by replacing e^{xz} by e^{-xz} . In [10], $B_n^{(k)}(z)$ are defined by

$$\frac{\text{Li}_k(1 - e^{-x})}{e^x - 1} e^{zx} = \sum_{n=0}^{\infty} B_n^{(k)}(z) \frac{x^n}{n!}$$

Concerning the poly-Bernoulli polynomials, for an integer k and a positive integer n we have

$$\frac{d}{dz} B_n^{(k)}(z) = n B_{n-1}^{(k)}(z)$$

([1, Theorem 1.4]). The poly-Cauchy polynomials $c_n^{(k)}$, however, are not Appell sequences. By differentiating $c_n^{(k)}$, we have

$$\frac{d}{dz} c_n^{(k)}(z) = (-1)^n n! \sum_{l=0}^{n-1} \frac{(-1)^l}{(n-l)l!} c_l^{(k)}(z) \quad (n \geq 1).$$

We have a recurrence formula for the poly-Cauchy polynomials $c_n^{(k)}(z)$ in terms of the poly-Cauchy numbers $c_n^{(k)}$ and the Cauchy polynomials $c_n(z)$.

Theorem 2. *For a positive integer k and a non-negative integer n we have*

$$c_n^{(k)} = (-1)^n n! \sum_{m=0}^n \frac{(-1)^m c_m^{(k-1)}}{m!} \sum_{l=0}^{n-m} \frac{(-1)^l c_l(z)}{(n-l+1)l!}.$$

Poly-Cauchy polynomials of the first kind can be also expressed explicitly in terms of the Stirling number of the first kind:

$$c_n^{(k)}(z) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} (-1)^{n-m} \sum_{i=0}^m \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}. \quad (4)$$

5 Poly-Cauchy numbers and polynomials of the second kind

The *poly-Cauchy polynomials of the second kind* $\hat{c}_n^{(k)}(z)$ are defined by

$$(1+x)^z \text{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty} \hat{c}_n^{(k)}(z) \frac{x^n}{n!}.$$

The first several polynomials are

$$\hat{c}_0^{(k)}(z) = 1,$$

$$\hat{c}_1^{(k)}(z) = -\frac{1}{2^k} + z,$$

$$\hat{c}_2^{(k)}(z) = \frac{1}{2^k} + \frac{1}{3^k} - \left(1 + \frac{2}{2^k}\right)z + z^2,$$

$$\hat{c}_3^{(k)}(z) = -\frac{2}{2^k} - \frac{3}{3^k} - \frac{1}{4^k} + \left(2 + \frac{6}{2^k} + \frac{3}{3^k}\right)z - \left(3 + \frac{3}{2^k}\right)z^2 + z^3,$$

$$\begin{aligned} \hat{c}_4^{(k)}(z) = & \frac{6}{2^k} + \frac{11}{3^k} + \frac{6}{4^k} + \frac{1}{5^k} - \left(6 + \frac{22}{2^k} + \frac{18}{3^k} + \frac{4}{4^k}\right)z \\ & + \left(11 + \frac{18}{2^k} + \frac{6}{3^k}\right)z^2 - \left(6 + \frac{4}{2^k}\right)z^3 + z^4. \end{aligned}$$

If $z = 0$, then $\hat{c}_n^{(k)}(0) = \hat{c}_n^{(k)}$ are the *poly-Cauchy numbers of the second kind*. If $k = 1$, then $\hat{c}_n^{(1)}(z) = \hat{c}_n(z)$ are the Cauchy polynomials given in [3]. The generating function of $c_n(z)$ is given by

$$\begin{aligned} (1+x)^z \text{Lif}_1(-\ln(1+x)) &= \frac{x(1+x)^z}{(1+x)\ln(1+x)} \\ &= \sum_{n=0}^{\infty} \hat{c}_n(z) \frac{x^n}{n!}. \end{aligned}$$

Note that x is replaced by $-x$ in the generating function in [3]. Under these definitions we call $c_n^{(k)}$ and $c_n^{(k)}(z)$ poly-Cauchy numbers of the first kind and poly-Cauchy polynomials of the first kind, respectively. In similar methods, we have the corresponding results to those in the previous sections.

Proposition 1.

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \hat{c}_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^y}{(1+x)^{e^y}}.$$

Theorem 3. For a positive integer k and a non-negative integer n we have

$$\begin{aligned} \hat{c}_n^{(k)}(z) = & (-1)^n n! \sum_{m=0}^n \frac{(-1)^m \hat{c}_m^{(k-1)}}{m!} \sum_{l=0}^{n-m} \frac{(-1)^l \hat{c}_l(z)}{(n-l+1)l!} \\ & + (-1)^n n! \sum_{m=0}^{n-1} \frac{(-1)^m \hat{c}_m^{(k-1)}}{m!} \sum_{l=0}^{n-m-1} \frac{(-1)^{l+1} \hat{c}_l(z)}{(n-l)l!}. \end{aligned}$$

Theorem 4.

$$\frac{d}{dz} \hat{c}_n^{(k)}(z) = (-1)^{n-1} n! \sum_{l=0}^{n-1} \frac{(-1)^l}{(n-l)l!} \hat{c}_l^{(k)}(z) \quad (n \geq 1).$$

6 Some generalizations of poly-Cauchy numbers and polynomials

The generating function of ordinary generalized poly-Bernoulli numbers $B_{n,\chi}^{(k)}$ ([10]) is given by

$$\frac{1}{f} \sum_{a=1}^f \chi(a) \frac{\text{Li}_k(1 - e^{-fx})}{e^{fx} - 1} e^{ax} = \sum_{n=0}^{\infty} B_{n,\chi}^{(k)} \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f}.$$

The generating function of generalized poly-Bernoulli polynomials $B_{n,\chi}^{(k)}(z)$ ([2]) is given by

$$\frac{1}{f} \sum_{a=1}^f \chi(a) \frac{\text{Li}_k(1 - e^{-fx})}{e^{fx} - 1} e^{(z+a)x} = \sum_{n=0}^{\infty} B_{n,\chi}^{(k)}(z) \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f}.$$

The generating function of poly-Bernoulli polynomials with a, b parameters $B_n^{(k)}(z; a, b)$ ([6]) is given by

$$\frac{\text{Li}_k(1 - (ab)^{-x})}{b^x - a^{-x}} e^{zx} = \sum_{n=0}^{\infty} B_n^{(k)}(z; a, b) \frac{x^n}{n!}.$$

The generating function of poly-Bernoulli polynomials with a, b, c parameters $B_n^{(k)}(z; a, b, c)$ ([6]) is given by

$$\frac{\text{Li}_k(1 - (ab)^{-x})}{b^x - a^{-x}} c^{zx} = \sum_{n=0}^{\infty} B_n^{(k)}(z; a, b, c) \frac{x^n}{n!}.$$

Mari Yokohama (Hirosaki University) proposes the following generalizations of Cauchy numbers. Let n and k be integers with $n \geq 0$ and $k \geq 1$. Let q be a real number with $q \neq 0$. Define the *poly-Cauchy numbers with q*

parameter (of the first kind) $c_{n,q}^{(k)}$ by

$$c_{n,q}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 x_2 \cdots x_k) (x_1 x_2 \cdots x_k - q) \cdots (x_1 x_2 \cdots x_k - (n-1)q) dx_1 dx_2 \cdots dx_k.$$

Then for a real number $q \neq 0$

$$c_{n,q}^{(k)} = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-q)^{n-m}}{(m+1)^k} \quad (n \geq 0, k \geq 1).$$

The generating function of $c_{n,q}^{(k)}$ is given by

$$\text{Lif}_k \left(\frac{\ln(1+qx)}{q} \right) = \sum_{n=0}^{\infty} c_{n,q}^{(k)} \frac{x^n}{n!} \quad (q \neq 0).$$

The generating function can be also written in the form of iterated integrals as that of the poly-Cauchy numbers. For $k \geq 2$ we have

$$\begin{aligned} & \underbrace{\frac{q}{\ln(1+qx)} \int_0^x \frac{q}{(1+qx) \ln(1+qx)} \int_0^x \cdots \frac{q}{(1+qx) \ln(1+qx)} \int_0^x}_{k-1} \\ & \frac{q((1+qx)^{1/q} - 1)}{(1+qx) \ln(1+qx)} \underbrace{dx dx \cdots dx}_{k-1} \\ & = \sum_{n=0}^{\infty} c_{n,q}^{(k)} \frac{x^n}{n!}. \end{aligned}$$

For $k = 1$ we have

$$\frac{q((1+qx)^{1/q} - 1)}{\ln(1+qx)} = \sum_{n=0}^{\infty} c_{n,q} \frac{x^n}{n!}.$$

Similarly, define the poly-Cauchy numbers of the second kind with q parameter $\hat{c}_{n,q}^{(k)}$ by

$$\hat{c}_{n,q}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (-x_1 x_2 \cdots x_k) (-x_1 x_2 \cdots x_k - q) \cdots (-x_1 x_2 \cdots x_k - (n-1)q) dx_1 dx_2 \cdots dx_k.$$

Then

$$\hat{c}_{n,q}^{(k)} = (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{q^{n-m}}{(m+1)^k}.$$

The generating function of the poly-Cauchy numbers of the second kind with q parameter $\hat{c}_{n,q}^{(k)}$ is given by

$$\text{Lif}_k \left(-\frac{\ln(1+qx)}{q} \right) = \sum_{n=0}^{\infty} \hat{c}_{n,q}^{(k)} \frac{x^n}{n!}.$$

For $k \geq 2$ we have

$$\begin{aligned} & \underbrace{\frac{q}{\ln(1+qx)} \int_0^x \frac{q}{(1+qx) \ln(1+qx)} \int_0^x \cdots \frac{q}{(1+qx) \ln(1+qx)} \int_0^x}_{k-1} \\ & \quad \frac{q(1 - (1+qx)^{-1/q})}{(1+qx) \ln(1+qx)} \underbrace{dx dx \cdots dx}_{k-1} \\ & = \sum_{n=0}^{\infty} \hat{c}_{n,q}^{(k)} \frac{x^n}{n!}. \end{aligned}$$

For $k = 1$ we have

$$\frac{q(1 - (1+qx)^{-1/q})}{\ln(1+qx)} = \sum_{n=0}^{\infty} \hat{c}_{n,q} \frac{x^n}{n!}.$$

Poly-Cauchy polynomials with q parameter of the first kind $c_{n,q}^{(k)}(z)$ and of the second kind $\hat{c}_{n,q}^{(k)}(z)$ are also similarly defined.

Even more generalizations are possible. For example, define $c_{n,q}^{(k)}(l_1, l_2, \dots, l_k)$, where $l_1, l_2, \dots, l_k$ are nonzero real numbers, by

$$\begin{aligned} c_{n,q}^{(k)}(l_1, l_2, \dots, l_k) &= \int_0^{l_1} \int_0^{l_2} \cdots \int_0^{l_k} (x_1 x_2 \cdots x_k) (x_1 x_2 \cdots x_k - q) \\ & \quad \cdots (x_1 x_2 \cdots x_k - (n-1)q) dx_1 dx_2 \cdots dx_k. \end{aligned}$$

Then, for a real number $q \neq 0$

$$c_{n,q}^{(k)}(l_1, l_2, \dots, l_k) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-q)^{n-m} (l_1 l_2 \cdots l_k)^{m+1}}{(m+1)^k} \quad (n \geq 0, k \geq 1).$$

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