



Doctoral Dissertation

De Sitter Holography through Quantum Entanglement

by

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Submitted on December 18, 2025

Abstract

In this Ph.D. thesis, we explore various approaches of holographic principle in de Sitter space. In the absence of a universally accepted framework for de Sitter holography, we examine each candidate model and introduce new ingredients . Basically, we introduce our original work regarding the dS/CFT correspondence, static patch holography and half de Sitter holography. After that, we also present our work on holography in Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes.

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Acknowledgements

First of all, I would like to thank Prof. Tadashi Takayanagi for supervising me as a Ph.D. candidate and also for teaching me a lot of things from my undergraduate days. Without him, I could not finish my Ph.D. I was lost during the undergraduate days, but I found motivation to study the profound beauty of string theory and the holographic principle. He always encouraged me and supported me whatever I needed. His deep insight into physics guides me to challenge new physics and I would like to continue the research with this passion.

I would also thank all the collaborators for wonderful works. Without help of them, I could not accumulate such knowledge.

Finally, I would like to convey my appreciation to my parents for supporting me. Also, I would appreciate Yuri and my friends so that I could survive Ph.D. life.

Chapter 1

Introduction and Summary

One of the central open problems in contemporary particle physics is that gravity, unlike the Standard Model, does not define a renormalizable quantum field theory [5]. The most promising candidate for a consistent quantum theory of gravity is string theory, whose dynamics emerge from the simple premise that distinct vibrational modes of a fundamental string realize particle excitations. At present, however, string theory is under perturbative control only in the regime $g_s \ll 1$, which obstructs a complete understanding of the quantum nature of gravity.

A complementary route toward quantum gravity is provided by the holographic principle. The holographic principle posits that quantum gravitational physics can be reformulated in terms of a dual quantum-mechanical system [6, 7]. A concrete realization was given by Maldacena within the framework of string theory and is now known as the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [8]. It asserts that quantum gravity in $(d + 1)$ -dimensional AdS spacetime is dual to a d -dimensional CFT. This is remarkable in that it relates theories living in different spacetime dimensions.

Another fundamental question raised in [6] concerns the number of degrees of freedom contained within a finite region. It is natural to expect that a black hole saturates the maximal number of degrees of freedom, and Bekenstein and Hawking derived the precise relation [9, 10]

$$S = \frac{A}{4G}, \tag{1.0.1}$$

where A is the area of the event horizon and G is Newton's constant. In AdS/CFT, a natural extension of this formula to entanglement entropy was proposed by Ryu and Takayanagi [11]. The Ryu-Takayanagi formula states that the entanglement entropy of a boundary CFT subsystem is given by the area of a bulk minimal surface anchored on the subsystem's boundary. Since entanglement entropy quantifies the information (quantum bits) lost by tracing out the complement, this formula is a conceptual continuation of the Bekenstein-Hawking entropy, where information inside a black hole is separated by the event horizon.

Following the discovery of the Ryu-Takayanagi formula, the interplay between quantum information and geometry has been extensively explored. For instance, at low temperature the two CFTs are essentially decoupled, whereas upon increasing the temperature an eternal black hole appears [12]. The corresponding state is the thermofield double (TFD), which is manifestly an entangled state of two CFTs.

Let us now turn to our universe. Observations indicate that our universe is well described at late times by de Sitter space [13–15], and its earliest epoch is also expected to be of de Sitter space (more precisely, inflationary space) [16, 17]. One of the ultimate goals of theoretical physics is to formulate a consistent quantum gravity theory for our universe. A natural application of the holographic principle to de Sitter space was proposed by Strominger [18], now referred to as the de Sitter/Conformal Field Theory (dS/CFT) correspondence, which claims that quantum gravity in $(d + 1)$ -dimensional de Sitter space is dual to a d -dimensional CFT living at future infinity, corresponding to the end of inflation. However, serious conceptual issues arise. In AdS/CFT, the boundary CFT has a real time direction and is unitary, as is the bulk theory. By contrast, in the dS/CFT proposal time evolution in the bulk de Sitter space is unitary (absent dissipation), while the Euclidean boundary CFT is defined on a constant-time (Euclidean) slice and thus lacks real-time evolution. This indicates non-unitarity of the Euclidean CFT. The same tension is visible in the bulk-boundary dictionary [19–21]: the conformal dimension of the operator dual to a massive free scalar in de Sitter space can be complex. Moreover, the central charge is generically complex; see [22] for a dimension-by-dimension discussion. These features thwart the direct application of standard CFT tools (e.g. energy boundedness from below) and motivate us to consider alternative holographic frameworks for de Sitter space.

We now classify several approaches to holography in de Sitter space. The first is the original dS/CFT correspondence. Within this, one popular strategy employs analytic continuation from AdS [21], which allows for calculational control. Nevertheless, we believe this continuation does not capture the physical spectrum, as it does not realize unitary irreducible representations of $SO(1, d + 1)$ [23]. To simplify the discussion, let us focus on three dimensions, where the isometry of dS_3 is $SL(2, \mathbb{C})$. Different choices of real form lead to different realizations of the Virasoro algebra; for example, analytic continuation from AdS/CFT suggests a highest-weight representation. To evade non-unitarity on the boundary, one may instead place the dual QFT on a timelike surface. One realization is “half de Sitter” holography [3], where one considers a de Sitter slice and imposes Dirichlet boundary conditions as in AdS/CFT. A related idea, proposed by Susskind, is the static patch holography [24], rooted in Bousso’s covariant entropy bound [25], which we will review later.

There is currently no consensus on which holographic description is most plausible for de Sitter space. In this thesis, we analyze several holographic models using new methods.

We briefly summarize each paper below. The first main paper is [26]. There we construct bulk local states from symmetry considerations; in particular, we employ a highest-weight representation of the Virasoro algebra via analytic continuation from AdS/CFT, building on the technique of [27]. In AdS, one exploits isometries to construct a bulk local state at the center of global AdS, where rotational symmetry about the angular direction is manifest and must be respected by the state. States at arbitrary bulk points are then generated by acting with isometries. This construction is equivalent to the HKLL reconstruction [28]. We adapt this method to de Sitter space and analyze its relation to bulk scalar dynamics in de Sitter.¹ The second paper is [2], where we propose a new notion of holographic entanglement entropy in de Sitter space. Holographic entanglement entropy in de Sitter is not yet fully understood. A key difference from AdS is that bulk path integrals are not localized to a single saddle, so the geodesic/minimal-area approximation fails in general. In the dS/CFT setup one usually chooses a subsystem on future infinity and attempts to connect its entangling surfaces; in three dimensions there are, however, no bulk geodesics connecting two points on future infinity. A proposed resolution in [30] invokes the Hartle-Hawking state and leads to a pseudo-entropy [31], which is complex and therefore obstructs the standard entropic inequalities. In our approach we follow the static patch holography of Susskind and introduce a stretched horizon where the dual QFT degrees of freedom localize. We define a subsystem on a constant-time slice along the angular direction and compute its entanglement entropy. Since the Ryu-Takayanagi-type entropy violates the Lorentz-boosted version of strong subadditivity [32, 33], we search for a new holographic prescription that fully satisfies strong subadditivity. In later sections, we instead assume the Ryu-Takayanagi prescription and discuss the requirement on background spacetime. Using the replica trick and twist operators, and incorporating the α -vacua—which preserve the $SO(1, d + 1)$ symmetry but arise from ambiguity in splitting modes into positive and negative ones—we propose a new holographic entanglement entropy and determine the parameter regime in which it is consistent. This analysis yields constraints on the position of the stretched horizon, which is crucial for static patch holography. The third paper summarized here is [3]. As mentioned above, the idea is to introduce a timelike boundary on which the dual QFT resides. We consider a subsystem on a constant-time slice of this surface and compute the holographic entanglement entropy via the Ryu-Takayanagi formula. Inside the lightcone determined by null geodesics, the entropy is real and obeys strong subadditivity, while it fails beyond that region. The resulting super-volume-law behavior indicates an overcounting of Hilbert-space degrees of freedom and signals nonlocality in the dual QFT, a consequence of imposing a finite cutoff surface. The final paper is [4]. We assume the Ryu-Takayanagi prescription in the de Sitter space and compute the holographic entanglement entropy in the FLRW spacetime. We take a timelike surface as a holographic dual and define a subsystem there. We discuss

¹A similar construction in flat space appears in [29].

the strong subadditivity. Although the paper [1] is not the main focus of this thesis, we also present it for completeness.

This thesis is organized as follows. Chapter 2 reviews AdS/CFT and entanglement entropy in both CFT and AdS/CFT. Section 2.3 presents our original derivation of the UV behavior of entanglement entropy: in contrast to the standard heat-kernel approach with an ad hoc cutoff, we show that a double-cone regularization of the metric suffices to reproduce the known free-theory formula. Chapter 3 reviews de Sitter spacetime, beginning with metrics and coordinate systems, then turning to thermodynamic properties; we also discuss the Ryu-Takayanagi formula in de Sitter. Chapter 4, based on [26], constructs bulk local states in de Sitter via analytic continuation: after reviewing the AdS/CFT case, we explain how the conjugation rule is modified in dS/CFT and show that the Bunch-Davies Green’s function is reproduced from these states. Chapter 5 concerns static patch holography: after reviewing Bousso’s covariant entropy bound and Susskind’s proposal, Section 5.1 presents our results [2] on a new entanglement entropy in de Sitter, imposing strong subadditivity and deriving consistency regions for the α -vacuum parameter, the twist-operator mass, and the stretched-horizon position. While our primary analysis uses principal-series twist operators, we also analyze the complementary series, where the outcome changes qualitatively—suggesting that complementary-series twists may be natural in a static patch setup. Chapter 6 presents our work [3]: introducing a timelike boundary in global de Sitter, imposing Dirichlet boundary conditions, placing the dual QFT there, and analyzing holographic entanglement entropy and strong subadditivity on constant-time slices. The observed super-volume law is interpreted as nonlocality in the dual. Chapter 7 focuses on our paper [4], which explores holography in Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes. We extend static patch holography by introducing a suitable timelike holographic boundary and consider two geometries for the dual QFT: a “half-holography” model and a “horizon-type” model. We compute holographic entanglement entropy and examine strong subadditivity, finding evidence in favor of the horizon-type model. We also study the dependence on the equation-of-state parameter $w = p/\epsilon$, with p and ϵ the pressure and energy density in a perfect fluid. We conclude in Chapter 8.

Chapter 2

Holography in Anti-de Sitter space

2.1 AdS/CFT correspondence

2.1.1 Derivation from the superstring theory

We first review the "derivation" of the AdS/CFT correspondence by Maldacena in [8] (see [34] for a comprehensive review).

Let us consider the type-IIB superstring theory in ten-dimensional Minkowski spacetime with a fixed string coupling constant g_s . We place N parallel D3-branes coincidently¹. We label the tangential and perpendicular coordinates to branes as $x^0 \cdots x^3$ and $x^4 \cdots x^9$, respectively. We can realize this configuration because we can fix the gauge so that each brane stays at $x^i = \text{constant}$, where $i = 4 \cdots 9$. We will denote the distance of each brane as $U = \sqrt{\sum_i \Delta x_i^2}$ and consider the $U \rightarrow 0$ case. However, if we naively take this limit, it boils down to a trivial theory and there are no excitations. To resolve this, we fix U/α' and take $\alpha' \rightarrow 0$ instead. At sufficiently low energies, we can only consider massless modes because we have to consider the energy below $1/l_s$. The mass of the string is given by

$$M^2 \propto \frac{N - a}{l_s^2}, \quad (2.1.1)$$

where N is the level of excitations and a is the renormalization constant arising from the vacuum. If we take the $\alpha' = l_s^2 \rightarrow 0$ limit, $M \rightarrow \infty$ and thus, only $N = 0$ (massless) mode survives. We can describe the excitation in different ways; from open string and closed string viewpoints.

Let us first consider open strings description. Since the open strings can end on the D3-branes, these excitations correspond to the propagation of open strings between almost coincident D-branes. However, we need to take into account the interaction between D3-branes and the ten-dimensional superstrings. Then, the effective action for massless modes

¹To regulate physics, we introduce a regulator and then take a coincident limit

is given by

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}, \quad (2.1.2)$$

where the bulk represents the ten-dimensional Minkowski space and S_{bulk} is the ten-dimensional supergravity action with some higher derivative corrections. The S_{brane} is the D-brane effective action called a DBI action. The S_{int} is the action representing the interaction between ten-dimensional bulk gravity modes and brane modes, which can be obtained by covariantizing the brane action [35].

Linearizing the metric $g_{\mu\nu} \sim \eta_{\mu\nu} + \sqrt{8\pi G} h_{\mu\nu}$, we can expand the S_{bulk} as

$$S_{\text{bulk}} = \frac{1}{16\pi G} \int d^{10}x R \sim \int (\partial h)^2 + \sqrt{8\pi G} h (\partial h)^2 \dots \quad (2.1.3)$$

Similarly, the S_{int} can be expanded in G and is indeed proportional to the positive powers of $\sqrt{8\pi G}$. In order to evaluate the low energy limit correctly, we will consider the double scaling limit, where we take $l_s \rightarrow 0$ with other dimensionless parameters such as g_s and N fixed. In this limit, the Newton constant $\sqrt{8\pi G} \sim g_s l_s^4 \rightarrow 0$ and hence the S_{int} vanishes since it contains just positive powers of G . Also, the higher derivative corrections to the action are suppressed by powers of l_s . In this way, there are only freely propagating massless modes.

Now, let us identify the worldvolume theory on N D3-branes. For the starting and ending points of the string, we have N^2 degrees of freedom. This describes the four-dimensional $\mathcal{N} = 4$ U(N) Super-Yang Mills (SYM) theory, which is a worldvolume theory on the D3-branes. More precisely, the trace part $U(1) \in U(N)$ is neutrally charged for all fields and decouples². Thus, dynamically we can think of the gauge group as SU(N). Notice that this four-dimensional $\mathcal{N} = 4$ SU(N) SYM can be obtained by the dimensional reduction from the ten-dimensional $\mathcal{N} = 1$ SYM theory, which admits 16 supercharges. The most important property of four-dimensional $\mathcal{N} = 4$ SU(N) SYM is that it admits a conformal symmetry even at the quantum level, believed to be in full orders. The four-dimensional $\mathcal{N} = 4$ SU(N) SYM has 16 supercharges and 16 emergent superconformal charges³. This is an important point why the AdS/CFT works.

This four-dimensional SYM admits six scalars and four Weyl fermions transforming in the adjoint representation of the gauge group SU(N). Also, there is a global symmetry called an R-symmetry, which interchanges the supercharges. This SO(6) symmetry reflects the Lorentz symmetry of the six-dimensional transverse directions that appear in the dimensional

²The generator of U(1) is proportional to the identity matrix and it commutes with all the other fields. Therefore, the U(1) part does not interact with SU(N) part and decouples.

³Existence of the supercharge is equivalent to that of covariant constant spinor. Roughly speaking, usually the covariant derivative is not a partial derivative and the constant spinor is not a solution. However, if the connection vanishes along some direction, the constant becomes a solution. That is why we study holonomy of manifold.

reduction from ten dimensions to four dimensions. The $SU(4)$ is in accordance with the R-symmetry of the SYM $SO(6) \sim SU(4)$. We briefly comment on the physical meaning of the scalar field. It is the $N \times N$ matrix describing the position of D-branes, where open strings attach. The eigenvalues of the scalar field ϕ_{ab}^i , where $i = 1 \cdots 6$ and $a, b = 1 \cdots N$, represent the position of the N D3-branes. For example, ϕ_{12}^3 represents the distance of the first and second D3-branes along the x_3 direction. Thus after taking the trace with the gauge group, it corresponds to the average position of the N D3-branes. This trace part corresponds to the open strings attaching the same D3-branes.

Summary of open string viewpoint. In the open string description, we have two decoupled systems. The one is the worldvolume theory, which is a four-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM theory and the other is supergravity theory (only containing massless modes) in the ten-dimensional bulk.

Next, we move on to the closed string picture. D-branes are massive and non-perturbative objects. Usually, amplitude is a perturbative expansion in the weak coupling limit of string theory $g_s \sim 0$. However, the D-brane action is proportional to the inverse power of g_s as $S_{\text{brane}} \sim g_s^{-1}$ and hence it contributes to the path integral in $e^{-\frac{1}{g_s}}$, which is clearly non-perturbative configuration such as instantons. They backreact the geometry and form the black p(= 3)-brane solutions⁴

$$\begin{aligned} ds^2 &= f^{-\frac{1}{2}}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2), \\ f &= 1 + \frac{R_{\text{AdS}}^4}{r^4}, \\ R_{\text{AdS}}^4 &= 4\pi\alpha'^2 g_s N. \end{aligned} \tag{2.1.4}$$

Now, we consider the low energy limit. Effectively, there are two modes, which survive in this limit. The first one is the near horizon modes. The net energy observed from infinity is

$$E_\infty = f^{-\frac{1}{4}} E(r). \tag{2.1.5}$$

The low energy limit corresponds to $r \rightarrow 0$ limit, which means that the near horizon modes survive. In this limit, f can be approximated as $f \sim \frac{R_{\text{AdS}}^4}{r^4}$ and the corresponding metric becomes

$$ds^2 = \frac{r^2}{R_{\text{AdS}}^2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R_{\text{AdS}}^2}{r^2} dr^2 + R_{\text{AdS}}^2 d\Omega_5^2, \tag{2.1.6}$$

which is indeed the metric of $AdS_5 \times S^5$. The AdS space has negative contributions to the scalar curvature and the sphere has the same amount, but positive contributions. Summing

⁴This is not a black hole solution since the horizon is not one dimensional. Here, it extends three dimensions.

up these, the total curvature should vanish since we start with the flat metric.

The others are massless degrees of freedom with long wavelength in almost flat bulk space. We can confirm that the massless modes and the near horizon degrees of freedom indeed decouple because the low energy absorption behaves like $\sigma \sim \omega^3 R_{\text{AdS}}^8$, where ω is the energy and R_{AdS} is the AdS radius [36, 37]. Therefore, we have the following result:

Closed string Viewpoint. There are two low-energy modes. One is the near horizon mode localized at $r = 0$ and the other is given by freely propagating (massless) supergravity modes in flat regions $r = \infty$, which are far away from D3-branes.

After identifying the free supergravity modes in the bulk, we obtain the final statement

Maldacena's conjecture

"The type-IIB superstring theory on $AdS^5 \times S^5$ is dual to (3+1)-dim $\mathcal{N}=4$ $SU(N)$ SYM theory."

Indeed, type-IIB superstring theory admits 32 supercharges, which is the same number of supercharges in (3+1)-dim $\mathcal{N}=4$ $SU(N)$ SYM theory.

Validity of Approximation

In the previous section, we take various low energy limits and we will confirm whether the approximations are suitably valid. Firstly, we can approximate superstring theory as the supergravity (just a low energy theory) if the stringy effects can be ignored or the AdS radius is much larger than the string scale,

$$\frac{R_{\text{AdS}}^4}{l_s^4} \sim g_{YM}^2 N = \lambda \gg 1, \quad (2.1.7)$$

which states that the super Yang-Mills theory should be strongly coupled⁵. λ is called a 't Hooft coupling. This equation states that the AdS radius is small compared to the string scale and the strings can be approximated by particle description. That is also equivalent to saying that we ignore the higher derivative terms in string theory (like in the DBI action) since the expansion parameter is $\alpha' = l_s^2$.

Secondly, to ignore the loop correction of the gravity we should require that the Newton constant should be small. To separate the above higher derivative corrections from the loop corrections, we rescale the Weyl factor in the metric as $ds^2 = R_{\text{AdS}}^2 ds'^2$, where the R_{AdS} is

⁵From the string scattering, we have $g_s = g_{YM}^2$.

the AdS radius, and represent all the quantities in terms of the new metric, which is given by

$$ds'^2 = r'^2(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{dr'^2}{r'^2} + d\Omega_5^2, \quad (2.1.8)$$

where we rescale the coordinates as

$$r = r' R_{\text{AdS}}, \quad x_i = x'_i R_{\text{AdS}}. \quad (2.1.9)$$

The ten-dimensional Einstein-Hilbert action transforms as

$$S = \frac{1}{G} \int R_{\text{AdS}}^{10} \left(\frac{R'}{R_{\text{AdS}}^2} \right) = \frac{R_{\text{AdS}}^8}{G} \int R'. \quad (2.1.10)$$

Then, the effective Newton constant is given by

$$G' = \frac{G}{R_{\text{AdS}}^8} \sim \frac{1}{N^2}. \quad (2.1.11)$$

Thus, the original condition is given by

$$N \gg 1. \quad (2.1.12)$$

Brief Check of the Correspondence

In this section, we will check how the correspondence works from the symmetry point of view.

Let us examine the symmetry of each theory. For gravity side, it has a maximal number of isometries with a negative cosmological constant, which is just an AdS space. The AdS space can be formulated as the following embedding. Given the following metric

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 - dx_5^2 \quad (2.1.13)$$

and define the hypersurface by

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 = -R_{\text{AdS}}^2, \quad (2.1.14)$$

where R_{AdS} is an AdS radius. We can see that these coordinates span the AdS_5 space. By definition, this space has $SO(2,4)$ symmetry. On the other hand, four-dimensional $\mathcal{N} = 4$ SYM is a conformally invariant even at the quantum level. The bosonic subgroup of superconformal group $SU(2,2|4)$ is $SO(2,4)$, which is the same as the isometry of the AdS space. Also, the internal space S^5 has a symmetry $SO(6) \sim SU(4)$ and this is reflected as the R-symmetry in $SU(2,2|4)$, which rotates the supercharges (also rotates the 6 adjoint scalars

and 4 Weyl fermions).

We do not go into details here, but we can confirm the correspondence also by other quantities partition functions, correlation functions, spectrum, Wilson lines, entropy and so on [19, 20, 34, 36–38].

2.2 Review on entanglement entropy

Since the late twentieth century, entanglement entropy has become a central diagnostic for classifying quantum many-body phases and quantum field theories, especially CFTs. In two dimensions, an early derivation was provided by Holzhey-Larsen-Wilczek [39], and the replica method was subsequently systematized by Calabrese-Cardy [40]. We also present a mathematically rigorous derivation due to Ohmori and Tachikawa [41]. Within the AdS/CFT correspondence, the holographic counterpart is given by the Ryu-Takayanagi prescription [11, 42], whose derivation from the gravitational path integral was established by Lewkowycz-Maldacena [43]. After reviewing these developments, we analyze quantum inequalities that entanglement entropy must satisfy. Finally, we provide an alternative field-theoretic computation employing a double-cone regularization introduced in our own work [1].

2.2.1 Entanglement entropy in CFT: Calabrese-Cardy prescription

We now compute the entanglement entropy in two-dimensional CFT. The standard route is the replica method of Calabrese and Cardy [40]. Consider the theory on $\mathbf{R}^2$ with metric

$$ds^2 = dx^2 + d\tau^2, \quad (2.2.1)$$

and introduce complex coordinates $z = x + i\tau$ and $\bar{z} = x - i\tau$. Define the subsystem by

$$\tau = 0, \quad u \leq x \leq v. \quad (2.2.2)$$

We split the Hilbert space into a bipartite tensor product⁶

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B. \quad (2.2.3)$$

The reduced density matrix is given by

$$\rho_A = \text{Tr}_B \rho. \quad (2.2.4)$$

⁶We assume that the Hilbert space becomes factorizable upon introducing a short-distance cutoff ϵ .

The entanglement entropy is the von Neumann entropy of ρ_A ,

$$S = -\text{Tr } A \rho_A \log \rho_A. \quad (2.2.5)$$

To evaluate it, we employ the replica trick,

$$S = \lim_{n \rightarrow 1} \frac{1}{1-n} \log \text{Tr}_A (\rho_A)^n. \quad (2.2.6)$$

The reduced density matrix can be constructed by path integral. Fix field configurations $\phi_+(x)$ and $\phi_-(x)$ at $\tau = 0$, where the subscripts denote the limits from above and below, respectively. The corresponding wavefunctionals are $\Psi[\phi_+]$ and $\Psi[\phi_-]$. The full density matrix is the product $\rho = \Psi[\phi_-(x)] \Psi^*[\phi_+(x)]$ ⁷. Tracing over the complement—or equivalently inserting delta-function constraints supported on the complement to fix boundary conditions—yields the reduced density matrix. Performing the trace over A as well gives

$$\text{Tr}_A \rho_A^n = \frac{\int_{\Sigma_n} \mathcal{D}\phi e^{-S(\phi)}}{Z_1^n}, \quad (2.2.7)$$

where Σ_n is the n -sheeted geometry obtained by gluing copies of the plane along the interval, and Z_1 denotes the one-sheet partition function. Equivalently, one can replicate the fields instead of the sheets, implementing branch-cut monodromies as

$$\phi^{(k)}(e^{2\pi i}(z-u)) = \phi^{(k+1)}(z-u), \quad \phi^{(k)}(e^{2\pi i}(z-v)) = \phi^{(k+1)}(z-v), \quad (2.2.8)$$

with k labeling the sheets. Note that in this formulation the replicated fields live on a single sheet.

In two-dimensional CFTs, imposing boundary conditions on the fields can be recast—via the state-operator correspondence—as inserting twist operators. We now determine their conformal weights. A conformal map from the n -sheeted surface to the plane is given by

$$w^n = \frac{z-u}{z-v}, \quad (2.2.9)$$

where w is the coordinate on Σ_n and z is the coordinate on the one-sheet geometry $\mathbf{R}^2$. From the vacuum expectation value (VEV) of the stress tensor we can read off the twist-operator dimension. The VEV on Σ_n is

$$\langle T(z) \rangle_{\Sigma_n} = \frac{c}{24} \left(1 - \frac{1}{n^2} \right) \frac{(v-u)^2}{(z-u)^2(z-v)^2}. \quad (2.2.10)$$

⁷For a qubit state, this is $\rho = |\Psi\rangle \langle \Psi|$.

Hence,

$$\frac{\langle T(z)\sigma(u)\bar{\sigma}(v)\rangle_{\mathbf{R}^2}}{\langle \sigma(u)\bar{\sigma}(v)\rangle_{\mathbf{R}^2}} = \frac{c}{24} \left(n - \frac{1}{n} \right) \frac{(v-u)^2}{(z-u)^2(z-v)^2}. \quad (2.2.11)$$

Independently, the OPE with the stress tensor fixes the three-point function as

$$\langle T(z)\sigma(u)\bar{\sigma}(v)\rangle_{\mathbf{R}^2} = \left(\frac{h}{(z-u)^2} + \frac{1}{z-u}\partial_u + \frac{h}{(z-v)^2} + \frac{1}{z-v}\partial_v \right) \langle \sigma(u)\bar{\sigma}(v)\rangle_{\mathbf{R}^2}. \quad (2.2.12)$$

Taking $z \rightarrow u$ or $z \rightarrow v$ then yields

$$\Delta = h + \bar{h} = \frac{c}{12} \left(n - \frac{1}{n} \right). \quad (2.2.13)$$

Finally, the n -sheet partition function $\text{Tr}_A \rho_A^n$ is

$$\text{Tr}_A \rho_A^n = \langle \sigma_n(0)\bar{\sigma}_n(l)\rangle_{\mathbf{R}^2} = \left(\frac{l}{\epsilon} \right)^{-\frac{c}{6}(n-\frac{1}{n})}, \quad (2.2.14)$$

which leads to the universal two-dimensional result

$$S_A = \frac{c}{3} \log \frac{l}{\epsilon}, \quad (2.2.15)$$

with $l = v - u$ being the length of the interval.

2.2.2 Entanglement Entropy in CFT: Ohmori-Tachikawa prescription

In the previous Calabrese-Cardy prescription, we assume that the Hilbert space of the whole system is splittable into those of subsystems by introducing a cutoff. We will review another method called Ohmori-Tachikawa prescription [41], which pays attention to the above issue.

Let us consider a two-dimensional CFT on $\mathbf{R}^2$, whose metric reads

$$ds^2 = dx^2 + d\tau^2. \quad (2.2.16)$$

The subsystem is defined by

$$\tau = 0, \quad u \leq x \leq v. \quad (2.2.17)$$

To factorize the Hilbert space, they introduce a circle-shaped cut with the radius ϵ around the entangling surfaces $x = u, v$. They attach a boundary state to make the factorization of the Hilbert space manifest.

The partition function of this system can be computed via conformal transformation to the cylinder

$$w = \log \frac{z}{l-z}, \quad (2.2.18)$$

where $l = v - u$ and $z = x + i\tau$. w is the complex coordinate of the cylinder. Applying this transformation, the circumference and the length of the cylinder are given by 2π and $2 \log \frac{l}{\epsilon} + O(\frac{l}{\epsilon})$, respectively. For the n -sheet geometry, the circumference and the length of the cylinder are given by $2\pi n$ and $2 \log \frac{l}{\epsilon} + O(\frac{l}{\epsilon})$, respectively. Utilizing the conformal transformation, we can rescale the cylinder so that the circumference and the length become 2π and $\frac{2}{n} \log \frac{l}{\epsilon} + O(\frac{l}{\epsilon})$, respectively. Thus, we find

$$\begin{aligned} \text{Tr } \rho_A^n &= \frac{Z_n}{Z_1^n} \\ &= \frac{\langle a_1^{(n)} | e^{\frac{2}{n} \log \frac{l}{\epsilon} (\frac{c+\bar{c}}{24} - L_0 - \bar{L}_0)} | a_2^{(n)} \rangle}{\left(\langle a_1^{(1)} | e^{2 \log \frac{l}{\epsilon} (\frac{c+\bar{c}}{24} - L_0 - \bar{L}_0)} | a_2^{(1)} \rangle \right)^n}, \end{aligned} \quad (2.2.19)$$

where $|a_i^{(n)}\rangle$ is the boundary state and i, n label the position of the boundary and replica number, respectively. In $\epsilon \rightarrow 0$ limit, only the vacuum amplitude contributes to the partition function and we obtain

$$\begin{aligned} \text{Tr } \rho_A^n &\sim \frac{\langle a_1^{(n)} | 0 \rangle e^{\frac{2}{n} \log \frac{l}{\epsilon} (\frac{c}{12} - \Delta_0)} \langle 0 | a_2^{(n)} \rangle}{\left(\langle a_1^{(1)} | 0 \rangle e^{2 \log \frac{l}{\epsilon} (\frac{c}{12} - \Delta_0)} \langle 0 | a_2^{(1)} \rangle \right)^n} \\ &\sim \frac{\log g_{a_1}^n \log g_{a_2}^n}{(\log g_{a_1}^1 \log g_{a_2}^1)^n} \left(\frac{l}{\epsilon} \right)^{\frac{c}{6} (\frac{1}{n} - n)}, \end{aligned} \quad (2.2.20)$$

where in the first line we only extrapolate the vacuum contribution $\Delta_0 = 0$ and set $c = \bar{c}$. At the leading order in $O(1 - n)$, the entanglement entropy reads

$$S = \frac{c}{3} \log \frac{l}{\epsilon} + O\left(\frac{l}{\epsilon}\right), \quad (2.2.21)$$

where the sub-leading terms depend on the higher excitations of closed string channel in the cylinder and also the choice of the boundary state.

2.3 Entanglement entropy in CFT: double cone regularization

Previous method

In this section, we present a new derivation of entanglement entropy in quantum field theories, following our original analysis [1].

We begin by revisiting the orbifold technique used to derive entanglement entropy [44]. Consider the half-space $\mathcal{A}$ specified by $x_1 > 0$ in Minkowski space $\mathbb{R}^{1,d-1}$,

$$ds^2 = -dx_0^2 + dx_1^2 + dx_{\mathbb{R}^{d-2}}^2. \quad (2.3.1)$$

To implement the replica construction, we perform a Wick rotation $x_0 \rightarrow i\tau$ and use the complex coordinate $X = x_1 + i\tau$.

Within the orbifold framework, the n -th Rényi entropy for this semi-infinite region is

$$S_{\mathcal{A}}^{(n)} = \frac{1}{1-n} \left[Z[\mathbb{C}/\mathbb{Z}_N \times \mathbb{R}^{d-2}] - \frac{1}{N} Z[\mathbb{C} \times \mathbb{R}^{d-2}] \right]_{N=\frac{1}{n}}, \quad (2.3.2)$$

where the $\mathbb{Z}_N$ action is

$$X = x_1 + i\tau \rightarrow X e^{\frac{2\pi i}{N}}, \quad (2.3.3)$$

namely a rotation in the complex plane. Here $Z[\mathcal{M}]$ denotes the logarithm of the first-quantized partition function of a QFT (e.g. a free scalar) on the manifold $\mathcal{M}$. We evaluate it using the heat-kernel representation. In flat space $\mathbb{R}^d$, the kernel is given in [45]

$$K_{\mathbb{R}^d}(x, x'; t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{r^2}{4t} - tm^2}, \quad r = |x - x'|, \quad (2.3.4)$$

with t the Schwinger parameter. On the orbifold, the kernel follows from the method of images:

$$\begin{aligned} K_{\mathbb{C}/\mathbb{Z}_N \times \mathbb{R}^{d-2}}(r, \theta, x_i; r', \theta', x'_i; t) \\ = \frac{1}{N} \sum_{k=0}^{N-1} K_{\mathbb{C} \times \mathbb{R}^{d-2}}\left(r, \theta, x_i; r', \theta' - \frac{2\pi k}{N}, x'_i; t\right). \end{aligned} \quad (2.3.5)$$

Substituting these expressions and integrating, the Rényi entropy becomes

$$\begin{aligned} S_{\mathcal{A}}^{(n)} &= \int_{\epsilon^2}^{\infty} \frac{dt}{2t} \int d^d x \sqrt{g} \left(K_{\mathbb{C}/\mathbb{Z}_N \times \mathbb{R}^{d-2}}(x, x; t) - K_{\mathbb{C} \times \mathbb{R}^{d-2}}(x, x; t) \right) \Big|_{N=1/n} \\ &= \frac{(n+1)\pi V_{d-2}}{6n} \int_{\epsilon^2}^{\infty} \frac{dt}{(4\pi t)^{\frac{d}{2}}} e^{-m^2 t} \\ &= \frac{(n+1)}{6n} \frac{\pi}{(4\pi)^{\frac{d}{2}}} \frac{V_{d-2}}{\epsilon^{d-2}} E_{\frac{d}{2}}(\epsilon^2 m^2), \end{aligned} \quad (2.3.6)$$

where $E_{\nu}(z)$ is the ν -th order exponential integral and ϵ is a short-distance cutoff. To expose the cutoff dependence, expand $S_{\mathcal{A}}^{(n)}$ in powers of ϵ^2 . The universal term is the constant (odd d) or logarithmic (even d) contribution:

$$S_{\mathcal{A}}^{(n)} \sim \begin{cases} \frac{(n+1)}{6n} \frac{\pi}{(4\pi)^{\frac{d}{2}}} \Gamma\left(\frac{2-d}{2}\right) \cdot V_{d-2} m^{d-2}, & d \text{ is odd} \\ \frac{(n+1)}{6n} \frac{2\pi}{(4\pi)^{\frac{d}{2}}} \frac{(-1)^{\frac{d}{2}-1}}{(\frac{d}{2}-1)!} \cdot V_{d-2} m^{d-2} \log\left(\frac{1}{m\epsilon}\right), & d \text{ is even.} \end{cases}$$

Up to this point, the UV cutoff ϵ in (2.3.6) was introduced on dimensional grounds. A natural question is whether this parameter admits a geometric realization. The double-cone

(or $i\epsilon$) prescription provides a transparent interpretation. As we show below, implementing this prescription smooths the conical fixed point and thereby resolves the orbifold singularity.

Our method: double-cone regularization

We revisit the entanglement entropy of the half-space $\mathcal{A}$ in the Minkowski background (2.3.1). As an alternative coordinate system on the full Minkowski spacetime, we employ Rindler coordinates,

$$ds^2 = dr^2 - r^2 dT_{\text{Rindler}}^2 + dx_{\mathbb{R}^{d-2}}^2, \quad (2.3.7)$$

where the Rindler radial coordinate r ranges over $-\infty < r < \infty$, with $r > 0$ the right wedge and $r < 0$ the left wedge. We then place the QFT on the Rindler background. As in QFT on black-hole geometries, a UV divergence appears at the horizon⁸. To tame this near-horizon divergence, we introduce a complex deformation of the background,

$$r \in \mathbb{R} \rightarrow r = \tilde{r} - i\epsilon, \quad \epsilon > 0, \quad \tilde{r} \in \mathbb{R}. \quad (2.3.8)$$

For the replica/orbifold construction, we also Wick-rotate the Rindler time, $T_{\text{Rindler}} \rightarrow i\theta$. Relabeling $\tilde{r}$ back to r yields

$$ds^2 = dr^2 + (r - i\epsilon)^2 d\theta^2 + ds_{\mathbb{R}^{d-2}}^2, \quad r \in (-\infty, \infty). \quad (2.3.9)$$

We denote this manifold by $\mathcal{M}_{\text{dc}}$. Such deformations have appeared in [46–48] in the context of JT gravity.

To compute the n -th Rényi entropy, we impose a $\mathbb{Z}_N$ orbifold around $r = 0$ (prior to the complex deformation), implemented by $\theta \sim \theta + \frac{2\pi}{N}$. The resulting geometry is

$$ds^2 = dr^2 + (r - i\epsilon)^2 d\theta^2 + ds_{\mathbb{R}^{d-2}}^2, \quad \theta \sim \theta + \frac{2\pi}{N}. \quad (2.3.10)$$

We refer to this space as $\mathcal{M}_{\text{dc}}^{\mathbb{Z}_N}$. The heat kernels on $\mathcal{M}_{\text{dc}}$ and $\mathcal{M}_{\text{dc}}^{\mathbb{Z}_N}$, $K_{\mathcal{M}_{\text{dc}}}(x, x'; t)$ and $K_{\mathcal{M}_{\text{dc}}^{\mathbb{Z}_N}}(x, x'; t)$, are obtained by analytic continuation from $K_{\mathbb{R}}(x, x'; t)$ and $K_{\mathbb{C}/\mathbb{Z}_N \times \mathbb{R}^{d-2}}(x, x'; t)$, respectively. In particular, the $i\epsilon$ prescription both smooths the curvature singularity and provides a geometric UV regulator—even if one does not initially introduce an imaginary shift by hand.

We now spell out the details. As stated, the n -th Rényi entropy is obtained from one half of the orbifold partition function (2.3.10):

$$S_{\mathcal{A}}^{(n)} = \frac{1}{2(1-n)} \left[Z[\mathcal{M}_{\text{dc}}^{\mathbb{Z}_N}] - \frac{1}{N} Z[\mathcal{M}_{\text{dc}}] \right]_{N=\frac{1}{n}}. \quad (2.3.11)$$

⁸The horizon is a fixed point of the Euclidean identification $T \sim T + \beta$, and the Green's function diverges there.

Although the subsystem $\mathcal{A}$ lives in $r > 0$, it may look as though we are “extending” spacetime from $r > 0$ to $-\infty < r < \infty$ and then deforming it complexly. In our double-cone regularization, however, (2.3.11) indeed computes the Rényi entropy. A heuristic rationale is as follows. Consider an entangled state ψ on the full space and its reduced density matrices on the right/left wedges, $\rho_{r>0}$ and $\rho_{r<0}$. Define

$$\rho = (\rho_{r>0}) \otimes (\rho_{r<0}). \quad (2.3.12)$$

Both $\text{Tr}_{r>0}[(\rho_{r>0})^n]$ and $\text{Tr}_{r<0}[(\rho_{r<0})^n]$ are UV-divergent; geometrically, ρ prepares two Euclidean cones. The corresponding partition function factorizes:

$$\text{Tr} [\rho^n] = \text{Tr}_{r>0}[(\rho_{r>0})^n] \text{Tr}_{r<0}[(\rho_{r<0})^n]. \quad (2.3.13)$$

Assuming a symmetry between the wedges (as in a thermofield-double-like setup),

$$\text{Tr}_{r>0}[(\rho_{r>0})^n] = \text{Tr}_{r<0}[(\rho_{r<0})^n], \quad (2.3.14)$$

so that

$$\text{Tr} [\rho^n] = \left[\text{Tr}_{r>0}[(\rho_{r>0})^n] \right]^2. \quad (2.3.15)$$

This argument merely indicates that (2.3.11) reproduces the standard result; a full justification requires care because Hilbert-space factorization may fail in the double-cone regularization.

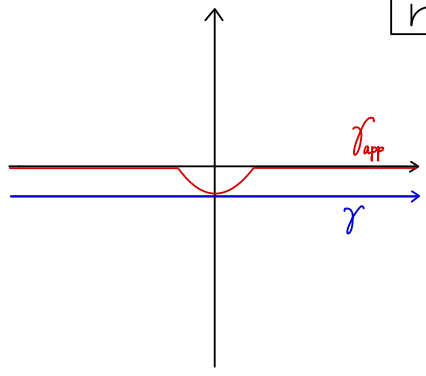
To regulate divergences, we introduce a geometric cutoff in the metric. Conceptually, the double-cone version of ρ glues the two Euclidean path integrals $\text{Tr}_{r>0}[(\rho_{r>0})^n]$ and $\text{Tr}_{r<0}[(\rho_{r<0})^n]$ through a complex “wormhole,” in parallel with [49, 50] (there, the wormhole is real Euclidean rather than complex).

Concretely, we use the orbifold method and evaluate a volume integral of the heat kernel:

$$\begin{aligned} Z[\mathcal{M}_{\text{dc}}^{\mathbb{Z}_N}] - \frac{1}{N} Z[\mathcal{M}_{\text{dc}}] &= \int_0^\infty \frac{dt}{2t} \int d^d x \sqrt{g} \left(K_{\mathcal{M}_{\text{dc}}^{\mathbb{Z}_N}}(x, x; t) - K_{\mathcal{M}_{\text{dc}}}(x, x; t) \right) \\ &= \frac{2\pi V_{d-2}}{N} \sum_{k=1}^{N-1} \int_\Gamma \frac{dt}{2t} \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_\gamma dr \sqrt{r^2} e^{-tm^2 - \frac{r^2}{t} \sin^2 \frac{\pi k}{N}}, \end{aligned}$$

where g is the determinant of the metric. The only difference from the earlier computation is the choice of contour. We take γ in the complex r -plane as in Fig. 2.3.1, and pick the negative branch of $\sqrt{r^2}$ on the segment with $\text{Re}[r] < 0$. Since the deformed contour is invariant under the $\mathbb{Z}_2$ action $\text{Re}[r] \rightarrow -\text{Re}[r]$, it suffices to keep the $\text{Re}[r] > 0$ contribution. This yields

$$Z[\mathcal{M}_{\text{dc}}^{\mathbb{Z}_N}] - \frac{1}{N} Z[\mathcal{M}_{\text{dc}}] = \frac{\pi V_{d-2}}{(4\pi)^{\frac{d}{2}} N} \sum_{k=1}^{N-1} \frac{1}{\sin^2 \left(\frac{k\pi}{N} \right)} \int_\Gamma dt t^{-\frac{d}{2}} e^{-tm^2 + \frac{\epsilon^2}{t} \sin^2 \left(\frac{k\pi}{N} \right)}. \quad (2.3.16)$$

Figure 2.3.1: The contour γ for our double cone regularization.

Next, we specify the integration contour Γ for the Schwinger parameter t . If we first take $\Gamma = [0, \infty)$ along the real axis, the integrand diverges near $t \rightarrow 0^+$ due to the exponential factor $\exp\left[\frac{\epsilon^2}{t} \sin\left(\frac{k\pi}{N}\right)\right]$. To obtain a finite answer, we must allow Γ to lie in the complex t -plane, i.e. consider a complex Schwinger parameter [51]. Concretely, we require that the contour approach $t = 0$ with $\arg t \sim \pi$ and, for simplicity, that it asymptote to $\text{Re}[t] \rightarrow \infty$ with $\text{Im}[t] \rightarrow 0$.

To determine an appropriate contour, we use the Green's-function identity [52]

$$\int_{\Gamma} dt K_{\mathcal{M}_{\text{dc}}}(x, x'; t) = G_{\mathcal{M}_{\text{dc}}}(x, x'), \quad (2.3.17)$$

where $G_{\mathcal{M}}(x, x')$ satisfies

$$(\square_x - m^2)G_{\mathcal{M}}(x, x') = \frac{1}{\sqrt{g(x)}}\delta(x - x'). \quad (2.3.18)$$

In particular, the flat-space Green's function is

$$G_{\mathbb{R}^d}(x, x') = \frac{1}{2\pi} \left(\frac{m}{2\pi r}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(mr), \quad (2.3.19)$$

with $K_\nu(z)$ the modified Bessel function of the second kind.

Within the double-cone regularization, we must ensure that (2.3.17) holds for separations with $\text{Re}[r] > 0$ and $\text{Im}[r] < 0$. This leads us to deform the t -contour off the real axis. One convenient choice meeting these criteria is shown in Fig. 2.3.2. The justification uses the integral representation of the Hankel function of the first kind $H_\nu^{(1)}(z)$,

$$H_\nu^{(1)}(z) = \frac{1}{i\pi} \int_{\Gamma'} \frac{dt'}{t'^{\nu+1}} e^{\frac{i}{2}(t' - \frac{1}{t'})}, \quad (2.3.20)$$

valid for $\text{Re}[z] > 0$, where Γ' is depicted in Fig. 2.3.2. Setting $z = imr$ with $\text{Re}[r] > 0$ and

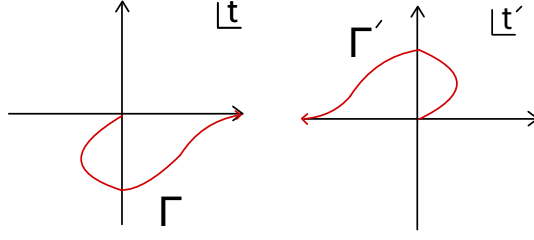


Figure 2.3.2: (Left) Contour for the heat-kernel integral. (Right) Contour for the Hankel-function representation.

$\text{Im}[r] < 0$ implies $\text{Re}[imr] > 0$ and $\text{Im}[imr] > 0$. Combining these formulae gives

$$K_\nu(mr) = \frac{i^\nu}{2} \int_{\Gamma'} \frac{dt'}{t'^{\nu+1}} e^{\frac{imr}{2}(t' - \frac{1}{t'})} = \frac{1}{2} \left(\frac{r}{2m}\right)^\nu \int_{\Gamma} \frac{dt}{t^{\nu+1}} e^{-m^2 t - \frac{r^2}{4t}}.$$

With $\nu = \frac{d}{2} - 1$ and $t' = -\frac{2m^2}{imr} t$, we recover (2.3.17) for $\text{Re}[r] > 0$ and $\text{Im}[r] < 0$, establishing that the chosen Γ is the correct contour in this domain.

With the above contour choice, we obtain

$$\int^{r_\infty} dr r e^{-\frac{r^2}{t} \sin^2 \frac{\pi k}{N}} = \frac{t}{2 \sin^2 \frac{\pi k}{N}} e^{-\frac{r_\infty^2}{t} \sin^2 \frac{\pi k}{N}}. \quad (2.3.21)$$

This integral diverges as $r_\infty \rightarrow \infty$ when $\text{Re}[t] < 0$. However, after performing the t -integral, $S_{\mathcal{A}}$ remains finite even in the limit $r_\infty \rightarrow \infty$. Indeed,

$$\begin{aligned} & \int_{\Gamma} \frac{dt}{t} \frac{1}{(4\pi t)^d} \frac{t}{2 \sin^2 \frac{\pi k}{N}} e^{-\frac{r_\infty^2}{t} \sin^2 \frac{\pi k}{N}} e^{-tm^2} \\ &= \frac{1}{(4\pi)^d \sin^2 \frac{\pi k}{N}} \left(\frac{m}{r_\infty \sin \frac{\pi k}{N}} \right)^{d-1} \cdot K_{d-1} \left(mr_\infty \sin \frac{\pi k}{N} \right) \\ &\sim (r_\infty)^{-d+\frac{1}{2}} e^{-mr_\infty \sin \frac{\pi k}{N}} \rightarrow 0, \quad \text{as } r_\infty \rightarrow \infty. \end{aligned}$$

Here, we used the integral representation of the modified Bessel function $K_\nu(z)$, whose large-argument behavior is

$$K_\nu(z) \sim \left(\frac{\pi}{2z} \right)^{\frac{1}{2}} e^{-z} \left(1 + O\left(\frac{1}{z} \right) \right). \quad (2.3.22)$$

Combining the above, we arrive at a finite expression for the Rényi entropy:

$$Z[\mathcal{M}_{\text{dc}}^{\mathbb{Z}_N}] - \frac{1}{N} Z[\mathcal{M}_{\text{dc}}] = \frac{i\pi^2 V_{d-2}}{(4\pi)^{\frac{d}{2}} N} \sum_{k=1}^{N-1} \frac{1}{\sin^2 \frac{k\pi}{N}} \left(-\frac{m}{\epsilon \sin \frac{k\pi}{N}} \right)^{\frac{d}{2}-1} H_{\frac{d}{2}-1}^{(1)} \left(2\epsilon m \sin \frac{k\pi}{N} \right).$$

To check this result, expand in ϵ . For even d , the n -th Rényi entropy behaves as

$$\begin{aligned}
S_{\mathcal{A}}^{(n)} &= \frac{\left(\frac{d}{2} - 1\right)!}{2} (-1)^{\frac{d}{2}-1} \frac{V_{d-2}}{\epsilon^{d-2}} \frac{1}{N} \sum_{k=1}^{N-1} \frac{1}{(\sin \frac{k\pi}{N})^{d-2}} + \dots \\
&+ \frac{n+1}{3n} \frac{2\pi}{(4\pi)^{\frac{d}{2}}} \frac{(-1)^{\frac{d}{2}-1}}{\left(\frac{d}{2} - 1\right)!} \cdot V_{d-2} m^{d-2} \log\left(\frac{1}{m\epsilon}\right) \\
&+ \frac{n+1}{3n} \frac{2\pi}{(4\pi)^{\frac{d}{2}}} \frac{(-1)^{\frac{d}{2}-1}}{\left(\frac{d}{2} - 1\right)!} \left(\psi\left(\frac{d}{2} - 1\right) - \gamma + \frac{i\pi}{2}\right) V_{d-2} m^{d-2} \\
&+ \dots
\end{aligned} \tag{2.3.23}$$

When d is odd, we find

$$\begin{aligned}
S_{\mathcal{A}}^{(n)} &= \frac{2\pi}{(4\pi)^{\frac{d}{2}}} \frac{1}{N} \sum_{k=1}^{N-1} \frac{1}{\left(\sin \frac{k\pi}{N}\right)^d} \frac{(-1)^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2} - 1\right)}{2} \frac{V_{d-2}}{\epsilon^{d-2}} \\
&+ \dots + \frac{n+1}{3n} \frac{\pi}{(4\pi)^{\frac{d}{2}}} \Gamma\left(\frac{2-d}{2}\right) \cdot V_{d-2} m^{d-2} + \dots
\end{aligned}$$

Note that for odd d the divergent terms are purely imaginary.

A key check is that, after dividing by two (extracting a half-cone), the universal part of the Rényi entropy from the double-cone regularization agrees with the momentum cutoff result in (2.3.7).

Double cone in two-dimensional CFTs

In two-dimensional conformal field theories, the single-interval entanglement entropy takes the universal form

$$S_A = \frac{c}{3} \log \frac{L}{\epsilon}, \tag{2.3.24}$$

where c is the central charge and L is the interval length. As verified in the previous section, our two-dimensional free-boson computation reproduces (2.3.24) with $c = 1$.

We now reinterpret the $i\epsilon$ prescription on the replica manifold. Consider an arbitrary unitary CFT on the flat plane $\mathbb{R}^2$ with metric

$$ds^2 = dx^2 + d\tau^2 = dr^2 + r^2 d\theta^2. \tag{2.3.25}$$

Choose the subsystem $\mathcal{A}$ to be $x \in [0, L]$ at $\tau = 0$. For convenience, let us introduce complex coordinates

$$z = x + i\tau, \quad \bar{z} = x - i\tau. \tag{2.3.26}$$

We then engineer a double-cone geometry—heuristically a “wormhole” with two throats—centered near $z = 0$ and $z = L$. To regulate the neighborhood of the entangling surface (the boundary of $\mathcal{A}$), we implement the $i\epsilon$ prescription locally. Concretely, take polar coordinates around the endpoints (see Fig.2.3.3); as in the flat-plane example, extend the radial coordinate to negative values by

$$z = (r - i\epsilon)e^{i\theta}. \quad (2.3.27)$$

An analogous construction can be carried out directly on the n -sheeted cover using

$$\zeta = \left(\frac{z - L}{z} \right)^{\frac{1}{n}}. \quad (2.3.28)$$

Pulled back to the one-sheet geometry, the locations of the “throats” need not coincide exactly with $z = 0$ and $z = L$. For example, using (2.3.28), a double-cone with the $i\epsilon$ shift maps to a throat near L at

$$z = \frac{L}{1 - (-i\epsilon e^{i\theta})^n}, \quad (2.3.29)$$

which differs from the original insertion point even in the $n \rightarrow 1$ limit,

$$z = L - i\epsilon e^{i\theta}. \quad (2.3.30)$$

While the precise throat positions (and in some schemes even their profiles) depend on the chosen regularization, these differences do not affect the leading terms of the entanglement entropy, as we will confirm below. The discussion parallels the prescription of Ohmori-Tachikawa [41].

After introducing the double-cone geometry with the $i\epsilon$ prescription on the one-sheet geometry, we map it to a torus via

$$w = \log \frac{z}{L - z}. \quad (2.3.31)$$

Because we are working with a complex manifold, the anti-holomorphic sector is independent. Below, we treat only the holomorphic part, with the understanding that the anti-holomorphic contribution is identical and will be included at the end by doubling. Under the above map, the torus length and circumference are $l = 4 \log \frac{L}{\epsilon} + \dots$ and 2π , respectively. The precise value of l depends on the details of the regularization discussed above, but such differences affect only subleading terms in the $\frac{L}{\epsilon}$ expansion. An analogous construction on the n -sheeted cover yields a torus with $l = 4 \log \frac{L}{\epsilon} + \dots$ and circumference $2\pi n$. After a modular transformation, one may rescale to $l = \frac{4}{n} \log \frac{L}{\epsilon} + \dots$ and circumference 2π . We also note that subleading corrections to l can be complex in general, depending on the chosen complex contour.

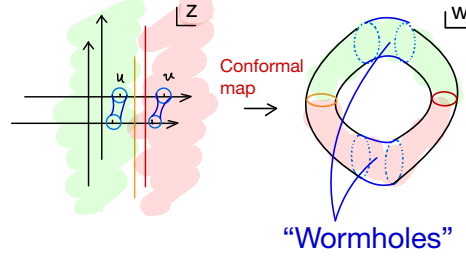


Figure 2.3.3: Construction of the “torus” geometry connected by “wormholes” using the $i\epsilon$ prescription. Left: the double-cone geometry, built from two planes describing the regions $\text{Re}[r] > 0$ and $\text{Re}[r] < 0$, respectively. Right: the resulting “torus” with throats providing the gluing. This figure is cited from [1].

The entanglement entropy is then

$$S_{\mathcal{A}} = \lim_{n \rightarrow 1} \frac{1}{1-n} \log \frac{Z_n}{(Z_1)^n}, \quad (2.3.32)$$

where Z_n and Z_1 are the partition functions on the n -sheet and one-sheet geometries, respectively. In the long-length limit ($\log \frac{L}{\epsilon} \rightarrow \infty$), the partition function is dominated by vacuum propagation, as in [41, 53]:

$$S_{\mathcal{A}} = \lim_{n \rightarrow 1} \frac{1}{1-n} \frac{c}{6} \frac{1-n^2}{n} \log \frac{L}{\epsilon} = \frac{c}{3} \log \frac{L}{\epsilon}. \quad (2.3.33)$$

Extracting the half-cone contribution (divide by two) and then doubling to include the anti-holomorphic sector reproduces

$$S_{\mathcal{A}} = \frac{c}{3} \log \frac{L}{\epsilon}, \quad (2.3.34)$$

up to subleading terms in $\frac{L}{\epsilon}$, which are generically complex-valued.

2.4 Entanglement entropy in AdS: Ryu-Takayanagi formula

Up to this point, we have analyzed entanglement entropy in two-dimensional CFTs. In higher dimensions there is, in general, no closed-form field-theoretic expression. The proposal of Ryu and Takayanagi provides a holographic definition that rewrites the problem in terms of gravitational data.

For clarity, restrict to a single time slice. The holographic entanglement entropy is then

$$S_A = \text{Ext}_{\gamma_A} \frac{\text{Area}[\gamma_A]}{4G}, \quad (2.4.1)$$

where γ_A runs over bulk surfaces anchored on the boundary of the chosen subsystem and we extremize the area. In the $\text{AdS}_3/\text{CFT}_2$ setting, the extremal surface is a geodesic, since the

relevant object is co-dimension two.

2.4.1 Computation in Poincaré coordinates

As an exercise, consider Poincaré AdS_{d+1} with metric

$$ds^2 = \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}. \quad (2.4.2)$$

Choose the subsystem at $t = 0$ with $-l \leq x_1 \leq l$. By symmetry the extremal surface depends only on x_1 , so we write $z = z(x_1)$. The induced metric on the surface is

$$ds^2| = \frac{(1 + z'^2)dx_1^2 + dx_i^2}{z^2}, \quad (2.4.3)$$

with $i = 2, \dots, d-1$ and $z' = \frac{\partial z}{\partial x_1}$. The corresponding Lagrangian is

$$L = \frac{1}{4G} \int d^{d-1}x \frac{1}{z^{d-1}} \sqrt{1 + z'^2}. \quad (2.4.4)$$

Since L has no explicit x_1 dependence, there is a conserved quantity,

$$\frac{\partial L}{\partial z'} z' - L = -\frac{1}{z^{d-1} \sqrt{z'^2 + 1}} = E. \quad (2.4.5)$$

Introducing the turning point by $z'(x_1)|_{z=z_*} = 0$ gives

$$E = -\frac{1}{z_*^{d-1}}. \quad (2.4.6)$$

Integrating, for $x_1 \geq 0$ one finds

$$\begin{aligned} x_1 &= \int \frac{dz}{\sqrt{\left(\frac{z_*}{z}\right)^{2d-2} - 1}} \\ &= \frac{z_*^d}{d z_*^{d-1}} \cdot {}_2F_1 \left[\frac{1}{2}, \frac{d}{2(d-1)}, \frac{3d-2}{2d-2}, \left(\frac{z}{z_*}\right)^{2d-2} \right] - \frac{\sqrt{\pi}}{d} \frac{\Gamma\left(\frac{3d-2}{2d-2}\right)}{\Gamma\left(\frac{2d-1}{2d-2}\right)}. \end{aligned} \quad (2.4.7)$$

The area of the extremal surface then evaluates to

$$S = \frac{1}{4G(d-2)} \left(\frac{2}{\epsilon^{d-2}} - \left(\frac{2}{\epsilon^{d-2}} \right)^{d-1} \frac{1}{l^{d-2}} \right). \quad (2.4.8)$$

2.4.2 Time-dependent case

For time-dependent backgrounds, two equivalent prescriptions are available at tree level [54, 55].

First, the HRT approach: take a boundary time slice and define the subsystem in the CFT. Consider bulk surfaces γ'_A at constant boundary time that connect the subsystem endpoints. Form the causal diamond associated with γ'_A , and within it allow arbitrary bulk surfaces γ_A anchored on the same boundary endpoints. The HRT prescription states

$$S_A = \text{Min} \frac{\gamma_A}{4G}, \quad (2.4.9)$$

i.e. minimize over such γ_A .

Second, the construction due to Wall [55]: choose an arbitrary Cauchy slice Σ in the bulk and find, on Σ , a minimal surface anchored on the subsystem boundary. Varying over all Σ in the time direction, select the configuration that maximizes this minimal area.

2.4.3 Proof of the Ryu-Takayanagi formula

The proof of the Ryu-Takayanagi formula within AdS/CFT was given by Lewkowycz and Maldacena [43]. In close analogy with the CFT replica construction, the reduced density matrix can be realized via a gravitational path integral, and the entanglement entropy can be written as

$$S_A = -\frac{\partial}{\partial n} (\log Z_{\Sigma_n} - n \log Z_{\Sigma_1})|_{n=1}, \quad (2.4.10)$$

where Z denotes the CFT partition function. Using the holographic dictionary, we instead evaluate the corresponding gravitational partition functions,

$$Z_{\Sigma_n} = Z_{\text{gravity}}(M_n), \quad (2.4.11)$$

with M_n the n -fold replicated bulk geometry (in $(d+1)$ -dimensional AdS) produced by the gravitational path integral; M_n solves the Einstein equations with smooth asymptotic boundary Σ_n . We shall work semiclassically. Introduce coordinates r and $0 \leq \tau \leq 2\pi n$ to parametrize the transverse and angular directions of the replica construction. The replica symmetry $\tau \sim \tau + 2\pi$ implements a $\mathbf{Z}_n$ action. As in the field-theory computation, we consider the “one-sheet” orbifolded geometry obtained by dividing by n ,

$$\log Z_{\text{gravity}}(M_n) = n \cdot \log Z_{\text{gravity}}(\hat{M}_n), \quad (2.4.12)$$

where $\hat{M}_n$ is defined by restricting to $0 \leq \tau \leq 2\pi$. Formally there is a conical singularity at $r = 0$, the fixed locus of the replica symmetry⁹. We define it this way and ignore the associated divergences because the original M_n is a smooth on-shell geometry. Although evaluating Z for arbitrary real n is intricate, the analysis simplifies in the $n \rightarrow 1$ limit.

To proceed, introduce an auxiliary geometry N_n that coincides with M_1 for $r > \delta$ and with M_n for $r \lesssim \delta$. This is an off-shell configuration chosen for bookkeeping. Then

$$\begin{aligned} \log Z_{\Sigma_n} - n \log Z_{\Sigma_1} &= \log Z_{\text{gravity}}(M_n) - n \cdot \log Z_{\text{gravity}}(M_1) \\ &= n \cdot \log Z_{\text{gravity}}(\hat{M}_n) - n \cdot \log Z_{\text{gravity}}(M_1) \\ &= n \cdot (\log Z_{\text{gravity}}(\hat{M}_n) - \log Z_{\text{gravity}}(\hat{N}_n)) - n \cdot (\log Z_{\text{gravity}}(M_1) - \log Z_{\text{gravity}}(\hat{N}_n)). \end{aligned} \quad (2.4.13)$$

Taking $n \rightarrow 1$ and keeping only $O(n-1)$ terms: since M_n and $\hat{M}_n$ are on-shell, the first difference is $O((n-1)^2)$. The second term receives support only from the cap region $r < \delta$. Because $\hat{N}_n$ is nonsingular, we may isolate the singular contribution from $n \cdot M_1$. Both carry a conical deficit $2\pi(1-n)$; one is singular, the other smooth. Evaluating the Einstein-Hilbert action, the Ricci scalar contributes

$$R_{n \cdot \hat{N}_n} - R_{n \cdot M_1} = 4\pi(1-n)\delta^{(2)}(r). \quad (2.4.14)$$

Label the remaining $(d-1)$ directions by γ_A . Then

$$\begin{aligned} S_A &= -\frac{\partial}{\partial n} \left(-\frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} \, 4\pi(1-n)\delta^{(2)}(r) \right) \Big|_{n=1} \\ &= -\frac{\partial}{\partial n} \left(\frac{1-n}{4G} \int_{\gamma_A} d^{d-1}x \sqrt{g} \right) \Big|_{n=1} \\ &= \frac{1}{4G} \int_{\gamma_A} d^{d-1}x \sqrt{g}. \end{aligned} \quad (2.4.15)$$

Finally, extremizing the action shows that γ_A obeys the minimal-surface condition. Note that γ_A is a co-dimension-two (i.e. $(d-1)$ -dimensional) surface in the bulk.

2.5 Entropic inequalities

In this section, we collect fundamental inequalities obeyed by entanglement entropy. For a subsystem A we write its entropy as S_A . To establish the inequalities, we employ the relative entropy

$$S(\rho||\sigma) = \text{Tr } \rho \log \rho - \text{Tr } \rho \log \sigma. \quad (2.5.1)$$

⁹Near $r \sim 0$, one has $ds^2 = dr^2 + r^2 d\tau^2/n^2$, exhibiting a conical deficit. For convenience we compute $Z_{\hat{M}_n}$ rather than Z_{M_n} .

The relative entropy is nonnegative and vanishes if and only if the two states coincide. A concise way to see this is to use the concavity of the function $f(x) = -x \log x$. Since

$$\frac{\partial^2 f}{\partial x^2} \leq 0, \quad (2.5.2)$$

we obtain

$$\text{Tr}(f(\rho) - f(\sigma) - f'(\sigma)(\rho - \sigma)) \leq 0. \quad (2.5.3)$$

Using $\text{Tr } \rho = \text{Tr } \sigma = 1$ this simplifies to

$$S(\rho || \sigma) \geq 0. \quad (2.5.4)$$

For monotonicity, we only sketch the idea and refer to [56, 57] for full proofs. The partial trace is a completely positive trace-preserving (CPTP) map, so it sends density matrices to density matrices. Relative entropy is monotone under CPTP maps:

$$S(\rho || \sigma) \geq S(\Phi(\rho) || \Phi(\sigma)), \quad (2.5.5)$$

with $\Phi = \text{Tr}_B$. A streamlined argument following [58] runs as follows.

Lieb's Theorem $f(A, B) = \text{Tr}(X^\dagger A^t X B^{1-t})$ is jointly concave for any linear operator X , for positive A, B and $0 \leq t \leq 1$. Joint concavity means

$$f(\lambda A_1 + (1 - \lambda)A_2, \lambda B_1 + (1 - \lambda)B_2) \geq \lambda f(A_1, B_1) + (1 - \lambda)f(A_2, B_2), \quad 0 \leq \lambda \leq 1.$$

Define

$$I_t(A, X) = \text{Tr}(X^\dagger A^t X A^{(1-t)}) - \text{Tr}(X^\dagger X A), \quad (2.5.6)$$

and

$$\begin{aligned} I(A, X) &= \frac{d}{dt} I_t|_{t=0}, \\ &= \text{Tr}(X^\dagger \log A X A) - \text{Tr}(X^\dagger X \log A A). \end{aligned} \quad (2.5.7)$$

Then

$$\begin{aligned} I(\lambda A_1 + (1 - \lambda)A_2, X) &= \lim_{\Delta \rightarrow 0} \frac{I_\Delta(\lambda A_1 + (1 - \lambda)A_2, X)}{\Delta} \\ &= \lambda \lim_{\Delta \rightarrow 0} \frac{I_\Delta(A_1, X)}{\Delta} + (1 - \lambda) \lim_{\Delta \rightarrow 0} \frac{I_\Delta(A_2, X)}{\Delta} \\ &= \lambda I(A_1, X) + (1 - \lambda)I(A_2, X). \end{aligned} \quad (2.5.8)$$

In particular, choose

$$A = \begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \quad (2.5.9)$$

which yields

$$I(A, X) = \rho \log \sigma - \rho \log \rho = -S(\rho || \sigma). \quad (2.5.10)$$

Thus the relative entropy is jointly convex. Using that partial trace can be represented via a unitary dilation,

$$\rho^A \otimes \frac{I}{d} = \sum_j p_j U_j \rho^{AB} U_j^\dagger, \quad (2.5.11)$$

with d the dimension of B , we obtain

$$\begin{aligned} S\left(\rho^A \otimes \frac{I}{d} \middle| \middle| \sigma^A \otimes \frac{I}{d}\right) &\leq \sum_j p_j S(U_j \rho_{AB} U_j^\dagger || U_j \sigma^{AB} U_j^\dagger) \\ &= \sum_j p_j S(\rho^{AB} || \sigma^{AB}) = S(\rho^{AB} || \sigma^{AB}). \end{aligned} \quad (2.5.12)$$

This establishes the desired monotonicity under partial trace.

2.5.1 Subadditivity

Subadditivity states that

$$S_A + S_B \geq S_{AB}. \quad (2.5.13)$$

This follows from the positivity of the relative entropy. Indeed,

$$\begin{aligned} S(\rho_{AB} || \rho_A \otimes \rho_B) &= \text{Tr}(\rho_{AB} \log \rho_{AB} - \rho_{AB} \log(\rho_A \otimes \rho_B)) \\ &= \text{Tr} \rho_{AB} \log \rho_{AB} - \text{Tr} \rho_A \log \rho_A - \text{Tr} \rho_B \log \rho_B \geq 0, \end{aligned} \quad (2.5.14)$$

which is equivalent to $S_A + S_B \geq S_{AB}$. The mutual information is defined by

$$I(A; B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \quad (2.5.15)$$

so subadditivity is precisely stated as $I(A; B) \geq 0$.

2.5.2 Araki-Lieb inequality

The Araki-Lieb inequality states

$$|S_A - S_B| \leq S_{AB}. \quad (2.5.16)$$

One way to see this starts from a purification. Introduce a reference system C and a pure state ρ_{ABC} with

$$S(\rho_{AB}) = S(\rho_C), \quad S(\rho_{BC}) = S(\rho_A), \quad S(\rho_{AC}) = S(\rho_B). \quad (2.5.17)$$

Apply subadditivity to the bipartitions $A|C$ and $B|C$:

$$S_{AC} \leq S_A + S_C, \quad S_{BC} \leq S_B + S_C. \quad (2.5.18)$$

Using the equalities above, these become

$$S_B \leq S_A + S_{AB}, \quad S_A \leq S_B + S_{AB}. \quad (2.5.19)$$

Rearranging yields the two one-sided bounds

$$S_B - S_A \leq S_{AB}, \quad S_A - S_B \leq S_{AB}, \quad (2.5.20)$$

which together imply

$$|S_A - S_B| \leq S_{AB}. \quad (2.5.21)$$

2.5.3 Strong subadditivity

Strong subadditivity requires

$$S_{AB} + S_{BC} \geq S_B + S_{ABC}. \quad (2.5.22)$$

It follows from the monotonicity of relative entropy. In particular,

$$S(\rho_{ABC} || \rho_A \otimes \rho_{BC}) \geq S(\rho_{AB} || \rho_A \otimes \rho_B). \quad (2.5.23)$$

Evaluating both sides using $S(\rho || \sigma) = \text{Tr } \rho \log \rho - \text{Tr } \rho \log \sigma$ gives

$$[-S(\rho_{ABC}) + S(\rho_A) + S(\rho_{BC})] \geq [-S(\rho_{AB}) + S(\rho_A) + S(\rho_B)], \quad (2.5.24)$$

which rearranges to (2.5.22), proving strong subadditivity.

2.5.4 Monogamy of the mutual information

Monogamy of the mutual information asserts

$$S(\rho_{AB}) + S(\rho_{BC}) + S(\rho_{AC}) \geq S(\rho_{ABC}) + S(\rho_A) + S(\rho_B) + S(\rho_C). \quad (2.5.25)$$

This inequality need not hold in generic quantum systems, but it is satisfied by holographic entanglement entropy at the classical (RT/HRT) level. Subleading quantum ($1/N$) corrections can introduce deviations.

2.5.5 Holographic proof of the inequality

We can show these inequalities diagrammatically at tree-level in $1/G$ using the Ryu-Takayanagi formula. Especially in the strip case, we just describe figures in Poincaré coordinates.

Subadditivity

We can prove

$$S_A + S_B \geq S_{AB} \quad (2.5.26)$$

as in figure 2.5.1.

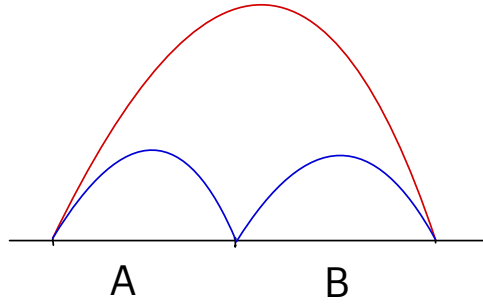


Figure 2.5.1: The red and blue curve describe S_{AB} and $S_A + S_B$, respectively. The red one is the geodesic connecting the boundaries of AB , thus we prove the subadditivity.

Strong subadditivity

For details, see [59]. We can prove

$$S_{AB} + S_{BC} \geq S_B + S_{ABC} \quad (2.5.27)$$

as in figure 2.5.2.

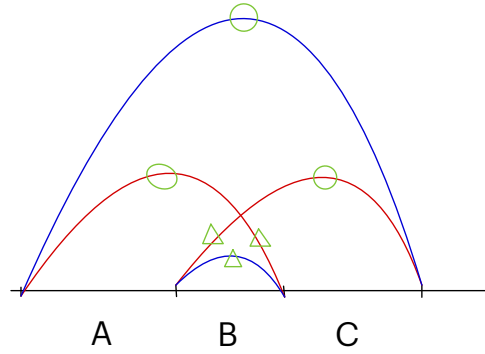


Figure 2.5.2: The red and blue curve describe the left and right hand side of (2.5.22). Comparing the circle and rectangular parts, we find blue ones are smaller since they are geodesics.

Monogamy of mutual information

For the original reference, see [60]. We can actually prove

$$S_{AB} + S_{BC} + S_{AC} \geq S_{ABC} + S_A + S_B + S_C \quad (2.5.28)$$

as in figure 2.5.3.

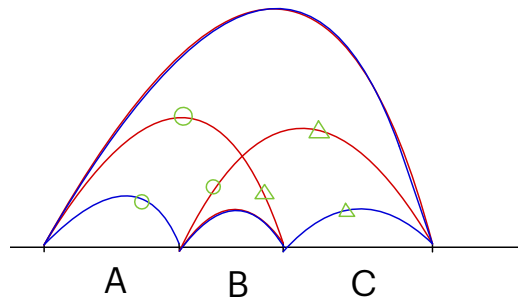


Figure 2.5.3: The red and blue curve describe the left and right hand side of (2.5.25). Comparing the circle and rectangular parts, we find blue ones are smaller since they are geodesics.

Chapter 3

Review on de Sitter space

3.1 de Sitter space

3.1.1 Metric and coordinates

$(d + 1)$ -dimensional de Sitter space is realized by the following embedding into the $(d + 2)$ -dimensional Minkowski space. The metric of the Minkowski space is given by

$$ds^2 = -dX_0^2 + dX_1^2 + \cdots + dX_{d+1}^2. \quad (3.1.1)$$

The $(d + 1)$ -dimensional de Sitter space is defined by the hypersurface

$$-X_0^2 + X_1^2 + \cdots + X_{d+1}^2 = R_{\text{dS}}^2. \quad (3.1.2)$$

For later use, we also define the AdS space similarly. We consider the $(d + 2)$ -dimensional Minkowski space with a metric

$$ds^2 = -d\tilde{X}_0^2 - d\tilde{X}_1^2 + \cdots + d\tilde{X}_{d+1}^2. \quad (3.1.3)$$

The $(d + 1)$ -dimensional AdS space is defined by

$$-\tilde{X}_0^2 - \tilde{X}_1^2 + \cdots + \tilde{X}_{d+1}^2 = -R_{\text{AdS}}^2. \quad (3.1.4)$$

Below, we introduce major coordinate systems in order.

Global coordinates

The global coordinates are given by the following embedding:

$$X_0 = R_{\text{dS}} \sinh \tau,$$

$$\begin{aligned}
X_1 &= R_{\text{dS}} \cosh \tau \cos \theta_1, \\
&\vdots \\
X_{d+1} &= R_{\text{dS}} \cosh \tau \sin \theta_1 \cdots \sin \theta_d.
\end{aligned} \tag{3.1.5}$$

The corresponding metric is given by

$$ds^2 = R_{\text{dS}}^2 (-d\tau^2 + \cosh \tau^2 d\Omega_d^2), \tag{3.1.6}$$

where $-\infty < \tau < \infty$ and $d\Omega_d^2$ is the line elements on S^d . This is called global coordinates since it spans the whole region in the de Sitter space.

Using coordinate transformations, we can move to the coordinates with conformal time:

$$ds^2 = \frac{R_{\text{dS}}^2}{\cos^2 T} (-dT^2 + d\Omega_d^2), \tag{3.1.7}$$

where $-\frac{\pi}{2} < T < \frac{\pi}{2}$. It is convenient for analyzing the causal structure.

Static patch coordinates

Let us introduce the static patch with an embedding:

$$\begin{aligned}
X_0 &= \sqrt{R_{\text{dS}}^2 - r^2} \sinh \left(\frac{t}{R_{\text{dS}}} \right), \\
X_1 &= \sqrt{R_{\text{dS}}^2 - r^2} \cosh \left(\frac{t}{R_{\text{dS}}} \right), \\
X_2 &= r \cos \theta_1, \\
&\vdots \\
X_{d+1} &= r \sin \theta_1 \cdots \sin \theta_{d-1},
\end{aligned} \tag{3.1.8}$$

which give the metric

$$ds^2 = - \left(1 - \frac{r^2}{R_{\text{dS}}^2} \right) dt^2 + \frac{dr^2}{1 - \frac{r^2}{R_{\text{dS}}^2}} + r^2 d\Omega_{d-1}^2. \tag{3.1.9}$$

Notice that these coordinates can be obtained by the analytic continuation from the global AdS. The embedding of the global coordinates in AdS space is given by

$$\begin{aligned}
\tilde{X}_0 &= \sqrt{R_{\text{AdS}}^2 + r^2} \sin \left(\frac{t}{R_{\text{AdS}}} \right), \\
\tilde{X}_1 &= \sqrt{R_{\text{AdS}}^2 + r^2} \cos \left(\frac{t}{R_{\text{AdS}}} \right),
\end{aligned}$$

$$\begin{aligned}
\tilde{X}_2 &= r \cos \theta_1, \\
&\vdots \\
\tilde{X}_{d+1} &= r \sin \theta_1 \cdots \sin \theta_{d-1},
\end{aligned} \tag{3.1.10}$$

where the metric takes

$$ds^2 = - \left(1 + \frac{r^2}{R_{\text{AdS}}^2} \right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{R_{\text{AdS}}^2}} + r^2 d\Omega_{d-1}^2. \tag{3.1.11}$$

Clearly we can obtain the metric in the static patch by setting $R_{\text{dS}} = iR_{\text{AdS}}$ from the one in the global AdS.

These coordinates are called static patch since the metric components do not depend on time t . There is a Killing vector $\partial/\partial t$, which generates a time translation $t \rightarrow t + \text{const}$. Notice that for $R_{\text{dS}} \leq r \leq \infty$, $\partial/\partial t$ becomes spacelike and thus we cannot define a timelike Killing vector globally in the de Sitter space. This leads to the fact that there is no positive conserved energy since if we construct a conserved quantity, it becomes negative in some regions. Also, there is another important remark: there is no unbroken supersymmetry in the de Sitter space. Since the commutation relation of the SUSY algebra suggests $\{Q, Q^\dagger\} \sim H$, where Q and H are the supercharge and the Hamiltonian, respectively. For the right hand side, it should be positive, but in the de Sitter case it is impossible.

We call the point $r = 0$ as the north pole (depending on a reference, it is sometimes called the south pole). At $r = R_{\text{dS}}$, there is a horizon called the cosmological horizon even though this is a vacuum solution of the Einstein equation. This can be naively understood as follows. Since the de Sitter space expands rapidly, the observer cannot see the whole region of de Sitter space physically (of course we can cover all the region by the global coordinates mathematically).

This is also clear from the Penrose diagram. A light ray sent from the north pole at $\mathcal{I}^-$ arrives at the south pole at $\mathcal{I}^+$. The spacetime region where a single observer at the north pole can affect and can be affected is the one which is surrounded by the cosmological horizon in the static patch.

Characteristic feature of the static patch is that they are in a thermal ensemble, which is also natural from the fact that there is a horizon. The temperature is fixed by the smoothness of the Wick-rotated metric as

$$T = \frac{|f'(R_{\text{dS}})|}{4\pi} = \frac{1}{2\pi R_{\text{dS}}}. \tag{3.1.12}$$

Flat slicing (Poincaré coordinates)

Next, we consider the flat slicing

$$\begin{aligned} X_0 &= R_{\text{dS}} \frac{\eta^2 - 1 - \vec{x}^2}{2\eta}, \\ X_1 &= R_{\text{dS}} \frac{\eta^2 + 1 - \vec{x}^2}{2\eta}, \\ X_i &= \frac{x_i}{\eta}, \end{aligned} \quad (3.1.13)$$

which leads to the metric

$$ds^2 = R_{\text{dS}}^2 \frac{-d\eta^2 + d\vec{x}^2}{\eta^2}, \quad (3.1.14)$$

called a flat slicing. In this case, we can also obtain the embedding from the AdS Poincaré coordinates. AdS Poincaré metric reads

$$ds^2 = R_{\text{AdS}}^2 \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}. \quad (3.1.15)$$

In the double Wick rotation,

$$R_{\text{AdS}} = iR_{\text{dS}}, \quad t = i\tau, \quad z = i\eta, \quad (3.1.16)$$

we obtain

$$ds^2 = R_{\text{dS}}^2 \frac{-d\eta^2 + d\tau^2 + d\vec{x}^2}{\eta^2}. \quad (3.1.17)$$

Note that flat slicing is also an example of the conformal coordinates. If we take $-\infty < \eta < 0$, it is called a expanding patch since $\mathcal{I}^+$ is located at $\eta = 0$ and the metric is literally expanding as η increases. On the other hand if $0 < \eta < \infty$, then it is called a contracting patch and the boundary is located at $\eta = 0$ or $\mathcal{I}^-$.

de Sitter slicing

We introduce a de Sitter slicing by

$$\begin{aligned} X_0 &= R_{\text{dS}} \sin \frac{\omega}{R_{\text{dS}}} \sinh \frac{t}{R_{\text{dS}}} \cosh \xi, \\ X_1 &= R_{\text{dS}} \cos \frac{\omega}{R_{\text{dS}}}, \\ X_2 &= R_{\text{dS}} \sin \frac{\omega}{R_{\text{dS}}} \cosh \frac{t}{R_{\text{dS}}}, \\ X_3 &= R_{\text{dS}} \sin \frac{\omega}{R_{\text{dS}}} \sinh \frac{t}{R_{\text{dS}}} \sinh \xi \cos \theta_1, \\ &\vdots \end{aligned}$$

$$X_{d+2} = R_{\text{dS}} \sin \frac{\omega}{R_{\text{dS}}} \sinh \frac{t}{R_{\text{dS}}} \sinh \xi \sin \theta_1 \cdots \sin \theta_{d-3}. \quad (3.1.18)$$

The metric reads

$$ds^2 = d\omega^2 + R_{\text{dS}}^2 \sin^2 \frac{\omega}{R_{\text{dS}}} (-dt^2 + R_{\text{dS}}^2 \sinh^2 \frac{t}{R_{\text{dS}}} (d\xi^2 + \sinh \xi^2 d\Omega_{d-2}^2)), \quad (3.1.19)$$

where $0 < \omega < \frac{\pi}{2}$. This metric foliates the one-dimensional lower de Sitter space.

hyperbolic slicing

We can consider the hyperbolic slicing as

$$\begin{aligned} X_0 &= R_{\text{dS}} \sinh \frac{t}{R_{\text{dS}}} \cosh \xi, \\ X_1 &= R_{\text{dS}} \cosh \frac{t}{R_{\text{dS}}}, \\ X_2 &= R_{\text{dS}} \sinh \frac{t}{R_{\text{dS}}} \sinh \xi \cos \theta_1, \\ &\vdots \\ X_{d+1} &= R_{\text{dS}} \sinh \frac{t}{R_{\text{dS}}} \sinh \xi \sin \theta_1 \sin \theta_{d-2}. \end{aligned} \quad (3.1.20)$$

The metric reads

$$ds^2 = -dt^2 + R_{\text{dS}}^2 \sinh^2 \frac{t}{R_{\text{dS}}} (d\xi^2 + \sinh^2 \xi d\Omega_{d-1}^2), \quad (3.1.21)$$

where $0 < t < \infty$. In this case, the one-dimensional lower hyperbolic space is foliated.

3.1.2 Symmetry

The de Sitter space is a solution of the vacuum Einstein equation with a positive cosmological constant and it has maximal symmetry. In $d + 1$ dimensions, this statement can be rephrased as there being $\frac{(d+1)(d+2)}{2}$ Killing vectors. The direction of the Killing vector ξ generates the isometry of the spacetime and this obeys the Killing equation

$$\mathcal{L}_\xi g = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \quad (3.1.22)$$

where $\mathcal{L}$ is the Lie derivative. This is symmetric with an exchange of indices and thus maximally there are $\frac{(d+1)(d+2)}{2}$ solutions. We can explicitly write down this symmetry in terms of the embedding coordinates. The embedding coordinates have a Lorentz symmetry,

whose generators are explicitly given by rotations and boosts

$$L_{\mu\nu} = i(X_\mu \partial_{X_\nu} - X_\nu \partial_{X_\mu}), \quad (3.1.23)$$

satisfying $L_{\mu\nu} = L_{\nu\mu}$, where $\mu, \nu = 0, \dots, d+1$. The commutation relations are given by

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\sigma} L_{\nu\rho}), \quad (3.1.24)$$

where $\eta_{\mu\nu} = \text{diag}(-1, +1, \dots, +1)$.

The d-dimensional conformal algebra can be embedded as

$$\begin{aligned} L_{ij} &= M_{ij} \quad (i, j = 1, \dots, d), \quad L_{0,d+1} = D, \\ L_{0i} + L_{d+1,i} &= P_i \quad L_{0i} - L_{d+1,i} = K_i, \end{aligned} \quad (3.1.25)$$

satisfying the following commutation relations

$$\begin{aligned} [M_{ij}, M_{kl}] &= i(\delta_{jk} M_{il} - \delta_{ik} M_{jl} + \delta_{il} M_{jk} - \delta_{jl} M_{ik}), \\ [M_{ij}, P_k] &= i(\delta_{jk} P_i - \delta_{ik} P_j), \quad [M_{ij}, K_k] = i(\delta_{jk} K_i - \delta_{ik} K_j), \\ [D, P_i] &= iP_i, \quad [D, K_i] = -K_i, \quad [D, M_{ij}] = 0, \\ [K_i, P_j] &= 2i(\delta_{ij} D - M_{ij}), \quad [P_i, P_j] = 0, \quad [K_i, K_j] = 0. \end{aligned} \quad (3.1.26)$$

Using this formulae, we can explicitly relate the Killing vectors in the de Sitter space with generators of the conformal algebra.

3.1.3 Black hole solutions

For a good review, refer to [61].

Three dimensions

The black hole solution in three-dimensional de Sitter space is given by

$$ds^2 = - \left(1 - 8GM - \frac{r^2}{R_{\text{dS}}^2} \right) dt^2 + \frac{dr^2}{1 - 8GM - \frac{r^2}{R_{\text{dS}}^2}} + r^2 d\phi^2. \quad (3.1.27)$$

Note that we have only one horizon $r = R_{\text{dS}} \sqrt{1 - 8GM}$ and if we take $M \rightarrow 0$, this reduces to the cosmological horizon. In this sense the black hole solution in three dimensions can be regarded as a conical deficit geometry rather than black hole geometry.

The temperature of this black hole is fixed by the smoothness of the Euclidean solution

as

$$T = \frac{\sqrt{1 - 8GM}}{2\pi R_{\text{dS}}}. \quad (3.1.28)$$

Schwarzschild-de Sitter solution in higher dimensions

The $(d+1)$ -dimensional Schwarzschild black hole solution in de Sitter space is given by

$$ds^2 = -\left(1 - \frac{2m}{r^{d-2}} - \frac{r^2}{R_{\text{dS}}^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r^{d-2}} - \frac{r^2}{R_{\text{dS}}^2}} + r^2 d\Omega_{d-1}^2. \quad (3.1.29)$$

In general we have two horizons, where $1 - \frac{2m}{r^{d-2}} - \frac{r^2}{R_{\text{dS}}^2} = 0$: the event horizon and the cosmological horizon.

Now, let us focus on the dS_4 case and take $R_{\text{dS}} = 1$. In this case the horizon is determined by the equation

$$1 - \frac{2m}{r} - r^2 = 0, \quad (3.1.30)$$

which gives two positive real solutions in general. For $m = 0$, the solution is degenerated as $r = 1$ and as we increase the mass there are two solutions corresponding to the event horizon and the cosmological horizon ($r_{\text{cos}} > r_{\text{ev}}$). For $m = 3^{-\frac{2}{3}}$, the solution again degenerates and gives the same radius $r_{\text{cos}} = r_{\text{ev}} = \frac{1}{\sqrt{3}}$.

Nariai solution

The solution with maximal mass $m = 3^{-\frac{2}{3}}$ is known as the Nariai black hole. Let us parametrize the radius as

$$r_{\text{cos}} = \frac{1}{\sqrt{3}} + \epsilon, \quad r_{\text{ev}} = \frac{1}{\sqrt{3}} - \epsilon, \quad (3.1.31)$$

where $\epsilon \ll 1$ since we consider the Nariai limit. By expanding the metric with the following choice of coordinates

$$r = \frac{1}{\sqrt{3}} + \epsilon\rho, \quad t = \frac{\tau}{\epsilon}, \quad (3.1.32)$$

to the leading order in ϵ we find

$$ds^2 = -3(1 - \rho^2)d\tau^2 + \frac{d\rho^2}{3(1 - \rho^2)} + \frac{1}{3}d\Omega_2^2, \quad (3.1.33)$$

which is $dS_2 \times S^2$.

3.1.4 Thermodynamics

Since there is a cosmological horizon in the static patch, we expect that there is thermodynamics in the static patch suggested by [62, 63].

One particular way to identify the thermodynamic quantities is to utilize the gravitational path integral. As we usually do, we Wick-rotate the geometry as

$$ds^2 = \left(1 - \frac{r^2}{R_{\text{dS}}^2}\right) d\tau^2 + \frac{dr^2}{1 - \frac{r^2}{R_{\text{dS}}^2}} + r^2 d\Omega_{d-1}^2, \quad (3.1.34)$$

or more generally

$$ds^2 = f(r) d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2. \quad (3.1.35)$$

In the gravitational path integral, we need a smooth saddle satisfying Einstein equation. We require that this geometry is smooth at $r = R_{\text{dS}}$. Using the expansion $f(r) \sim f(R_{\text{dS}}) + f'(R_{\text{dS}})(r - R_{\text{dS}}) = f'(R_{\text{dS}})(r - R_{\text{dS}})$, the metric reads

$$ds^2 = f'(R_{\text{dS}})(r - R_{\text{dS}}) d\tau^2 + \frac{dr^2}{f'(R_{\text{dS}})(r - R_{\text{dS}})} + r^2 d\Omega_{d-1}^2. \quad (3.1.36)$$

Using a coordinate transformation $R^2 = \frac{2(r - R_{\text{dS}})}{f'(R_{\text{dS}})}$, we obtain

$$ds^2 = \frac{f'(R_{\text{dS}})^2}{4} R^2 d\tau^2 + dR^2 + R_{\text{dS}}^2 d\Omega_{d-1}^2. \quad (3.1.37)$$

From the periodicity of the τ direction, we obtain $\tau \sim \tau + 2\pi R_{\text{dS}}$. Thus, the temperature is fixed as its inverse:

$$T = \frac{1}{2\pi R_{\text{dS}}}. \quad (3.1.38)$$

Note that substituting the current observed value of the cosmological constant gives extremely low temperature.

To look closer at the thermodynamics, let us compute the Euclidean action. Since the de Sitter space has no spatial boundary, the Euclidean geometry just gives a sphere. The action is given by the Euclidean Einstein-Hilbert action:

$$\begin{aligned} I_E &= -\frac{1}{16\pi G} \int d\tau \int_0^{R_{\text{dS}}} dr \int d\Omega_{d-1} \sqrt{g} (R - 2\Lambda) \\ &= -\frac{1}{4G} \frac{2R_{\text{dS}}^{d-1} \pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \\ &= -\frac{A_H}{4G}, \end{aligned} \quad (3.1.39)$$

where we use

$$R = \frac{d(d+1)}{R_{\text{dS}}^2}, \quad \Lambda = \frac{d(d-1)}{2R_{\text{dS}}^2}, \quad \beta = 2\pi R_{\text{dS}}, \quad (3.1.40)$$

and A_H is the area of the cosmological horizon. Thermodynamic relation suggests

$$I_E = \frac{F}{T} = \frac{E}{T} - S. \quad (3.1.41)$$

De Sitter static patch does not admit the boundary and thus no notion of energy. We simply set $E = 0$, we find

$$S = \frac{A_H}{4G}, \quad (3.1.42)$$

where $A_H = \frac{2R_{\text{dS}}^d \pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$. Particularly, in the dS_3 case $A_H = 2\pi R_{\text{dS}}$ as expected.

It is true not only in the Nariai black hole but also in the Schwarzschild de Sitter black hole that the entropy of them is always less than the entropy of the static patch

$$S_{\text{BH}} < S_{\text{static}}. \quad (3.1.43)$$

This means that the vacuum solution is the most entropically favored configuration, surprisingly. This can also be seen from the first law of black hole thermodynamics. Let us consider the three-dimensional black hole. The entropy is given by

$$S = \frac{A}{4G} = \frac{2\pi R_{\text{dS}} \sqrt{1 - 8GM}}{4G}. \quad (3.1.44)$$

We confirm that this satisfies the first law of thermodynamics

$$\frac{dS}{dM} = -\frac{2\pi R_{\text{dS}}}{\sqrt{1 - 8\pi GM}} = -\frac{1}{T}. \quad (3.1.45)$$

Thus, as we throw some object with small mass M into the cosmological horizon, the entropy decreases. In other words, the black holes in the de Sitter space have negative capacity of heat.

3.2 Quantum field theory on de Sitter (Free scalar)

3.2.1 Free scalar

Let us consider the free scalar field in dS_{d+1} . The action reads

$$S = -\frac{1}{2} \int d^{d+1}x \sqrt{-g} \left((\nabla\phi)^2 + m^2 \phi^2 \right). \quad (3.2.1)$$

The most interesting quantity is the Green's function (Wightman function). For now, we just consider classical Green's function without $i\epsilon$. The Green's function is defined by

$$(\nabla^2 - m^2)G(x, x') = \frac{1}{\sqrt{-g}}\delta^{(d+1)}(x). \quad (3.2.2)$$

For simplicity, let us focus on the vacuum solution without sources. From the translational invariance, the Green's function should be a function of the geodesic distance D_{dS} , which is related to the embedding coordinates by

$$\cos \frac{D_{\text{dS}}}{R_{\text{dS}}} = \frac{1}{R_{\text{dS}}^2}(-X_0 \cdot X'_0 + X_1 \cdot X'_1 + \cdots X_{d+1} \cdot X'_{d+1}). \quad (3.2.3)$$

If $\cos \frac{D_{\text{dS}}}{R_{\text{dS}}} > 1$, it is timelike geodesic, having an imaginary geodesic distance and similarly $\cos \frac{D_{\text{dS}}}{R_{\text{dS}}} < 1$ and $\cos \frac{D_{\text{dS}}}{R_{\text{dS}}} = 1$ for spacelike and null geodesics, respectively. By calculating the Laplacian the equation (3.2.2) can be rewritten as

$$(1 - P^2)\frac{d^2 G}{dP^2} - (d+1)P\frac{dG}{dP} - m^2 R_{\text{dS}}^2 G = 0, \quad (3.2.4)$$

where $P = \cos \frac{D_{\text{dS}}}{R_{\text{dS}}}$. The solution is given by

$$G = c_{m,d} {}_2F_1 \left[h_+, h_-, \frac{d+1}{2}, \frac{1+P}{2} \right], \quad (3.2.5)$$

where

$$h_{\pm} = \frac{1}{2}(d \pm \sqrt{d^2 - 4m^2 R_{\text{dS}}^2}). \quad (3.2.6)$$

This hypergeometric function has a singularity at $D_{\text{dS}} = 0$ and a branch cut for $1 < \cos \frac{D_{\text{dS}}}{R_{\text{dS}}} < \infty$. This singularity corresponds to the coincident limit of the two points in the Green's function. At short distances, it is natural to expect that we not capture the curvature of de Sitter space or we just obtain the propagator in the Minkowski space, which fixes the constant as in [18]

$$c_{m,d} = 4^{1-\frac{d+1}{2}} \frac{\Gamma(h_+)\Gamma(h_-)}{4\pi^{\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right)}. \quad (3.2.7)$$

This is called the Green's function in the Euclidean vacuum (Bunch-Davies vacuum or Hartle-Hawking vacuum). Later, we will see that it is natural choice since it has a minimum energy as in the quantum field theory in the flat space. In curved space, the positive and negative modes usually are mixed and hence the choice of the vacuum is not unique.

As we mentioned above the choice of the vacuum is not unique. Then, how can we define the other ones? The point is that the Laplace equation for the Green's function (3.2.2) is

invariant under the replacement $P \leftrightarrow -P$. Thus we can find a linearly independent solution

$${}_2F_1 \left[h_+, h_-, \frac{d+1}{2}, \frac{1-P}{2} \right]. \quad (3.2.8)$$

The singularity is at $P = -1$, which means that the Green's function diverges at the antipodal point separations, where the antipodal point is defined by $X_i \rightarrow -X_i$ in the embedding coordinates. The antipodal point is always separated by the horizon and this solution can be regarded as a mirror image through the horizon. By taking the linear combinations of the solutions, we have one complex parameter family of vacua in general. We will later see in the form of mode expansions.

3.2.2 Mode expansions

The scalar field can be expanded as

$$\phi(x) = \sum_n (a_n \phi_n(x) + a_n^\dagger \phi_n^*(x)). \quad (3.2.9)$$

We can also define the vacuum by

$$a_n |\Omega\rangle = 0, \quad (3.2.10)$$

where

$$[a_n, a_m^\dagger] = \delta_{mn}. \quad (3.2.11)$$

The mode function ϕ_n satisfies the wave equation

$$(\nabla^2 - m^2)\phi = 0. \quad (3.2.12)$$

In Minkowski space, we can carry out a simple calculation and define the energy as a VEV of the Hamiltonian

$$E_k = |\dot{\phi}_k|^2 + k^2 |\phi_k|^2, \quad (3.2.13)$$

with a normalization

$$\dot{\phi}_k \phi_k^* - \phi_k \dot{\phi}_k^* = -i, \quad (3.2.14)$$

where the dot represents the derivative with time. If we minimize the energy, the solution is

$$\phi_k = \frac{1}{\sqrt{2k}} e^{-ikt}. \quad (3.2.15)$$

We can play the same game for the de Sitter space. In the de Sitter case, we define the Green's function in the Euclidean vacuum as the one matching with the one in the flat space limit. Thus, it is enough to believe that the Euclidean vacuum has a minimum energy if there are

no phase transitions in the flat space limit. Physically, this approximation tells us that all the modes are inside the horizon ($k \gg aH$) at sufficiently early times in the beginning of the inflation.

As we mentioned before, there is one parameter family of vacua in de Sitter space respecting the symmetry. In the language of the mode expansions, it is described by the Bogoliubov transformations

$$\tilde{\phi}_n = \frac{1}{\sqrt{1 - e^{\alpha + \alpha^*}}}(\phi_n + e^{\alpha} \phi_n^*), \quad \tilde{a}_n = \frac{1}{\sqrt{1 - e^{\alpha + \alpha^*}}}(a_n - e^{\alpha^*} a_n^\dagger). \quad (3.2.16)$$

The vacuum defined by

$$\tilde{a}_n |\alpha\rangle = 0, \quad (3.2.17)$$

is called α -vacua. Correspondingly, the Green's function in α -vacua reads

$$G_\alpha(x, x') = \frac{1}{1 - e^{\alpha + \alpha^*}}(G_E(x, x') + e^{\alpha + \alpha^*} G_E(x_A, x'_A) + e^{\alpha} G_E(x_A, x') + e^{\alpha^*} G_E(x, x'_A)). \quad (3.2.18)$$

Since the vacuum is time dependent in the de Sitter space, we cannot define an S-matrix. In some vacuum, there are no particle productions, but as we change the vacuum, we can create particles arbitrarily [64].

Finally, let us derive the temperature of the static patch from the Green's function. Suppose that we consider an observer at the north pole (or $r = 0$) and the Green's function at different times on the pole reads

$$G = G\left(\cos \frac{D_{\text{dS}}}{R_{\text{dS}}}\right) = G\left(\cosh \frac{t - t'}{R_{\text{dS}}}\right), \quad (3.2.19)$$

where t denotes the time coordinate in the static patch. This Green's function is invariant under $t \rightarrow t + 2\pi i R_{\text{dS}}$. This means that the static patch has a temperature

$$T = \frac{1}{2\pi R_{\text{dS}}}, \quad (3.2.20)$$

since we can read off the periodicity from the KMS condition after the Wick rotation.

3.3 RT formula in de Sitter space

It is natural to ask whether the Ryu-Takayanagi (RT) prescription extends to de Sitter space, since the original proof does not explicitly depend on a particular background geometry [43]. In practice, however, serious obstacles arise. For example, in the dS_3/CFT_2 setting there are no bulk spacelike geodesics that connect the endpoints of an interval anchored at future infinity. One proposed workaround is to prepare a Hartle-Hawking state [65] and allow complex

geodesics that pass into the Euclidean continuation of the geometry [30]; generically, this produces complex-valued holographic entanglement entropies.

A second challenge is that the entanglement entropy computed via an RT-prescription in de Sitter violates (strong) subadditivity. Within the RT framework this forces a choice of extremization rule. It has been argued that the appropriate rule is the *min-max* procedure—maximize along spatial directions and minimize along time—rather than the *max-min* ordering in AdS/CFT [66–69]. In AdS/CFT, the max-min prescription [55] is equivalent to the Hubeny-Rangamani-Takayanagi (HRT) proposal [54]. By contrast, de Sitter spatial slices are compact, so minimizing over spatial directions trivially yields zero, effectively selecting degenerate contributions localized at the sphere’s poles. To obtain a nontrivial answer, one is naturally led to expect contributions from the cosmological horizon instead (see, *e.g.*, [70, 71]). Altogether, these considerations indicate that the RT prescription, in its straightforward form, is ill-suited to de Sitter holography in the static patch case.

3.4 dS/CFT correspondence

The dS/CFT correspondence was originally conjectured by Strominger in [18]. This proposal posits that quantum gravity in $(d + 1)$ -dimensional de Sitter space is dual to a d -dimensional CFT defined at future infinity, *i.e.*, the asymptotic boundary of dS spacetime. Analogous to the holographic dictionary in the AdS/CFT correspondence [19, 20], correlation functions on both sides can be derived from the equivalence between the bulk wavefunction of the universe in de Sitter space and the partition function of the boundary CFT. In [21], it is suggested that this correspondence may be understood via an analytic continuation of the curvature radius from AdS to dS space. In particular, for the dS₃/CFT₂ case, the central charge c of the boundary CFT is proposed to be

$$c = i \frac{3L}{2G}, \quad (3.4.1)$$

where L denotes the dS radius. This expression implies that the central charge is purely imaginary at tree level, suggesting that the boundary CFT is non-unitary. This non-unitarity is partially attributed to the absence of a well-defined time direction on the future boundary where the CFT resides. One realization of the dS/CFT correspondence was proposed in [72], wherein the analytic continuation ($N \rightarrow -N$) of the $O(N)$ vector model leads to a CFT with $\text{Sp}(N)$ symmetry in CFT. Another realization derived in [30, 73] involves performing an analytic continuation of the level k in the Wess-Zumino-Witten model.

Chapter 4

Bulk local state in dS_3

A bulk-local state is a natural probe of excitations because its propagation is tightly constrained by the background geometry. In AdS/CFT, reconstructing a bulk local operator from boundary CFT operators is known as the HKLL reconstruction [28]. An alternative AdS/CFT construction was introduced in [74] and further developed in [75]. Exploiting symmetries—such as rotational and translational invariance—to constrain the excitation profile, these works first construct a bulk-local operator at the AdS center and then obtain operators at arbitrary bulk points via isometries. It was later shown that this symmetry-based construction is equivalent to HKLL [76].

In what follows, we present a symmetry-driven construction of bulk-local states in de Sitter space; see [26] for details. Throughout this section, we do not assume the dS/CFT correspondence.

Notations

We summarize some of the important notations that appear in this paper for the readers' convenience. Precise definitions of these states will be explained in later sections.

Wick-rotated bulk local states in de Sitter space

$$\begin{aligned}
 |\Psi_{dS,W}^{\Delta_+}(x)\rangle &= e^{(L_0+\tilde{L}_0)t} e^{-i\phi(L_0-\tilde{L}_0)} e^{i\theta J_1} \sum c_k (L_{-1}\tilde{L}_{-1})^k |\Delta_+\rangle, \\
 \langle\Psi_{dS,W}^{\Delta_\pm}(x')| &= \langle\Delta_\pm| \sum_{k=0}^{\infty} c_k(\Delta_\pm) (L_1\tilde{L}_1)^k e^{-i\theta J_1} e^{i\phi J_3} e^{-(L_0+\tilde{L}_0)t}.
 \end{aligned} \tag{4.0.1}$$

Conjugations of primary states

$$\begin{aligned} |\Delta_{\pm}\rangle^{\dagger} &\equiv \langle \widehat{\Delta_{\pm}} |, & \langle \Delta_{\pm} |^{\dagger} &\equiv | \widehat{\Delta_{\pm}} \rangle, \\ \langle \widehat{\Delta_{\pm}} | \Delta_{\pm} \rangle &= 0, & \langle \Delta_{\pm} | \Delta_{\pm} \rangle &\equiv 1. \end{aligned} \quad (4.0.2)$$

Conjugations of bulk local states

$$\begin{aligned} \nu_{\pm} \langle \Psi_{\text{dS},W}^{\Delta_{\pm}}(x'_A) | \Psi_{\text{dS},W}^{\Delta_{\pm}}(x) \rangle &= \langle \widehat{\Psi_{\text{dS},W}^{\Delta_{\mp}}}(x') | \Psi_{\text{dS},W}^{\Delta_{\pm}}(x) \rangle = \frac{e^{\mp\mu(\pi - D_{\text{dS}}(x'_A, x))}}{2i \sin D_{\text{dS}}}, \\ \langle \Psi_{\text{dS},W}^{\Delta_{\pm}}(x') | \Psi_{\text{dS},W}^{\Delta_{\pm}}(x) \rangle &= \frac{e^{\pm\mu D_{\text{dS}}}}{2i \sin D_{\text{dS}}}, \\ |\Psi_{\text{dS},W}^{\Delta_{+}}(x) \rangle^{\dagger} &\equiv \langle \widehat{\Phi_{\text{dS},W}^{\Delta_{+}}}(x) | = \langle \Phi_{\text{dS},W}^{\Delta_{-}}(x_A) |. \end{aligned} \quad (4.0.3)$$

4.0.1 Bulk local states in AdS₃

In this section, we review the construction of bulk-local states in AdS, concentrating on the AdS₃ case [27, 75].

4.0.2 Global $\text{SL}(2, \mathbf{R})_{\text{L}} \times \text{SL}(2, \mathbf{R})_{\text{R}}$ Virasoro generators from AdS₃

Embed AdS₃ into four-dimensional Minkowski space:

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + dX_2^2. \quad (4.0.4)$$

The AdS hypersurface is specified by

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 = -1, \quad (4.0.5)$$

with the AdS radius set to $R_{\text{AdS}} = 1$. Adopting global coordinates,

$$\begin{aligned} X_{-1} &= \cosh \rho \sin \tau, & X_0 &= \cosh \rho \cos \tau, \\ X_1 &= \sinh \rho \cos \phi, & X_2 &= \sinh \rho \sin \phi, \end{aligned} \quad (4.0.6)$$

one obtains the global AdS₃ metric

$$ds^2 = -\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\phi^2, \quad (4.0.7)$$

with ranges $0 \leq \rho < \infty$, $-\infty < \tau < \infty$, and $0 \leq \phi < 2\pi$. Throughout we take $R_{\text{AdS}} = 1$ unless stated otherwise.

The dual CFT is naturally defined on the cylinder. The AdS_3 isometry group decomposes as $\text{SL}(2, \mathbf{R})_{\text{L}} \times \text{SL}(2, \mathbf{R})_{\text{R}}$ and is represented on the boundary by the global Virasoro generators $(L_0, L_{\pm 1})$ and $(\tilde{L}_0, \tilde{L}_{\pm 1})$. Their explicit bulk Killing-vector realizations are

$$\begin{aligned} L_0 &= \frac{i}{2}(\partial_\tau + \partial_\phi), & \tilde{L}_0 &= \frac{i}{2}(\partial_\tau - \partial_\phi), \\ L_{\pm 1} &= \frac{i}{2}e^{\pm i(\tau+\phi)} \left[\frac{\sinh \rho}{\cosh \rho} \partial_\tau + \frac{\cosh \rho}{\sinh \rho} \partial_\phi \mp i \partial_\rho \right], \\ \tilde{L}_{\pm 1} &= \frac{i}{2}e^{\pm i(\tau-\phi)} \left[\frac{\sinh \rho}{\cosh \rho} \partial_\tau - \frac{\cosh \rho}{\sinh \rho} \partial_\phi \mp i \partial_\rho \right], \end{aligned} \quad (4.0.8)$$

which satisfy the global Virasoro (i.e. $\mathfrak{sl}(2, \mathbb{R})$) algebras

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [\tilde{L}_m, \tilde{L}_n] = (m - n)\tilde{L}_{m+n}. \quad (4.0.9)$$

In Lorentzian signature the Hermitian conjugation acts as

$$(L_n)^\dagger = L_{-n}, \quad (\tilde{L}_n)^\dagger = \tilde{L}_{-n}. \quad (4.0.10)$$

4.0.3 Bulk local states in global AdS_3

We begin by constructing a bulk-local operator at $\tau = 0$, since operators at arbitrary positions can then be generated by isometries. At $\tau = 0$, define the combinations

$$\begin{aligned} l_0 &= L_0 - \tilde{L}_0 = i\partial_\phi, \\ l_1 &= \tilde{L}_{-1} - L_1 = -ie^{i\phi} \left[\frac{1 + \cosh 2\rho}{\sinh 2\rho} \partial_\phi - i\partial_\rho \right], \\ l_{-1} &= \tilde{L}_1 - L_{-1} = ie^{-i\phi} \left[-\frac{1 + \cosh 2\rho}{\sinh 2\rho} \partial_\phi - i\partial_\rho \right], \end{aligned} \quad (4.0.11)$$

which close into an $\text{SL}(2, \mathbf{R})$ subgroup of $\text{SL}(2, \mathbf{R})_{\text{L}} \times \text{SL}(2, \mathbf{R})_{\text{R}}$:

$$[l_m, l_n] = (m - n) l_{m+n}. \quad (4.0.12)$$

Exploiting this $\text{SL}(2, \mathbf{R})$ symmetry, place the bulk-local state at the AdS center $(\tau, \rho) = (0, 0)$ and denote it by $|\Psi_{\text{AdS}}^{(\Delta)}(\tau = 0, \rho = 0)\rangle$, where Δ is the conformal dimension (the eigenvalue of $L_0 + \tilde{L}_0$)¹. Invariance of the point $\tau = \rho = 0$ under the stabilizer (e.g. rotations in ϕ) imposes

$$(L_0 - \tilde{L}_0) |\Psi_{\text{AdS}}^{(\Delta)}(0, 0)\rangle = (L_1 + \tilde{L}_{-1}) |\Psi_{\text{AdS}}^{(\Delta)}(0, 0)\rangle = (L_{-1} + \tilde{L}_1) |\Psi_{\text{AdS}}^{(\Delta)}(0, 0)\rangle = 0. \quad (4.0.13)$$

¹Again, we do not have to specify the details of CFT. We just rewrite the isometry as global conformal algebra.

A solution to these constraints is

$$|\Psi_{\text{AdS}}^{(\Delta)}(\tau = 0, \rho = 0)\rangle = e^{\frac{i\pi}{2}(L_0 + \tilde{L}_0 - \Delta)} |I_\Delta\rangle, \quad (4.0.14)$$

with $|I_\Delta\rangle$ the global Ishibashi state for $\text{SL}(2, \mathbf{R})_L \times \text{SL}(2, \mathbf{R})_R$, satisfying

$$(L_n - \tilde{L}_{-n}) |I_\Delta\rangle = 0, \quad n = 0, \pm 1. \quad (4.0.15)$$

A convenient representation is

$$|I_\Delta\rangle = \sum_{k=0}^{\infty} |k\rangle_L |k\rangle_R. \quad (4.0.16)$$

Here the orthonormal descendants are

$$|k\rangle = \prod_{j=1}^k \sqrt{\frac{1}{j^2 + (\Delta - 1)j}} (L_{-1})^k |\Delta\rangle, \quad (4.0.17)$$

obeying $\langle k_1 | k_2 \rangle = \delta_{k_1, k_2}$, and the primary $|\Delta\rangle$ is defined by

$$L_0 |\Delta\rangle = \tilde{L}_0 |\Delta\rangle = \frac{\Delta}{2} |\Delta\rangle, \quad L_1 |\Delta\rangle = 0. \quad (4.0.18)$$

By acting with the $\text{SL}(2, \mathbf{R})$ isometries and time evolution, we obtain bulk-local operators at arbitrary points,

$$|\Psi_{\text{AdS}}^{(\Delta)}(\tau, \rho, \phi)\rangle = e^{-i(L_0 + \tilde{L}_0)\tau} e^{i\phi l_0} e^{-\frac{\rho}{2}(l_1 - \tilde{l}_1)} e^{\frac{i\pi}{2}(L_0 + \tilde{L}_0)} |I_\Delta\rangle. \quad (4.0.19)$$

It is straightforward to verify that this state satisfies the scalar equation of motion:

$$\left[\frac{1}{2}(L_{-1}L_1 + L_1L_{-1}) - L_0^2 + \frac{1}{2}(\tilde{L}_{-1}\tilde{L}_1 + \tilde{L}_1\tilde{L}_{-1}) - \tilde{L}_0^2 + \frac{M^2}{2} \right] |\Psi_{\text{AdS}}^{(\Delta)}(\tau, \rho, \phi)\rangle = 0, \quad (4.0.20)$$

where the scalar mass M is related to the conformal dimension via the standard $\text{AdS}_3/\text{CFT}_2$ relation $M^2 = \Delta^2 - 2\Delta$. The quadratic Casimir coincides with the Laplacian of a free scalar in AdS,

$$\square = \frac{1}{2}(L_{-1}L_1 + L_1L_{-1}) - L_0^2 + \frac{1}{2}(\tilde{L}_{-1}\tilde{L}_1 + \tilde{L}_1\tilde{L}_{-1}) - \tilde{L}_0^2. \quad (4.0.21)$$

Thus the state (4.0.19) indeed represents a localized excitation in AdS_3 . Writing the bulk scalar field operator as $\hat{\Phi}^{(\text{AdS})}(\tau, \rho, \phi)$, the bulk-local state can be expressed as

$$|\Psi_{\text{AdS}}^{(\Delta)}(\tau, \rho, \phi)\rangle = \hat{\Phi}^{(\text{AdS})}(\tau, \rho, \phi) |0\rangle. \quad (4.0.22)$$

4.0.4 Two-point functions

Following [27], we now extract the two-point function from the bulk-local state. The scalar wavefunction in AdS_3 is obtained by taking inner products of $|\Psi_{\text{AdS}}^{(\Delta)}(\tau, \rho, \phi)\rangle$ with the descendant basis $|k\rangle_{\text{L}} |\bar{k}\rangle_{\text{R}}$, yielding (for $k \geq \bar{k}$)

$$\Phi_{k, \bar{k}}^{(\text{AdS})}(\tau, \rho, \phi) = e^{-i(\Delta+k+\bar{k})\tau} e^{i(k-\bar{k})\phi} (\tanh \rho)^{|k-\bar{k}|} (\cosh \rho)^{-\Delta} \cdot P_{\bar{k}}^{(k-\bar{k}, \Delta-1)} \left(\frac{2}{\cosh^2 \rho} - 1 \right), \quad (4.0.23)$$

where the Jacobi polynomial P is defined by

$$P_n^{(\alpha, \beta)}(x) = \frac{(n+\alpha)!}{n! \alpha!} \cdot {}_2F_1 \left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2} \right). \quad (4.0.24)$$

For $k < \bar{k}$, simply swap k and $\bar{k}$ in the P -function.

The Green's function is

$$\begin{aligned} G_{\text{AdS}}(\tau, \rho, \phi) &= \langle \Psi_{\text{AdS}}^{(\Delta)}(\tau, \rho, \phi) | \Psi_{\text{AdS}}^{(\Delta)}(0, 0, 0) \rangle \\ &= \langle \Psi_{\text{AdS}}^{(\Delta)}(\tau, 0, 0) | e^{-\rho(L_1 - L_{-1})} | \Psi_{\text{AdS}}^{(\Delta)}(0, 0, 0) \rangle. \end{aligned} \quad (4.0.25)$$

Inserting the resolution of identity

$$1 = \sum_{k, \bar{k}=0}^{\infty} |k\rangle_{\text{L}} |\bar{k}\rangle_{\text{R}} \langle k|_{\text{L}} \langle \bar{k}|_{\text{R}}, \quad (4.0.26)$$

we obtain

$$G_{\text{AdS}}(\tau, \rho, \phi) = \sum_{k=0}^{\infty} (-1)^k \langle \Psi_{\text{AdS}}^{(\Delta)}(\tau, 0, 0) | e^{-\rho(L_1 - L_{-1})} | k\rangle_{\text{L}} | k\rangle_{\text{R}}. \quad (4.0.27)$$

Moreover,

$$L_{-1} \tilde{L}_{-1} \Phi_{k,k}^{(\text{AdS})} = -(k+1)(k+\Delta) \Phi_{k+1,k+1}^{(\text{AdS})}. \quad (4.0.28)$$

Using this relation, one finds

$$\langle \Psi_{\text{AdS}}^{(\Delta)}(\tau, 0, 0) | e^{\rho(L_1 - L_{-1})} | k\rangle_{\text{L}} | k\rangle_{\text{R}} = (-1)^k \Phi_{k,k}^{(\text{AdS})}. \quad (4.0.29)$$

Consequently, the Green's function evaluates to

$$\begin{aligned} G_{\text{AdS}}(\tau, \rho, \phi) &= \sum_{k=0}^{\infty} \Phi_{k,k}^{(\text{AdS})}(\tau, \rho, \phi) \\ &= \sum_{k=0}^{\infty} \frac{e^{-i(2k+\Delta)\tau}}{\cosh^{\Delta} \rho} P_k^{(0, \Delta-1)}(1 - 2 \tanh^2 \rho) \end{aligned}$$

$$= \frac{e^{-(\Delta-1)D_{\text{AdS}}}}{2 \sinh D_{\text{AdS}}}, \quad (4.0.30)$$

where the geodesic distance is

$$\cosh D_{\text{AdS}} = \cos(\tau - \tau') \cosh \rho \cosh \rho' - \cos(\phi - \phi') \sinh \rho \sinh \rho'. \quad (4.0.31)$$

4.0.5 Bulk local states in static dS₃

Next, we construct bulk local states in dS₃ via analytic continuation from AdS₃. We first summarize the coordinate systems and then discuss the construction of bulk local states in dS₃.

4.0.6 Global SL(2, C) Virasoro generators

Let us start with the embedding in Minkowski space

$$ds^2 = -dX_{-1}^2 + dX_0^2 + dX_1^2 + dX_2^2. \quad (4.0.32)$$

dS₃ is defined by the hypersurface

$$-X_{-1}^2 + X_0^2 + X_1^2 + X_2^2 = 1, \quad (4.0.33)$$

where we set $R_{\text{dS}} = 1$. Embed the coordinates as

$$\begin{aligned} X_{-1} &= \cos \theta \sinh t, & X_0 &= \cos \theta \cosh t, \\ X_1 &= \sin \theta \cos \phi, & X_2 &= \sin \theta \sin \phi, \end{aligned} \quad (4.0.34)$$

which yields the static patch of dS₃

$$ds^2 = -\cos^2 \theta dt^2 + d\theta^2 + \sin^2 \theta d\phi^2, \quad (4.0.35)$$

where $0 < \theta < \pi$, $-\infty < t < \infty$, and $0 < \phi < 2\pi$. If we Wick-rotate the time $t \rightarrow i\tau$, the geometry becomes a sphere.

The static patch is obtained by replacing the AdS₃ coordinates as follows:

$$\tau = it, \quad \rho = i\theta, \quad \phi = \phi, \quad R_{\text{AdS}}^2 = -R_{\text{dS}}^2. \quad (4.0.36)$$

The isometries of dS₃ form an $\mathfrak{sl}(2, \mathbb{C})$ algebra, generated by the global Virasoro

generators $(L_0, L_{\pm 1})$ and $(\tilde{L}_0, \tilde{L}_{\pm 1})$, which can be written explicitly as

$$\begin{aligned} L_0 &= \frac{1}{2}(\partial_t + i\partial_\phi), & \tilde{L}_0 &= \frac{1}{2}(\partial_t - i\partial_\phi), \\ L_{\pm 1} &= \frac{i}{2}e^{\pm(-t+i\phi)} \left[\frac{\sin \theta}{\cos \theta} \partial_t - i \frac{\cos \theta}{\sin \theta} \partial_\phi \mp i\partial_\rho \right], \\ \tilde{L}_{\pm 1} &= \frac{i}{2}e^{\pm(-t-i\phi)} \left[\frac{\sin \theta}{\cos \theta} \partial_t + i \frac{\cos \theta}{\sin \theta} \partial_\phi \mp i\partial_\rho \right], \end{aligned} \quad (4.0.37)$$

which satisfy

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [\tilde{L}_m, \tilde{L}_n] = (m - n)\tilde{L}_{m+n}. \quad (4.0.38)$$

From (4.0.37) and by an analysis analogous to section 4.0.3 for AdS_3 , their Hermitian conjugates take the following nonstandard form:

$$\begin{aligned} (L_0)^\dagger &= -\tilde{L}_0, & (L_{\pm 1})^\dagger &= \tilde{L}_{\pm 1}, \\ (\tilde{L}_0)^\dagger &= -L_0, & (\tilde{L}_{\pm 1})^\dagger &= L_{\pm 1}. \end{aligned} \quad (4.0.39)$$

This contrasts sharply with the standard conjugation (4.0.10) in $\text{AdS}_3/\text{CFT}_2$.

The geodesic length $D_{\text{dS}}(x, x')$ between two points $x = (t, \theta, \phi)$ and $x' = (t', \theta', \phi')$ is

$$\cos D_{\text{dS}}(x, x') = \cosh(t - t') \cos \theta \cos \theta' + \cos(\phi - \phi') \sin \theta \sin \theta'. \quad (4.0.40)$$

The antipodal map is defined by flipping the signs of the embedding coordinates,

$$(X_{-1}, X_0, X_1, X_2) \rightarrow (-X_{-1}, -X_0, -X_1, -X_2), \quad (4.0.41)$$

which corresponds, via (4.0.34), to $x = (t, \theta, \phi) \rightarrow x_A = (t, \pi - \theta, \phi + \pi)$. We also have

$$\cos D_{\text{dS}}(x, x') = -\cos D_{\text{dS}}(x, x'_A), \quad (4.0.42)$$

implying

$$D_{\text{dS}}(x, x'_A) = \pi - D_{\text{dS}}(x, x'). \quad (4.0.43)$$

4.0.7 Bulk local states in the static coordinates of dS_3

Let us construct a bulk-local state in the static patch of dS_3 , following the AdS_3 analysis and exploiting symmetry. We introduce the $SU(2)$ generators:

$$\begin{aligned} J_1 &= \frac{1}{2} [(L_1 - \tilde{L}_{-1}) + (\tilde{L}_1 - L_{-1})], \\ J_2 &= \frac{i}{2} [(L_1 - \tilde{L}_{-1}) - (\tilde{L}_1 - L_{-1})], \\ J_3 &= L_0 - \tilde{L}_0, \end{aligned} \quad (4.0.44)$$

which satisfy the $\mathfrak{su}(2)$ algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad (4.0.45)$$

with Hermitian generators $J_i^\dagger = J_i$.

We first excite the origin $(t, \theta, \phi) = (0, 0, 0)$ in (4.0.35) and denote the corresponding dual CFT state by $|\Psi_\Delta\rangle$, where Δ is the conformal dimension of the primary state in the dual 2D CFT. The point $\theta = t = 0$ is invariant under the actions of $L_0 - \tilde{L}_0$ and $L_{\pm 1} + \tilde{L}_{\mp 1}$. Hence we impose

$$\begin{aligned} (L_0 - \tilde{L}_0)|\Psi_\Delta\rangle &= 0, \\ (L_1 + \tilde{L}_{-1})|\Psi_\Delta\rangle &= 0, \\ (L_{-1} + \tilde{L}_1)|\Psi_\Delta\rangle &= 0. \end{aligned} \quad (4.0.46)$$

These constraints (4.0.46) are solved by taking

$$|\Psi_\Delta\rangle = e^{i\frac{\pi}{2}(L_0 + \tilde{L}_0 - \Delta)}|I_\Delta\rangle. \quad (4.0.47)$$

Here $|I_\Delta\rangle$ is the Ishibashi state, and we can write the bulk-local state explicitly as

$$|\Psi_\Delta\rangle = \sum_{k=0}^{\infty} c_k(\Delta) L_{-1}^k \tilde{L}_{-1}^k |\Delta\rangle, \quad (4.0.48)$$

where $|\Delta\rangle$ is the primary state of dimension Δ (4.0.18), and c_k is

$$c_k(\Delta) = \frac{e^{i\pi k}}{\prod_{j=1}^k (j^2 + (\Delta - 1)j)}. \quad (4.0.49)$$

To obtain a generic bulk-local state representing an excitation at an arbitrary point (t, θ, ϕ) ,

note that the $t = 0$ slice is preserved by the $SU(2)$ generators in (4.0.44):

$$\begin{aligned} J_1 &= i \left(\frac{\sin \phi \cos \theta}{\sin \theta} \partial_\phi - \cos \phi \partial_\theta \right), \\ J_2 &= i \left(\frac{\cos \phi \cos \theta}{\sin \theta} \partial_\phi + \sin \phi \partial_\theta \right), \\ J_3 &= i \partial_\phi. \end{aligned} \tag{4.0.50}$$

The isometry that maps $(t, \theta, \phi) = (0, 0, 0)$ to $(0, \theta_0, \phi_0)$ is $e^{-i\phi_0 J_3} e^{i\theta_0 J_1}$. The bulk-local state at an arbitrary point is then obtained by time evolution:

$$|\Psi_{\text{dS},W}^\Delta(t, \theta, \phi)\rangle = e^{(L_0 + \tilde{L}_0)t} e^{-i\phi J_3} e^{i\theta J_1} |\Psi_\Delta\rangle, \tag{4.0.51}$$

where the subscript W indicates Wick rotation. The equation of motion for a scalar of mass m reads

$$\left[\frac{1}{2} (L_{-1} L_1 + L_1 L_{-1}) - L_0^2 + \frac{1}{2} (\tilde{L}_{-1} \tilde{L}_1 + \tilde{L}_1 \tilde{L}_{-1}) - \tilde{L}_0^2 - \frac{m^2}{2} \right] |\Psi_{\text{dS},W}^\Delta(x)\rangle = 0, \tag{4.0.52}$$

with $x = (t, \theta, \phi)$, which is the Laplace equation. This fixes the conformal dimensions to

$$\Delta_\pm = 1 \pm \sqrt{1 - m^2} \equiv 1 \pm i\mu. \tag{4.0.53}$$

We therefore obtain two bulk-local states

$$|\Psi_{\text{dS},W}^{\Delta_\pm}(x)\rangle = e^{(L_0 + \tilde{L}_0)t} e^{-i\phi J_3} e^{i\theta J_1} \sum_{k=0}^{\infty} c_k(\Delta_\pm) L_{-1}^k \tilde{L}_{-1}^k |\Delta_\pm\rangle. \tag{4.0.54}$$

However, the dS_3 bulk-local states constructed above are not physical: they do not reproduce the correct two-point function. We will construct physical states later in this section. Before that, let us compute their inner products.

4.0.8 Two-point functions via Wick rotation from AdS_3

We now compute the inner product of the locally excited state. We Wick-rotate the AdS_3 two-point function to obtain

$$\langle \Psi_{\text{dS},W}^{\Delta_\pm}(x') | \Psi_{\text{dS},W}^{\Delta_\pm}(x) \rangle = \frac{e^{\pm \mu D_{\text{dS}}(x,x')}}{2i \sin D_{\text{dS}}}, \tag{4.0.55}$$

where the conformal dimension is given by (4.0.53). The relation between geodesic distances is

$$D_{\text{AdS}}(x, x') = iD_{\text{dS}}(x, x'), \quad (4.0.56)$$

so that, for $x = (0, \theta, 0)$ and $x' = (0, 0, 0)$, we have $D_{\text{dS}}(x, x') = \theta$.

The bra state $\langle \Psi_{\text{dS,W}}^{\Delta_{\mp}}(x') |$ is defined by analytic continuation from the AdS case as

$$\langle \Psi_{\text{dS,W}}^{\Delta_{\pm}}(x') | = \langle \Delta_{\pm} | \left(\sum_{k=0}^{\infty} c_k(\Delta_{\pm}) L_1^k \tilde{L}_1^k \right) e^{-i\theta J_1} e^{i\phi J_3} e^{-(L_0 + \tilde{L}_0)t}, \quad (4.0.57)$$

and we note that the coefficients $c_k(\Delta_{\pm})$ in (4.0.48) become complex for (4.0.53) and satisfy $c_k^*(\Delta_{\pm}) = c_k(\Delta_{\mp})$. This choice ensures that the inner product reproduces the Wick-rotated AdS_3 result:

$$\langle \Psi(t, \theta) | \Psi(0, 0) \rangle = \frac{1}{(\cos \theta)^\Delta} \sum_{k=0}^{\infty} e^{-(2k+\Delta)t} P_k^{(0, \Delta-1)} \left(\frac{2}{\cos^2 \theta} - 1 \right). \quad (4.0.58)$$

It is not clear whether this bra coincides with the one obtained by applying the nonstandard dS_3 conjugation (4.0.39) to the corresponding ket. As we will show shortly, that conjugation maps $|\Psi_{\text{dS,W}}^{\Delta_{\pm}}(x)\rangle$ not to $\langle \Psi_{\text{dS,W}}^{\Delta_{\pm}}(x') |$, but instead to $\langle \Psi_{\text{dS,W}}^{\Delta_{\mp}}(x'_A) |$ up to an overall constant.

4.0.9 Antipodal map

In de Sitter space, every point has an antipode because the space is compact. We can describe the antipodal map $x \rightarrow x_A$ (4.0.41) by the action $(t, \theta, \phi) \rightarrow (t, \theta + \pi, \phi)^2$. The bulk local state at $t = \theta = 0$ (4.0.48) transforms as

$$|\Psi_{\Delta_{\pm}}\rangle \rightarrow e^{i\pi J_1} |\Psi_{\Delta_{\pm}}\rangle. \quad (4.0.59)$$

Since this map $e^{i\pi J_1}$ acts on the Virasoro generators as

$$\begin{aligned} e^{i\pi J_1} L_0 e^{-i\pi J_1} &= -L_0, \\ e^{i\pi J_1} L_{\pm 1} e^{-i\pi J_1} &= -L_{\mp 1}, \\ e^{i\pi J_3} L_{\pm 1} e^{-i\pi J_3} &= -L_{\pm 1}, \end{aligned} \quad (4.0.60)$$

it follows that the state obtained by the antipodal action still satisfies the constraints (4.0.46). Hence the antipodal state is proportional to the original state:

$$|\Psi_{\text{dS,W}}^{\Delta_{\pm}}(t, \theta, \phi)\rangle \propto |\Psi_{\text{dS,W}}^{\Delta_{\pm}}(t, \theta + \pi, \phi)\rangle. \quad (4.0.61)$$

²In embedding coordinates, the antipodal map acts as $X_i \rightarrow -X_i$.

At the point $t = \theta = \phi = 0$, we can therefore write

$$e^{i\pi J_1} |\Psi_{\Delta_{\pm}}\rangle = \lambda_{\pm} |\Psi_{\Delta_{\pm}}\rangle, \quad (4.0.62)$$

where $\lambda_{\pm}$ are proportionality constants. Since (4.0.55) implies

$$\langle \Psi_{\text{dS,W}}^{\Delta_{\pm}}(0, \pi + \theta, 0) | \Psi_{\text{dS,W}}^{\Delta_{\pm}}(0, 0, 0) \rangle = -e^{\pm\pi\mu} \langle \Psi_{\text{dS,W}}^{\Delta_{\pm}}(0, \theta, 0) | \Psi_{\text{dS,W}}^{\Delta_{\pm}}(0, 0, 0) \rangle, \quad (4.0.63)$$

we obtain

$$\lambda_{\pm} = -e^{\pm\pi\mu}. \quad (4.0.64)$$

Using (4.0.46), we can further rewrite (4.0.62) as

$$e^{i\pi(L_1 - L_{-1})} |\Psi_{\Delta_{\pm}}\rangle = \lambda_{\pm} |\Psi_{\Delta_{\pm}}\rangle. \quad (4.0.65)$$

Because $e^{i\pi(L_1 - L_{-1})}$ commutes with L_0 and $L_{\pm 1}$, this identity is equivalent to the corresponding one for the primary state:

$$e^{i\pi(L_1 - L_{-1})} |\Delta_{\pm}\rangle = \lambda_{\pm} |\Delta_{\pm}\rangle. \quad (4.0.66)$$

Thus, we arrive at

$$\langle \Delta_{\pm} | e^{-i\pi(L_1 - L_{-1})} | \Delta_{\pm} \rangle = \langle \Delta_{\pm} | e^{-i\pi(\tilde{L}_1 - \tilde{L}_{-1})} | \Delta_{\pm} \rangle = -e^{\pm\pi\mu}. \quad (4.0.67)$$

Notation is summarized in section 4.

4.0.10 de Sitter conjugation

To clarify the definition of the bra states in (4.0.57), we first examine how states transform under the conjugation rule (4.0.39). In particular, the conjugates of the primary states $|\Delta_{\pm}\rangle$, which satisfy (4.0.18), obey

$$\begin{aligned} |\Delta_{\pm}\rangle^{\dagger} L_0^{\dagger} &= \frac{\Delta_{\mp}}{2} |\Delta_{\pm}\rangle^{\dagger}, & |\Delta_{\pm}\rangle^{\dagger} L_1^{\dagger} &= 0, \\ L_0^{\dagger} \langle \Delta_{\pm}|^{\dagger} &= \langle \Delta_{\pm}|^{\dagger} \frac{\Delta_{\mp}}{2}, & L_{-1}^{\dagger} \langle \Delta_{\pm}|^{\dagger} &= 0. \end{aligned} \quad (4.0.68)$$

The same identities hold for the anti-chiral sector. Accordingly, if we define

$$|\Delta_{\pm}\rangle^{\dagger} \equiv \widehat{\langle \Delta_{\pm}|}, \quad \langle \Delta_{\pm}|^{\dagger} = \widehat{|\Delta_{\pm}\rangle}, \quad (4.0.69)$$

then the following non-standard relations hold:

$$\begin{aligned}\langle \widehat{\Delta}_{\pm} | L_0 &= -\frac{\Delta_{\mp}}{2} \langle \widehat{\Delta}_{\pm} |, \quad \text{and} \quad \langle \widehat{\Delta}_{\pm} | L_1 = 0, \\ L_0 | \widehat{\Delta}_{\pm} \rangle &= -\frac{\Delta_{\mp}}{2} | \widehat{\Delta}_{\pm} \rangle, \quad \text{and} \quad L_{-1} | \widehat{\Delta}_{\pm} \rangle = 0.\end{aligned}\tag{4.0.70}$$

These relations differ markedly from the AdS case [27], and can be understood by treating the CFT as a non-Hermitian system. Within that viewpoint we adopt the normalization.

$$\begin{aligned}\langle \Delta_i | \Delta_j \rangle &= \langle \widehat{\Delta}_i | \widehat{\Delta}_j \rangle = \delta_{ij}, \\ \langle \Delta_i | \widehat{\Delta}_j \rangle &= \langle \widehat{\Delta}_i | \Delta_j \rangle = 0.\end{aligned}\tag{4.0.71}$$

With this normalization, the conjugates of the bulk-local states $|\Psi_{\text{dS},W}^{\Delta_{\pm}}(x)\rangle$ can be written as

$$\begin{aligned}|\Psi_{\text{dS},W}^{\Delta_{\pm}}(x)\rangle^{\dagger} &\equiv \widehat{\langle \Psi_{\text{dS},W}^{\Delta_{\pm}}(x) |} \\ &= \langle \widehat{\Delta}_{\pm} | \sum_k c_k(\Delta_{\mp}) L_{-1}^k \tilde{L}_{-1}^k e^{-i\theta J_1} e^{i\phi J_3} e^{-(L_0 + \tilde{L}_0)t},\end{aligned}\tag{4.0.72}$$

where in the first line we introduced the conjugated state $\widehat{\langle \Psi_{\text{dS},W}^{\Delta_{\mp}}(x) |}$. Next, insert $1 = e^{i\pi J_1} \cdot e^{-i\pi J_1}$ and simplify:

$$\begin{aligned}\widehat{\langle \Psi_{\text{dS},W}^{\Delta_{\pm}}(x) |} &= \langle \widehat{\Delta}_{\pm} | \sum_k c_k(\Delta_{\pm}) L_{-1}^k \tilde{L}_{-1}^k e^{i\pi J_1} \cdot e^{-i\pi J_1} e^{-i\theta J_1} e^{i\phi J_3} e^{-(L_0 + \tilde{L}_0)t} \\ &= \langle \widehat{\Delta}_{\pm} | e^{i\pi J_1} \sum_k c_k(\Delta_{\mp}) L_1^k \tilde{L}_1^k \cdot e^{-i(\theta+\pi)J_1} e^{i\phi J_3} e^{-(L_0 + \tilde{L}_0)t}.\end{aligned}\tag{4.0.73}$$

From (4.0.70) we further note

$$\begin{aligned}\langle \widehat{\Delta}_{\pm} | e^{i\pi J_1} L_0 &= \frac{\Delta_{\mp}}{2} \cdot \langle \widehat{\Delta}_{\pm} | e^{i\pi J_1}, \\ \langle \widehat{\Delta}_{\pm} | e^{i\pi J_1} L_{-1} &= 0,\end{aligned}\tag{4.0.74}$$

so $\langle \widehat{\Delta}_{\pm} | e^{i\pi J_1}$ behaves as a standard primary bra. Since we do not expect any primary other than $|\Delta_{\mp}\rangle$ to couple to the scalar, it must be proportional to $\langle \Delta_{\mp} |$:

$$\langle \widehat{\Delta}_{\pm} | e^{i\pi J_1} = \nu_{\mp} \langle \Delta_{\mp} |.\tag{4.0.75}$$

Taking conjugates and using (4.0.69) gives

$$e^{-i\pi J_1} | \Delta_{\pm} \rangle = \nu_{\mp}^* | \widehat{\Delta}_{\mp} \rangle.\tag{4.0.76}$$

Consequently,

$$\nu_{\pm} \cdot \langle \Delta_{\pm} | e^{-2i\pi J_1} | \Delta_{\pm} \rangle = \nu_{\mp}^*. \quad (4.0.77)$$

Using (4.0.67)—including both chiral and anti-chiral contributions—yields $\langle \Delta_{\pm} | e^{-2i\pi J_1} | \Delta_{\pm} \rangle = e^{\pm 2\pi\mu}$. A minimal choice solving (4.0.77) is

$$\nu_{\pm} = e^{\mp\pi\mu}. \quad (4.0.78)$$

Thus, we find that the conjugated states $|\Psi_{\pm}(x)\rangle^{\dagger}$ are obtained by the antipodal transform of the bra states in (4.0.57):

$$\begin{aligned} \langle \widehat{\Psi}_{\text{dS,W}}^{\Delta_{\pm}}(x) | &= \nu_{\mp} \langle \Delta_{\mp} | \sum_k c_k(\Delta_{\mp}) L_1^k \tilde{L}_1^k \cdot e^{-i(\theta+\pi)J_1} e^{i\phi J_3} e^{-(L_0+\tilde{L}_0)t} \\ &= \nu_{\mp} \langle \Psi_{\text{dS,W}}^{\Delta_{\mp}}(t, \theta + \pi, \phi) |. \end{aligned} \quad (4.0.79)$$

Note that $\langle \widehat{\Psi}_{\pm}(x) | \Psi_{\pm}(x) \rangle = 0$, whereas $\langle \widehat{\Psi}_{\mp}(x) | \Psi_{\pm}(x) \rangle \neq 0$. This follows from the normalization (4.0.71) and reflects the non-Hermitian character of the CFT dual to dS_3 .

Finally, the inner product between bulk-local states and their conjugates evaluates to

$$\langle \widehat{\Psi}_{\text{dS,W}}^{\Delta_{\mp}}(x) | \Psi_{\text{dS,W}}^{\Delta_{\pm}}(x') \rangle = \frac{e^{\mp\mu(\pi - D_{\text{dS}}(x_A, x'))}}{2i \sin D_{\text{dS}}(x_A, x')} = \frac{e^{\mp\mu D_{\text{dS}}(x, x')}}{2i \sin D_{\text{dS}}(x, x')}, \quad (4.0.80)$$

while $\langle \widehat{\Psi}_{\text{dS,W}}^{\Delta_{\pm}}(x) | \Psi_{\text{dS,W}}^{\Delta_{\pm}}(x') \rangle = 0$.

To obtain a nontrivial result, we must mix the two primaries Δ_+ and Δ_- . In the next subsection we reproduce the expected dS_3 two-point function by combining them and using (4.0.80). The correct Wightman function in the Euclidean (Bunch-Davies) vacuum [65, 77] is (3.2.5)

$$\begin{aligned} G_{\text{E}}(x, x') &= \frac{\Gamma[\Delta_+] \Gamma[\Delta_-]}{(4\pi)^{\frac{3}{2}} \Gamma[\frac{3}{2}]} {}_2F_1\left(\Delta_+, \Delta_-, \frac{3}{2}; \cos^2 \frac{D_{\text{dS}}}{2}\right) \\ &= \frac{\sinh \mu(\pi - D_{\text{dS}}(x, x'))}{4\pi \sinh \mu\pi \sin D_{\text{dS}}(x, x')}. \end{aligned} \quad (4.0.81)$$

This function diverges only when $x \rightarrow x'$, whereas the naive Wick-rotated AdS expression also diverges at the antipode $x' = x_A$.

4.0.11 Two-point functions in the dS Euclidean vacuum

To describe physical excitations in dS_3 , we take a linear combination of bulk-local states with dimensions Δ_+ and Δ_- .

First, recall the relation $\phi_{\text{AdS}} = \pm \sqrt{i} \phi_{\text{dS}}$ for the scalar fields. This follows from the kinetic term

$$S = R_{\text{AdS}} \int \partial^\mu \phi_{\text{AdS}} \partial_\mu \phi_{\text{AdS}} = R_{\text{dS}} \int \partial^\mu \phi_{\text{dS}} \partial_\mu \phi_{\text{dS}}, \quad (4.0.82)$$

together with $R_{\text{dS}}^2 = -R_{\text{AdS}}^2$. Motivated by this, we propose that the physical local excitation of the scalar in the Euclidean vacuum is dual to the CFT state

$$|\Psi_E(x)\rangle = \frac{1}{2\sqrt{\pi \sinh \pi \mu}} \left[\frac{1}{\sqrt{i}} |\Psi_{\text{dS},W}^{\Delta_+}(x)\rangle + \sqrt{i} |\Psi_{\text{dS},W}^{\Delta_-}(x_A)\rangle \right], \quad (4.0.83)$$

with the overall constant chosen to match the normalization of (4.0.81). See also [78–80] for closely related bulk-field combinations. This state obeys the scalar equation of motion:

$$\left[\frac{1}{2}(L_{-1}L_1 + L_1L_{-1}) - L_0^2 + \frac{1}{2}(\tilde{L}_{-1}\tilde{L}_1 + \tilde{L}_1\tilde{L}_{-1}) - \tilde{L}_0^2 - \frac{m^2}{2} \right] |\Psi_E(x)\rangle = 0. \quad (4.0.84)$$

Moreover, $|\Psi_E(x)\rangle$ is invariant under CPT, which maps $x \rightarrow x_A$ and exchanges $\Delta_\pm = 1 \pm i\mu$ with their complex conjugates $\Delta_\mp = 1 \mp i\mu$. Accordingly, CPT acts as $|\Psi_{\text{dS},W}^{\Delta_\pm}(x)\rangle \rightarrow |\Psi_{\text{dS},W}^{\Delta_\mp}(x_A)\rangle$. We also note that gauging CPT in de Sitter space seems to play an important role in recent work [22, 81, 82].

The conjugate is

$$\langle \Psi_E(x) | = \frac{1}{2\sqrt{\pi \sinh \pi \mu}} \left[\sqrt{i} \langle \widehat{\Psi_{\text{dS},W}^{\Delta_+}}(x) | + \frac{1}{\sqrt{i}} \langle \widehat{\Psi_{\text{dS},W}^{\Delta_-}}(x_A) | \right]. \quad (4.0.85)$$

Finally, the inner product of (4.0.83) and (4.0.85) reproduces the expected two-point function (4.0.81) or (3.2.5):

$$\begin{aligned} \langle \Psi_E(x) | \Psi_E(x') \rangle &= \frac{i}{4\pi \sinh \pi \mu} \left[\langle \widehat{\Psi_{\text{dS},W}^{\Delta_+}}(x) | \Psi_{\text{dS},W}^{\Delta_-}(x'_A) \rangle - \langle \widehat{\Psi_{\text{dS},W}^{\Delta_-}}(x_A) | \Psi_{\text{dS},W}^{\Delta_+}(x') \rangle \right] \\ &= \frac{i}{4\pi \sinh \pi \mu} \left[\frac{e^{\mu(\pi - D_{\text{dS}}(x, x'))}}{2i \sin D_{\text{dS}}(x, x')} - \frac{e^{-\mu(\pi - D_{\text{dS}}(x, x'))}}{2i \sin D_{\text{dS}}(x, x')} \right] \\ &= \frac{\sinh \mu (\pi - D_{\text{dS}}(x, x'))}{4\pi \sinh \mu \pi \sin D_{\text{dS}}(x, x')}. \end{aligned} \quad (4.0.86)$$

We close with a remark on real conformal dimensions. When the conformal weights are real, analytic continuation $i\mu \rightarrow \gamma$ yields the Euclidean-vacuum Wightman function

$$G_E(x', x) = \frac{\sin \gamma (\pi - D_{\text{dS}})}{4\pi \sin \gamma \pi \sin D_{\text{dS}}}, \quad (4.0.87)$$

with $\gamma = \sqrt{m^2 - 1} \in \mathbf{R}$. In the usual dS/CFT setup [18], real dimensions lead to fast and slow falloff modes, and—analogously to AdS/CFT—one might expect to keep only normalizable modes. However, (4.0.87) indicates that both modes are needed, due to the sine factor in the

numerator. Thus the same reasoning applies with $i\mu = \gamma$.

4.0.12 α -vacua

It is well known that the vacuum state invariant under the symmetries of de Sitter space is not unique. The symmetry-invariant vacua are parametrized by a real parameter α ³ and are referred to as the α -vacua. Here we present a bulk-local state that reproduces the Green's function of the α -vacua given in [64].

The bulk-local state in an α -vacuum is

$$|\Psi_\alpha(x)\rangle = \frac{1}{\sqrt{1 - e^{2\alpha}}} (|\Psi_E(x)\rangle + e^\alpha |\Psi_E(x_A)\rangle). \quad (4.0.88)$$

It is immediate that this state satisfies the symmetry constraints (4.0.46) and is therefore invariant under the de Sitter isometries. The corresponding two-point function is

$$\begin{aligned} & \langle \Psi_\alpha(x) | \Psi_\alpha(x') \rangle \\ &= \frac{1}{1 - e^{2\alpha}} \left(G_E(x, x') + e^\alpha G_E(x, x'_A) + e^\alpha G_E(x_A, x') + e^{2\alpha} G_E(x_A, x'_A) \right), \end{aligned} \quad (4.0.89)$$

which reproduces the result of [64] and (3.2.18).

4.0.13 dS₃ from the information metric

Finally we would like to examine how the holographic spacetime looks from the viewpoint of the bulk local state $|\Psi_E(x)\rangle$ as a probe. We will eventually find that the metric it probes is given by the dS₃ metric, which is a dS generalization of the result for AdS₃ in [27]. Even though this result naturally follows from the conformal symmetry of our CFT construction, we will briefly present this argument below as another supporting evidence of the emergence of dS₃ spacetime from the 2D CFT.

Since this state $|\Psi_E(x)\rangle$ is singular in that it is localized at a point x and has infinite energy and a divergent inner product (indeed the two-point function (4.0.86) becomes divergent when $x = x'$), we regularize the divergence by smearing the bulk local excitation around the point x over a region with a small size δ . By introducing such a short scale cutoff δ that preserves the conformal symmetry, we expect that the two-point function to have the following behavior in

³In general a complex value is allowed, but CPT invariance requires α to be real.

the limit $x \rightarrow x'^4$:

$$\langle \Psi_E(x) | \Psi_E(x') \rangle \simeq \frac{\delta}{\sqrt{D_{\text{dS}}(x, x')^2 + \delta^2}}, \quad (4.0.90)$$

Here we normalized the bulk local state so that we have $\langle \Psi_E(x) | \Psi_E(x) \rangle = 1$.

The information metric G_{ij} (or Bures distance) is defined by

$$G_{ij} dx^i dx^j = 1 - |\langle \Psi_E(x) | \Psi_E(x + dx) \rangle|. \quad (4.0.91)$$

Since we have

$$D_{\text{dS}}^2(x, x + dx) \simeq -\cos^2 \theta dt^2 + d\theta^2 + \sin^2 \theta d\phi^2, \quad (4.0.92)$$

we obtain the information metric

$$ds_{\text{inf}}^2 = \frac{1}{2\delta^2} \left(-\cos^2 \theta dt^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right). \quad (4.0.93)$$

In this way we find that the information metric is proportional to that of dS_3 . Notice that we successfully obtain a timelike metric for the t -coordinate. The crucial reason why we find the timelike coordinate from bulk local states is because of the unusual conjugation operation (4.0.39) that flips the sign of energy $L_0 + \tilde{L}_0$.

Quantum estimation theory tells us that the inverse of information metric G provides the lower bound of quantum uncertainty of the coordinates x such that $\langle \delta x_i \delta x_j \rangle \geq (G^{-1})_{ij}$, when we try to estimate their values from quantum measurements. Refer to [83] for more details of the application of quantum estimation theory to AdS/CFT. This means that the UV cutoff scale δ describes the minimal distance where we can trust the dS_3 metric (4.0.93). For scales smaller than δ , the uncertainty washes out the classical spacetime. If we assume that the estimation done for AdS/CFT in [27] can also be applied to our dS_3/CFT_2 , it is natural to identify δ as $\frac{1}{\delta} \sim \frac{R_{\text{dS}}}{G_{\text{N}}}$, where R_{dS} is the dS_3 radius and G_{N} is the three-dimensional Newton constant. This might reproduce the expected dS_3 metric up to an $\mathcal{O}(1)$ factor.

⁴Though we do not explicitly construct the regularized state, it is well approximated by the state $e^{-\delta(L_0 + \tilde{L}_0)} |\Psi_E(x)\rangle$. Indeed this was used in [27] to reproduce the AdS_3 metric. The same argument can be applicable to the current dS_3 case. We expect that the localization of this state is smeared around the region with size δ .

Chapter 5

Static Patch Holography

Motivated by the AdS/CFT correspondence, it is natural to seek a holographic framework in which the boundary theory retains a Lorentzian time direction—rather than Strominger’s dS/CFT with a Euclidean boundary. Several proposals with timelike holographic boundaries have been advanced. In this subsection we review the “static patch holography” programs of [66,67,84,85] and clarify why the cosmological horizon emerges as the preferred holographic screen.

Bousso’s covariant entropy bound

As a prelude, we briefly review Bousso’s covariant entropy bound [25], later proven in [86,87] (see also [88] for a comprehensive account). This proposal serves as a covariant refinement of the holographic principle originally put forward in [6,7]. It constrains how many holographic degrees of freedom can be encoded on a horizon that encloses a given gravitational region.

Consider a d -dimensional Lorentzian manifold $\mathcal{M}$ and a spacelike codimension-two surface $\mathcal{B}$ with area $A(\mathcal{B})$. As sketched in figure 5.0.1, there are four congruences of null geodesics orthogonal to $\mathcal{B}$: future- and past-directed, each split into ingoing and outgoing. To distinguish “ingoing” from “outgoing,” take a null congruence with affine parameter λ . An infinitesimal shift $d\lambda$ moves $\mathcal{B}$ to $\mathcal{B}(\lambda + d\lambda)$. If the area decreases, $A(\mathcal{B}(\lambda + d\lambda)) \leq A(\mathcal{B}(\lambda))$, we call the direction ingoing; otherwise, outgoing. Equivalently, define the expansion

$$\theta(\lambda) = \frac{1}{A(\mathcal{B})} \frac{dA(\mathcal{B})}{d\lambda}. \quad (5.0.1)$$

A null congruence is ingoing when $\theta \leq 0$. At least one of the future-directed congruences satisfies this inequality.

By the Raychaudhuri equation, if $\theta \leq 0$ initially and suitable energy conditions hold, the expansion decreases monotonically along the congruence. Under the null energy condition

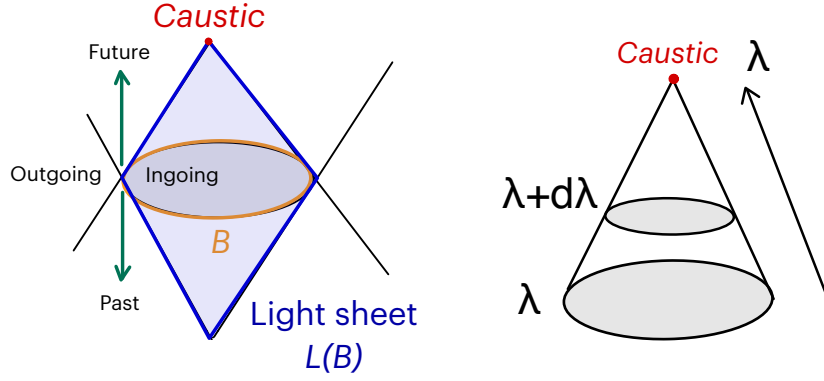


Figure 5.0.1: Left: the light sheet $L(\mathcal{B})$ (purple) associated with a spacelike codimension-two surface $\mathcal{B}$. Orthogonal null geodesics split into future- (past-) directed ingoing or outgoing congruences; the light sheet uses the ingoing ones. Right: evolution of a null congruence under an infinitesimal change of the affine parameter λ . This figure is cited from [2].

(NEC), one obtains

$$\dot{\theta} \leq -\frac{\theta^2}{2}. \quad (5.0.2)$$

Thus a caustic inevitably forms at finite affine parameter, where the null generators focus and intersect. We define the codimension-one null hypersurface with $\theta \leq 0$ as the light sheet of $\mathcal{B}$, denoted $L(\mathcal{B})$. The covariant entropy bound asserts

$$S[L(\mathcal{B})] \leq \frac{A(\mathcal{B})}{4G}, \quad (5.0.3)$$

where $S[L(\mathcal{B})]$ is the entropy of matter fields crossing the light sheet; in cosmological settings this is obtained by integrating the local entropy density along $L(\mathcal{B})$.

The bound has been checked in general Friedmann-Robertson-Walker universes with equation-of-state parameter $-1 \leq w \leq 1$, and a general proof was later furnished in [86, 87] using modular-Hamiltonian methods.

Static patch holography: Susskind's suggestion

Motivated by the holographic principle and Bousso's covariant entropy bound, Susskind advanced a program of static-patch holography [67, 84, 85]. In static coordinates the de Sitter geometry reads

$$ds^2 = -\left(1 - \frac{r^2}{L^2}\right) dt^2 + \frac{dr^2}{1 - \frac{r^2}{L^2}} + r^2 d\Omega_{d-2}^2, \quad (5.0.4)$$

where L denotes the de Sitter curvature radius. These coordinates cover a single quadrant of the global de Sitter manifold. The point $r = 0$ represents the north pole, and the region $0 < r < L$ is precisely the static patch accessible to an observer located there. The surface at $r = L$ is the cosmological horizon; beyond it ($r > L$), an observer at either pole can neither

send nor receive signals. In contrast to the eternal AdS black hole, de Sitter space is compact and lacks a spatially asymptotic boundary. The (connected) isometry group of global dS_{d+1} is $SO(1, d+1)$. Owing to the presence of the cosmological horizon, the symmetry preserved within the static patch is reduced to $SO(d) \times \mathbb{R}$, generated by spatial rotations together with time translations along the Killing vector $\xi^\mu = \partial_t$.¹ Bousso's entropy bound suggests that the holographic screen should be placed so that its area suffices to encode all information contained in the spacetime region of interest. To illustrate, first consider AdS. If the screen were located near a black hole horizon, the boundary degrees of freedom would capture only the physics behind the horizon, missing information from the asymptotic region. This motivates the standard AdS/CFT setup, where the holographic screen sits near the asymptotic boundary and encodes the full set of bulk degrees of freedom.

Turning to de Sitter space, if the screen is placed near the north or south pole, it captures only the degrees of freedom in the small region between the pole and the screen. In contrast, placing the screen at the cosmological horizon ensures access to all degrees of freedom within the entire static patch. Thus, the cosmological horizon naturally emerges as the appropriate location for the holographic screen in de Sitter space. On this basis, Susskind proposed the framework of static patch holography, which may be summarized as follows:

Quantum gravity in the static patch is dual to a quantum mechanical system defined on the cosmological horizon.

To regulate the physics near the horizon, which is a null surface, it is natural to define the boundary theory on a stretched horizon located at $r = L - \epsilon < L$.

5.1 Toward consistent definition of entanglement entropy in dS

5.1.1 Holographic dictionary of correlation functions

Before turning to the subtleties of de Sitter holography, we first recapitulate the AdS/CFT correspondence, which provides the canonical template for holographic duality. Lessons distilled from AdS/CFT will serve as guidance in formulating a prospective holographic dictionary for de Sitter space.

The AdS/CFT correspondence [8, 19, 20] asserts an equivalence between a d -dimensional CFT and a $(d+1)$ -dimensional gravitational theory on AdS. A core statement is the GKPW prescription [19, 20], equating the AdS bulk partition function with the CFT generating

¹Introducing a spatial boundary at a stretched horizon $r = r_*$ inside the static patch does not modify this reduced symmetry; the preserved isometries remain $SO(d) \times \mathbb{R}$.

functional:

$$Z_{\text{AdS}}[\phi_0] = \left\langle e^{\int_{\partial\text{AdS}} \phi_0(x) \mathcal{O}(x)} \right\rangle_{\text{CFT}}, \quad (5.1.1)$$

where $\phi_0(x)$ denotes the boundary value of a bulk field $\phi(x, z)$ and $\mathcal{O}(x)$ is the dual operator. Differentiating functionally with respect to $\phi_0(x)$ produces the connected correlators of $\mathcal{O}(x)$:

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \cdots \mathcal{O}(x_n) \rangle_{\text{CFT},c} = \frac{\delta^n \ln Z_{\text{AdS}}[\phi_0]}{\delta \phi_0(x_1) \delta \phi_0(x_2) \cdots \delta \phi_0(x_n)} \Big|_{\phi_0=0}. \quad (5.1.2)$$

An equivalent formulation, the extrapolate dictionary [89–91], identifies near-boundary AdS correlators with CFT correlators:

$$\lim_{r \rightarrow \infty} r^{n\Delta} \langle \phi(r, x_1) \phi(r, x_2) \cdots \phi(r, x_n) \rangle_{\text{AdS}} = \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \cdots \mathcal{O}(x_n) \rangle_{\text{CFT}}, \quad (5.1.3)$$

with r the AdS radial coordinate in the global metric $ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega^2$. The equivalence between this extrapolate dictionary and GKPW was established in [92]. For instance, at the level of two-point functions, the CFT correlator is reproduced by the boundary-to-boundary propagator of a bulk scalar, as follows from the near-boundary expansion

$$\phi(x, z \sim 0) = \phi_0(x) z^{d-\Delta} + \tilde{\phi}(x) z^\Delta, \quad (5.1.4)$$

with z the Poincaré radial coordinate. In particular, up to normalization,

$$G_{\partial\partial}(x_1, x_2) = \langle \phi(x_1, r_1 \rightarrow \infty) \phi(x_2, r_2 \rightarrow \infty) \rangle_{\text{AdS}} \sim \langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) \rangle_{\text{CFT}}. \quad (5.1.5)$$

In contrast to the well-tested AdS/CFT dictionary, a universally accepted holographic dictionary for static patch holography in de Sitter space has yet to emerge. Nevertheless, it is natural to expect that the principles underpinning AdS/CFT retain relevance. One appealing possibility is a GKPW-like relation,

$$Z_{\mathcal{O}}[\Phi_0]_{\text{bdy}} = Z_\Phi[\Phi_0]_{\text{dS}}. \quad (5.1.6)$$

Here $Z_{\mathcal{O}}$ is the boundary partition function with source Φ_0 , and Z_Φ is the corresponding bulk partition function in de Sitter; the boundary theory is taken to live on the stretched horizon. While such a relation would furnish a direct bulk-boundary dictionary, we will not assume it in this work, in view of the absence of a rigorous derivation.

Instead, guided by the extrapolate dictionary, we adopt a pragmatic working hypothesis: boundary correlators in static patch holography are read off directly from de Sitter bulk correlators. This stance aligns with our primary goal—an analysis of holographic entanglement

entropy within the static patch. Concretely, we posit

$$\boxed{\text{Conjectured Holographic Dictionary: } \langle O(x_1)O(x_2) \rangle_{\text{bdy}} \sim \lim_{r \rightarrow r_*} \langle \Phi(X_1)\Phi(X_2) \rangle_{\text{dS}} .} \quad (5.1.7)$$

Here r_* denotes the stretched-horizon radius in the static patch, and $\sim$ indicates equality up to an overall normalization. The relative factor between bulk and boundary correlators will be fixed by a normalization condition imposed later (we choose it so that the entanglement entropy vanishes at the cutoff scale). This bulk-to-boundary relation supplies a minimal operational framework for static patch holography and the input needed for our construction of holographic entanglement entropy.

5.1.2 Green's functions in de Sitter space

Below, we use the following linear combination of the two branches:

$$\boxed{G_{\text{dS}}(x, x') = N_G \cdot G_G(x, x') + N_A \cdot G_A(x, x') ,} \quad (5.1.8)$$

with arbitrary constants N_G, N_A . $G_G(x, x')$ is the Green's function in the Euclidean vacuum and $G_A(x, x')$ is its antipodal cousin. The Bunch-Davies and antipodal choices correspond to $N_A \rightarrow 0$ and $N_G \rightarrow 0$, respectively. Within static patch holography such linear combinations naturally emerge from boundary conditions (Dirichlet/Neumann/Robin) imposed on bulk fields at the stretched horizon.

5.2 Constraints of holographic entanglement entropy in dS space

Within AdS/CFT, entanglement entropy is captured geometrically by the Ryu-Takayanagi (RT) prescription, and on the CFT side the same quantity follows from the twist-operator formalism. A direct transplant of RT to de Sitter, however, is inconsistent: the resulting quantity fails to satisfy basic entropic inequalities. In view of this issue—and the lack of a settled holographic dictionary for dS—we take a different route. We define holographic entanglement entropy in de Sitter via the replica trick and twist operators, and recast the answer in terms of bulk Green's functions. This construction furnishes a de Sitter candidate for holographic entanglement entropy that is distinct from the RT formula.

5.2.1 Holographic entanglement entropy in the static patch

We assume throughout that the putative holographic dual living on the timelike screen in de Sitter has a positive-definite inner product and yields bona fide (non-negative) density

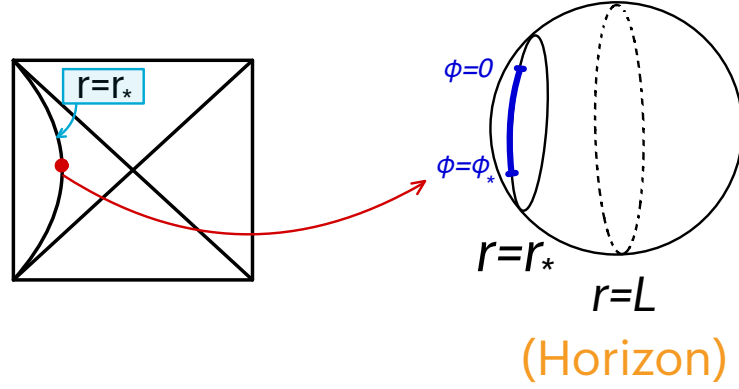


Figure 5.2.1: A spatial interval A (blue) on the stretched horizon at $r = r_*$ within a constant-time slice of the static patch. This figure is cited from [2].

matrices. For a spatial subregion A on the holographic screen, we insert twist operators at the endpoints of A and implement the replica construction as in the previous section. This motivates the proposal that the entanglement entropy of A is encoded by the logarithm of a two-point function of twist operators:

$$S_{\text{EE}} = -C \cdot \log G_{\text{dS}}(x, x') . \quad (5.2.1)$$

Here x and x' label the endpoints of A , and C is a dimensionless coefficient determined by the $n \rightarrow 1$ replica limit. Because an explicit twist operator in static patch holography is not at our disposal, we treat C as a positive but otherwise undetermined prefactor; the overall minus sign reflects the standard continuation from $n > 1$ to $n = 1$. It is natural to model the twist operator by a scalar primary, so its two-point function is taken to be the Green's function of a massive scalar of mass m in de Sitter space. Crucially, we will not attempt to match m to a specific operator dimension in the boundary theory, nor will we fix C a priori. Instead, we analyze the most general form of (5.2.1) and test consistency with basic entropic constraints. Since the boundary QFT is defined at a finite cutoff inside de Sitter, we assume the twist-operator formalism remains applicable even in the presence of mild nonlocality [3]. Any overall normalization leaves entropic inequalities unchanged, so S_{EE} may be rescaled by a constant without affecting our conclusions. Moreover, because the construction depends only on de Sitter geodesic distance, it is manifestly covariant.²

For definiteness we specialize to dS_3 , whose static patch is

$$ds^2 = - \left(1 - \frac{r^2}{L^2} \right) dt^2 + \frac{dr^2}{1 - \frac{r^2}{L^2}} + r^2 d\phi^2 , \quad (5.2.2)$$

²In higher dimensions, one expects genuinely surface-like data to enter, rather than curve data alone—precisely as in the RT setting.

with coordinate ranges

$$0 \leq r \leq L, \quad -\infty < t < \infty, \quad 0 \leq \phi \leq 2\pi. \quad (5.2.3)$$

An explicit embedding into Minkowski space is

$$\begin{aligned} X_0 &= \sqrt{L^2 - r^2} \sinh\left(\frac{t}{L}\right), & X_2 &= r \cos \phi, \\ X_1 &= \sqrt{L^2 - r^2} \cosh\left(\frac{t}{L}\right), & X_3 &= r \sin \phi, \end{aligned} \quad (5.2.4)$$

subject to

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 = L^2, \quad ds^2 = -dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2. \quad (5.2.5)$$

From this, the invariant geodesic separation $\mathcal{D}$ for two points in the same static patch follows as

$$\cos(\mathcal{D}(x, x')) = \sqrt{\left(1 - \frac{r^2}{L^2}\right) \left(1 - \frac{r'^2}{L^2}\right)} \cosh\left(\frac{t - t'}{L}\right) + \frac{rr'}{L^2} \cos(\phi - \phi'). \quad (5.2.6)$$

In dS₃ there are two independent scalar Green's-function branches,

$$G_G(x, x') = \frac{\sinh \mu (\pi - \mathcal{D})}{4\pi \sinh(\mu\pi) \sin \mathcal{D}}, \quad G_A(x, x') = \frac{\sinh(\mu\mathcal{D})}{4\pi \sinh(\mu\pi) \sin \mathcal{D}}, \quad (5.2.7)$$

with singular behavior at coincidence $\mathcal{D} = 0$ and at antipodal separation $\mathcal{D} = \pi$, respectively. Consequently, the most general scalar Green's function in dS₃ is the linear combination

$$\boxed{G_{\text{dS}}(x, x') = N_G \cdot \frac{\sinh \mu (\pi - \mathcal{D})}{4\pi \sinh \mu\pi \sin \mathcal{D}} + N_A \cdot \frac{\sinh(\mu\mathcal{D})}{4\pi \sinh \mu\pi \sin \mathcal{D}},} \quad (5.2.8)$$

with constants N_G, N_A . In what follows we focus on heavy scalars with $mL \geq 1$, so that $\mu = \sqrt{m^2 L^2 - 1} \in \mathbf{R}^+$. The light-field regime $mL \leq 1$ is obtained by the analytic continuation $\mu \rightarrow i\nu$ and will be discussed subsequently.

Although the cosmological horizon at $r = L$ is the most natural location for the holographic screen, it is convenient to regulate this null surface by introducing a stretched horizon inside the static patch, e.g.

$$r = r_* \in (0, L). \quad (5.2.9)$$

For later use we define a dimensionless parameter α by

$$\sin \alpha \equiv \frac{r_*}{L}, \quad \text{with} \quad \alpha \in \left(0, \frac{\pi}{2}\right), \quad (5.2.10)$$

which streamlines many expressions below. The induced metric on this screen is simply flat:

$$ds^2|_{\text{bdy}} = -\left(1 - \frac{r_*^2}{L^2}\right) dt^2 + r_*^2 d\phi^2 = -dT^2 + dX^2, \quad (5.2.11)$$

where we have introduced Minkowski coordinates

$$T = t \sqrt{1 - \frac{r_*^2}{L^2}}, \quad X = r_* \phi. \quad (5.2.12)$$

Consider now an arbitrary interval on the stretched horizon with endpoints $x = (t_1, r_*, \phi_1)$ and $x' = (t_2, r_*, \phi_2)$. The bulk geodesic length between these points reduces to

$$\mathcal{D} = \arccos \left(\sin^2 \alpha \cos(\Delta\phi) + \cos^2 \alpha \cosh(\Delta\tilde{t}) \right), \quad (5.2.13)$$

where $\Delta\phi = \phi_1 - \phi_2$ and $\Delta\tilde{t} = \frac{t_1 - t_2}{L}$. Note that the bulk geodesic distance is not generally invariant under boundary boosts of the flat metric (5.2.11), whose invariant interval is

$$\Delta L = \sqrt{\Delta X^2 - \Delta T^2} = L \sqrt{\sin^2 \alpha (\Delta\phi)^2 - \cos^2 \alpha (\Delta\tilde{t})^2}. \quad (5.2.14)$$

The mismatch stems from the periodicity of the boundary circle, which obstructs global Lorentz boosts. In the short-distance regime $\Delta\phi, \Delta t \rightarrow 0$, the two notions coincide:

$$D \stackrel{\Delta t, \Delta\phi \rightarrow 0}{\approx} L \sqrt{\sin^2 \alpha (\Delta\phi)^2 - \cos^2 \alpha (\Delta\tilde{t})^2} + \mathcal{O}(\Delta\phi^3, \Delta t^3) \approx \Delta L. \quad (5.2.15)$$

There are two bulk geodesics connecting the same endpoints; we choose the one with smaller length, i.e. the branch with $\mathcal{D} \geq \pi$. Consequently, such spacelike geodesics whose endpoints both lie in the static patch are free of conjugate points and are globally minimizing among spacelike curves with the same anchors. The allowed range is

$$0 \leq \mathcal{D} \leq 2\alpha. \quad (5.2.16)$$

The lower endpoint $\mathcal{D} = 0$ corresponds to a bulk null separation with

$$\Delta\tilde{t} \leq \Delta\tilde{t}|_{\text{bulk-null}} = \text{arccosh} \left(\frac{1 - \sin^2 \alpha \cos(\Delta\phi)}{\cos^2 \alpha} \right). \quad (5.2.17)$$

For fixed angular size $\Delta\phi$, a spacelike geodesic exists only prior to this critical time. The upper endpoint $\mathcal{D} = 2\alpha$ is reached at

$$\Delta\phi = \pi, \quad \Delta t = 0. \quad (5.2.18)$$

With these ingredients, one evaluates the entanglement entropy of a chosen interval A by inserting the geodesic length above into the bulk Green's functions (5.2.7) and then into the definition (5.2.1). Once the constants C and the mass parameter μ are specified, the result follows directly.

large mass limit and RT formula

We first explain why we adopt the generic two-point function (5.2.1) as our candidate for holographic entanglement entropy in de Sitter space, rather than the standard Ryu-Takayanagi (RT) construction. In fact, one can check that (5.2.1) reproduces the RT result in the large-dimension limit $\mu \rightarrow \infty$, in close analogy with AdS/CFT.³ To make this precise we restrict to the Bunch-Davies branch by setting

$$N_A = 0, \quad \text{for Bunch-Davies vacuum.} \quad (5.2.19)$$

Sending the twist-operator “dimension” large, i.e. $\mu \rightarrow \infty$,⁴ the Green's function is controlled by

$$G_{\text{ds}}(x, x') \Big|_{N_A \rightarrow 0, \mu \rightarrow \infty} \sim e^{-\mu \mathcal{D}}, \quad (5.2.20)$$

mirroring the geodesic dominance in holographic correlators on AdS [94–97]. If, in addition, the $n \rightarrow 1$ replica limit is taken such that

$$\lim_{n \rightarrow 1} (C \times \mu) = \frac{L}{4G_N}, \quad (5.2.21)$$

then (5.2.1) yields the de Sitter analogue of RT,

$$S_A = -C \cdot \log G_{\text{ds}}(x, x') \sim \frac{D_A}{4G_N} + \dots. \quad (5.2.22)$$

Here D_A stands for the length of the bulk geodesic anchored to the endpoints of the interval A , playing the role of an extremal surface in dS₃.

A serious obstacle is that the de Sitter RT expression fails basic entropic constraints. In particular, geodesic length in dS violates strong subadditivity (SSA),

$$S(AB) + S(BC) - S(B) - S(ABC) \geq 0, \quad (5.2.23)$$

originally proven in [32, 33]. Since subadditivity and SSA hold for any quantum state (positive semidefinite density operator on a Hilbert space with positive-definite inner product), the naive formula (5.2.22) cannot represent the entanglement entropy of a physical density matrix.

³For a discussion of geodesic approximations in de Sitter, see [93].

⁴In AdS/CFT an analogous limit appears because the twist dimension scales as $\Delta_n \sim c \sim 1/G_N \rightarrow \infty$ in the classical (large-central-charge) regime.

More generally, the violation already appears for the global branch $G_G(x, x')$ in (5.2.7) at any finite μ (Bunch-Davies vacuum). A natural interpretation is that the would-be boundary state on a finite stretched horizon is not the global Euclidean vacuum, and the discrepancy manifests as an SSA violation when one tries to use the purely global branch.

These considerations suggest that RT is not a consistent definition of holographic entanglement entropy in de Sitter space. They also point toward including the antipodal branch in the Green's function. Motivated by this, we now analyze (5.2.1) using the most general de Sitter two-point function (5.2.7), leaving N_G and N_A free. By enforcing basic entropic requirements—non-negativity, subadditivity, and strong subadditivity—we will delineate the allowed region in the (N_G, N_A) parameter space that yields a consistent notion of holographic entanglement entropy in the static patch.

5.2.2 Non-negativity of entanglement entropy

A basic requirement of entanglement entropy in any quantum system is that it never becomes negative. For a state with density matrix ρ_A on a subsystem A , the von Neumann entropy is

$$S_{\text{EE}} = -\text{Tr}(\rho_A \log \rho_A) . \quad (5.2.24)$$

This quantity measures how strongly A is entangled with its complement. Since ρ_A is positive semidefinite with unit trace, its eigenvalues can be interpreted as probabilities, and therefore

$$S_{\text{EE}} \geq 0 , \quad (5.2.25)$$

with equality precisely when ρ_A is pure. This non-negativity holds irrespective of dimension or microscopic dynamics.

Any holographic definition intended to reproduce field-theory entanglement must respect this property. In the static-patch setup, we proposed in (5.2.1) to compute the entropy via the bulk two-point function as $S_{\text{EE}} = -C \log G_{\text{dS}}(x, x')$. For S_{EE} to be non-negative, the correlator must obey

$$G_{\text{dS}}(x, x') \leq 1. \quad (5.2.26)$$

To test this, let us use the explicit form

$$G_{\text{dS}}(x, x') = N_G \cdot \frac{\sinh \mu (\pi - \mathcal{D})}{4\pi \sinh(\mu\pi) \sin \mathcal{D}} + N_A \cdot \frac{\sinh(\mu\mathcal{D})}{4\pi \sinh(\mu\pi) \sin \mathcal{D}} . \quad (5.2.27)$$

Evidently, $G_{\text{dS}}(x, x')$ is unbounded: it diverges at coincident and antipodal separations. This reflects the universal short-distance singularity at $\mathcal{D} \rightarrow 0$, familiar from flat space, together

with an antipodal singularity at $\mathcal{D} \rightarrow \pi$:

$$\begin{aligned} \lim_{\mathcal{D} \rightarrow 0} G_{\text{dS}}(x, x') &\sim \frac{N_G}{4\pi\mathcal{D}} \longrightarrow \infty, \\ \lim_{\mathcal{D} \rightarrow \pi} G_{\text{dS}}(x, x') &\sim \frac{N_A}{4\pi(\pi - \mathcal{D})} \longrightarrow \infty \end{aligned} \quad (5.2.28)$$

Hence for sufficiently small (or suitably chosen large) intervals one can have $G_{\text{dS}}(x, x') > 1$, making S_{EE} negative if left unregulated.

This mirrors the situation in AdS, where bulk-to-bulk propagators also blow up at short distances. In $(d+1)$ -dimensional AdS, a scalar bulk propagator may be written as

$$G_{\text{AdS}}^\Delta(x; x') = G_{\text{AdS}}^\Delta(\xi) = \frac{C_\Delta}{2^\Delta(2\Delta - d)} \xi^\Delta \cdot {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta - \frac{d}{2} + 1; \xi^2\right), \quad (5.2.29)$$

in terms of the chordal distance ξ related to the (dimensionless) geodesic length $\mathcal{D}(x, x')$ by

$$\mathcal{D}(x; x') = \ln\left(\frac{1 + \sqrt{1 - \xi^2}}{\xi}\right), \quad \xi = \frac{1}{\cosh(\mathcal{D}(x; x'))}. \quad (5.2.30)$$

Its short-distance behavior ($\mathcal{D} \rightarrow 0$, i.e. $\xi \rightarrow 1$) is

$$\lim_{\mathcal{D} \rightarrow 0} G_{\text{AdS}}^\Delta(x; x') \approx \frac{1}{4\pi} \left(\frac{1}{\pi\mathcal{D}^2}\right)^{\frac{d-1}{2}} \Gamma\left(\frac{d-1}{2}\right), \quad (5.2.31)$$

matching flat/dS behavior. In AdS/CFT, this is handled by a UV cutoff and an appropriate normalization of correlators. Consequently, the entropy of an interval at the cutoff scale can be set to vanish.⁵

By analogy, we introduce a bulk cutoff $\mathcal{D}_\epsilon$ in de Sitter. While the detailed geometric meaning of $\mathcal{D}_\epsilon$ in dS is not yet fixed, it is required both by the universal UV divergence and by entropy positivity. We then impose

$$S_{\text{EE}}|_{\mathcal{D}=\mathcal{D}_\epsilon} = 0, \quad (5.2.33)$$

so that intervals at the cutoff (or the entire boundary) are pure. Equivalently, this fixes the

⁵As a concrete illustration in CFT_2 on a circle, an interval of size ΔL has

$$S_{\text{EE}} = \frac{c}{3} \log\left(\frac{L_{\text{bdy}}}{\pi\epsilon} \sin\left(\frac{\pi\Delta L}{L_{\text{bdy}}}\right)\right), \quad (5.2.32)$$

with circumference L_{bdy} . Fixing the leading cutoff by $S_{\text{EE}}(\Delta L = \epsilon) = 0$ sets the normalization.

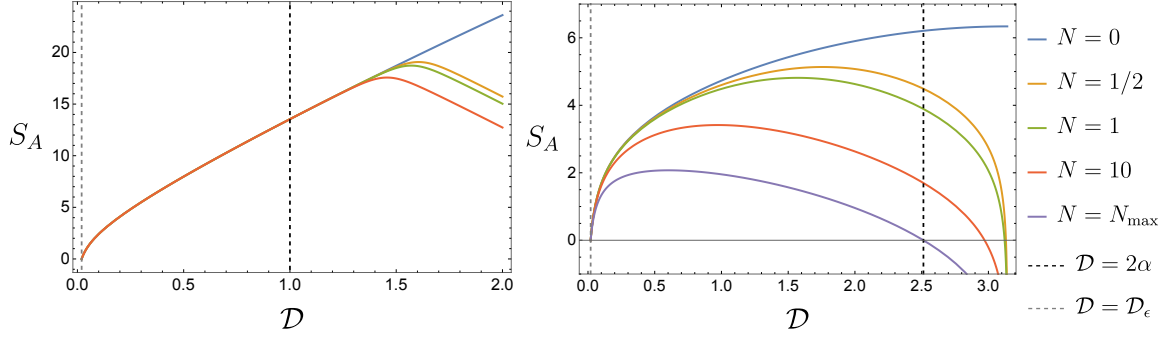


Figure 5.2.2: Entropy S_A (with $C = 1$) versus the de Sitter geodesic length in dS_3 . For a screen at $r_*/L = \sin \alpha$, the spacelike geodesic length available on the stretched horizon lies in $\mathcal{D} \in (\mathcal{D}_\epsilon, 2\alpha)$. Parameters: $\alpha = 1/2$, $\mathcal{D}_\epsilon = 1/50$. Left: large-dimension limit $\mu = 10$. Right: $\mu = 1$. This graph is cited from [2].

normalization of G_{dS} :

$$N_G = \frac{4\pi \sinh(\mu\pi) \sin \mathcal{D}_\epsilon}{\sinh(\mu(\pi - \mathcal{D}_\epsilon)) + N \sinh(\mu\mathcal{D}_\epsilon)} \approx 4\pi\mathcal{D}_\epsilon, \quad (5.2.34)$$

$$N_A = N \times \frac{4\pi \sinh(\mu\pi) \sin \mathcal{D}_\epsilon}{\sinh(\mu(\pi - \mathcal{D}_\epsilon)) + N \sinh(\mu\mathcal{D}_\epsilon)},$$

with the ratio $N_A/N_E = N$ held fixed; N quantifies deviations from the Bunch-Davies branch and reduces one free parameter in (5.2.1). The bulk cutoff also induces a boundary cutoff. Since the stretched-horizon geodesic distance satisfies

$$\cos \mathcal{D} = \cos^2 \alpha \cosh\left(\frac{\Delta t}{L}\right) + \sin^2 \alpha \cos \Delta\phi, \quad (5.2.35)$$

the limit $\mathcal{D}_\epsilon \rightarrow 0$ corresponds to $\Delta \rightarrow 0$ on a constant-time slice. Comparing with the bulk-null threshold (5.2.17), the constraint $\mathcal{D} \geq \mathcal{D}_\epsilon$ tightens it to

$$\Delta \tilde{t} \leq \Delta \tilde{t}|_{\text{zero-entropy}} = \text{arccosh}\left(\frac{\cos(\mathcal{D}_\epsilon) - \sin^2 \alpha \cos(\Delta\phi)}{\cos^2 \alpha}\right), \quad (5.2.36)$$

and enforces a minimal angular size

$$\Delta\phi \geq \arccos\left(\frac{(\cos \mathcal{D}_\epsilon - 1)}{\sin^2 \alpha} + 1\right) \approx \frac{\mathcal{D}_\epsilon}{\cos \alpha} + O(\mathcal{D}_\epsilon^3), \quad (5.2.37)$$

for nonzero $\mathcal{D}_\epsilon$.

While the short-distance singularity parallels AdS, the antipodal divergence at $\mathcal{D} = \pi$ is specific to de Sitter. From (5.2.13), this occurs only when

$$\arcsin \frac{r_*}{L} \equiv \alpha \rightarrow \frac{\pi}{2} \quad \text{and} \quad \Delta\phi \rightarrow \pi, \quad (5.2.38)$$

i.e. with the screen on the cosmological horizon and the interval covering half the circle. Thus a stretched screen ($r_* < L$) automatically avoids the antipodal pole. Nevertheless, if the antipodal branch dominates ($N \rightarrow \infty$), S_{EE} may again become negative. Imposing non-negativity bounds the ratio $N = N_A/N_G$ from above. The maximal allowed value follows from $S_A(\mathcal{D} = 2\alpha) = 0$:

$$N_{\text{max}} = \frac{\sin(2\alpha) \sinh((\mathcal{D}_\epsilon - \pi)\mu) + \sin(\mathcal{D}_\epsilon) \sinh((\pi - 2\alpha)\mu)}{\sin(2\alpha) \sinh(\mathcal{D}_\epsilon\mu) - \sin(\mathcal{D}_\epsilon) \sinh(2\alpha\mu)} \quad (5.2.39)$$

$$\approx \frac{\sin(2\alpha) \sinh(\pi\mu)}{\sinh(2\alpha\mu) - \mu \sin(2\alpha)} \frac{1}{\mathcal{D}_\epsilon} + O(\mathcal{D}_\epsilon^0),$$

Putting everything together, with the cutoff $\mathcal{D}_\epsilon$ the entropy becomes

$$S_{\text{EE}} = -C \cdot \log \left(\frac{\sin \mathcal{D}_\epsilon}{\sin \mathcal{D}} \frac{\sinh \mu(\pi - \mathcal{D}) + N \sinh(\mu \mathcal{D})}{\sinh \mu(\pi - \mathcal{D}_\epsilon) + N \sinh(\mu \mathcal{D}_\epsilon)} \right). \quad (5.2.40)$$

As illustrated in figure 5.2.2, this guarantees

$$S_{\text{EE}} \geq 0, \quad (5.2.41)$$

provided the relative weight satisfies

$$\text{Positivity condition:} \quad N \leq N_{\text{max}} \approx \frac{\sin(2\alpha) \sinh(\pi\mu)}{\sinh(2\alpha\mu) - \mu \sin(2\alpha)} \frac{1}{\mathcal{D}_\epsilon}. \quad (5.2.42)$$

This bound is a minimal consistency requirement for static-patch holography; stronger constraints will emerge from subadditivity and strong subadditivity, to which we now turn.

5.2.3 Constraints from entropic inequalities

Beyond mere non-negativity, the von Neumann entropy of a reduced density matrix ρ_A obeys a collection of structural inequalities that codify universal features of quantum correlations. These constraints are central in quantum information theory and, in our context, provide stringent consistency checks for any proposed holographic notion of entropy.

A first benchmark is *subadditivity* as we introduced before. For a bipartite decomposition into subsystems A and B , one has

$$S(A \cup B) \leq S(A) + S(B), \quad (5.2.43)$$

with equality if and only if the global state factorizes across the A - B interface. Intuitively, the total correlation contained in $A \cup B$ cannot exceed the sum of the correlations attributed to the individual parts.

A stronger and more informative constraint is *strong subadditivity* (SSA). For three adjacent regions A, B, C it states

$$S(A \cup B) + S(B \cup C) \geq S(B) + S(A \cup B \cup C). \quad (5.2.44)$$

SSA underlies, for example, the monotonicity of mutual information and organizes the entanglement structure of multipartite states. It was rigorously established by Lieb and Ruskai [32, 33]. Moreover, SSA strictly strengthens subadditivity: setting $B = \emptyset$ in (5.2.44) immediately reproduces (5.2.43). Conceptually, SSA follows from an even deeper principle—the data-processing inequality for relative entropy—namely the monotonic decrease of $S(\rho||\sigma)$ under partial trace, which expresses that discarding degrees of freedom cannot increase distinguishability.

Within quantum field theory, these implications have been extensively explored. In the AdS/CFT setting, both the Ryu-Takayanagi (RT) prescription [59] and its covariant Hubeny-Rangamani-Takayanagi (HRT) extension [55] are known to satisfy SSA. We will therefore treat SSA as a non-negotiable benchmark for any candidate definition of holographic entanglement entropy in de Sitter space (see also [4] for applications to FLRW backgrounds). As shown below, imposing SSA (together with related constraints) sharply restricts the admissible entropy functionals built from de Sitter Green’s functions.

Before turning to dS holography, it is instructive to recall a cautionary lesson from AdS with a finite radial cutoff: standard HRT constructions then need *not* obey SSA in full generality. In particular, for holographic theories deformed by $T\bar{T}$, Ref. [98] demonstrated violations of a *boosted* version of SSA involving regions on distinct time slices, even though the purely equal-time (static) form remains intact (see also [99, 100]). Since SSA descends from the monotonicity of relative entropy, such failures indicate that the corresponding bulk geometric functional is no longer computing the von Neumann entropy of a positive semidefinite density operator; instead, it encodes nonlocal cutoff-induced interactions. Analogous breakdowns have been observed in half-dS holography [3, 101] and in the dS/dS correspondence [101, 102], where extremal (or minimal) surfaces anchored to a finite timelike boundary similarly need not respect (strong) subadditivity. A further potential source of tension is the time dependence of the scalar-field vacuum in de Sitter: within the twist-operator approach, relying solely on the Euclidean (Bunch-Davies) branch for bulk propagators can be insufficient when approximating boundary correlators, and this can feed directly into violations of SSA.

The takeaway is straightforward: once locality is relinquished, the area of bulk extremal surfaces no longer furnishes the von Neumann entropy. By contrast, strong subadditivity (SSA) is a theorem for von Neumann entropy and can fail only when the Euclidean path integral that prepares the state for a boundary subregion A does not yield a positive semi-definite operator—e.g., if one inadvertently incorporates negative modes. The objective of

this work is therefore to formulate a holographic dual that *does* reproduce the entropy of a general density matrix from the perspective of static-patch observers in de Sitter space.

More concretely, one is led to expect that the de Sitter entropy S_{dS} —the area of the cosmological horizon—captures the entanglement entropy of a maximal-entropy state. This expectation has received compelling support: [103] demonstrated that explicitly including an observer in the description of the dS static patch transmutes the underlying von Neumann algebra from Type III to Type II_1 , thereby introducing a well-defined trace and, with it, a bona fide notion of entropy. Motivated by these developments, we will adopt an alternative definition of holographic entanglement entropy, eq. (5.2.40), constructed from bulk two-point functions, and subject it to a comprehensive battery of entropic tests, with SSA serving as our principal criterion.

Static SSA

We next show how standard entropy inequalities provide sharp diagnostics for candidate holographic entropy functionals in de Sitter holography. As a starting point, consider boundary intervals lying on a fixed time slice $t = t_0$ of the stretched horizon. In this static setting the entropy depends only on the interval length, which we denote by $S(A) = S(l_A)$. To probe strong subadditivity, take two overlapping intervals A and B . Their union has length $l + \delta l$, while the overlap $A \cap B$ has length l . Using the equivalent form of SSA (5.2.44),

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B), \quad (5.2.45)$$

one arrives at

$$2S\left(l + \frac{\delta l}{2}\right) \geq S(l + \delta l) + S(l). \quad (5.2.46)$$

Assuming $S(l)$ is twice differentiable, expand about l :

$$S\left(l + \frac{\delta l}{2}\right) = S(l) + \frac{\delta l}{2} S'(l) + \frac{(\delta l)^2}{8} S''(l) + \mathcal{O}((\delta l)^3), \quad (5.2.47)$$

$$S(l + \delta l) = S(l) + \delta l S'(l) + \frac{(\delta l)^2}{2} S''(l) + \mathcal{O}((\delta l)^3). \quad (5.2.48)$$

Inserting these series into (5.2.46) and simplifying yields the infinitesimal, static version of SSA [98]:

$$\boxed{S''(l) \leq 0}. \quad (5.2.49)$$

It is useful to stress that (5.2.46) is equivalent to the statement that the entropy must be a concave function of the size parameter that characterizes the subsystem [104]. Concavity here enforces that the incremental growth of S slows as the interval enlarges, thereby encoding

subadditive behavior. Equivalently, concavity can be written as

$$S(\lambda l_1 + (1 - \lambda)l_2) \geq \lambda S(l_1) + (1 - \lambda)S(l_2), \quad \forall \lambda \in [0, 1], \quad (5.2.50)$$

for any pair of lengths l_1, l_2 . A standard analysis result shows that for C^2 functions this condition holds iff $d^2S/dl^2 \leq 0$. Physically, this means the rate at which entropy increases must diminish with subsystem size. If there exists a region where $S''(l) > 0$, $S(l)$ grows faster than linearly there, opening the door to violations of subadditivity or SSA. Ensuring concavity of the entropy functional is therefore a minimal consistency requirement—one that geometric functionals arising in holographic constructions need not satisfy automatically without further input.

Returning to the dS bulk geometry, it is straightforward to verify that the geodesic length anchored on the stretched horizon satisfies

$$\frac{d\mathcal{D}}{d\Delta\phi} = \mathcal{D}' = \frac{\sin \alpha \cos\left(\frac{\Delta\phi}{2}\right)}{\sqrt{1 - \sin^2 \alpha \sin^2\left(\frac{\Delta\phi}{2}\right)}}, \quad (5.2.51)$$

and

$$\frac{d^2\mathcal{D}}{(d\Delta\phi)^2} = \mathcal{D}'' = -\frac{\sin \alpha \cos^2 \alpha \sin\left(\frac{\Delta\phi}{2}\right)}{2\left(1 - \sin^2 \alpha \sin^2\left(\frac{\Delta\phi}{2}\right)\right)^{3/2}} \leq 0, \quad (5.2.52)$$

so the resulting geodesic length—precisely the RT candidate in de Sitter space—obeys the infinitesimal, spatial form of strong subadditivity. This indicates that the contribution associated purely with the global vacuum is compatible with the concavity required by static SSA. Once the antipodal sector is included, however, a nontrivial competition appears and can drive violations of static SSA.

To make this interplay explicit, let us separate the contributions from the global branch G_G and the antipodal branch G_A via

$$S_G = -\log(G_G), \quad S_A = -\log(G_A). \quad (5.2.53)$$

One can prove that S_G is concave by checking

$$\frac{d^2S_G}{d\Delta\phi^2} = \underbrace{\mathcal{D}''}_{<0} \times \underbrace{(\mu \coth((\pi - \mathcal{D})\mu) + \cot \mathcal{D})}_{>0} + \underbrace{\mathcal{D}'^2 \left(\frac{\mu^2}{\sinh^2((\pi - \mathcal{D})\mu)} - \frac{1}{\sin^2 \mathcal{D}} \right)}_{<0} < 0, \quad (5.2.54)$$

where each factor is indicated to show that the total is negative. By contrast, the curvature of

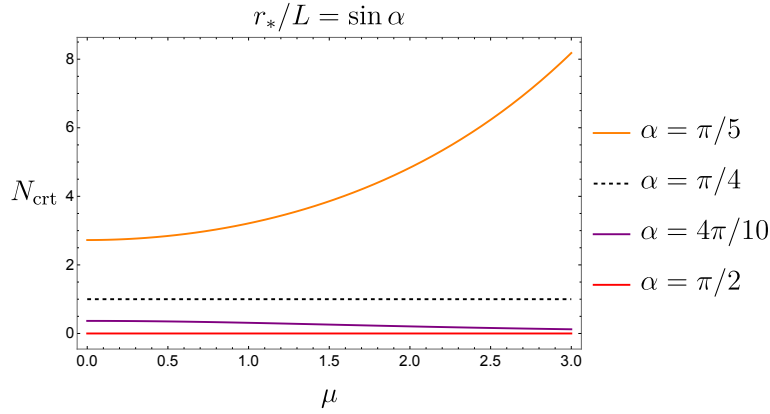


Figure 5.2.3: Upper bound on N from the static SSA condition in (5.2.60). The qualitative behavior differs for $\alpha > \frac{\pi}{4}$ versus $\alpha < \frac{\pi}{4}$. This figure is from [2].

the antipodal piece shows more nuanced behavior,

$$\frac{d^2 S_A}{d\Delta\phi^2} = \mathcal{D}''(\cot \mathcal{D} - \mu \coth(\mathcal{D}\mu)) + (\mathcal{D}')^2 \left(\frac{\mu^2}{\sinh^2(\mathcal{D}\mu)} - \frac{1}{\sin^2 \mathcal{D}} \right). \quad (5.2.55)$$

Depending on μ and $\mathcal{D}$, this can take either sign. At the endpoints one has

$$\begin{aligned} \mathcal{D}(\Delta\phi = 0) &= 0, & \mathcal{D}''(\Delta\phi = 0) &= 0, \\ \mathcal{D}(\Delta\phi = \pi) &= 2\alpha, & \mathcal{D}'(\Delta\phi = \pi) &= 0. \end{aligned} \quad (5.2.56)$$

Accordingly, near the boundaries the second derivative of S_A becomes

$$\frac{d^2 S_A}{d\Delta\phi^2} = \begin{cases} \mathcal{D}''(\cot \mathcal{D} - \mu \coth(\mathcal{D}\mu)) > 0, & \Delta\phi = \pi \\ -\frac{1}{3}(\mu^2 + 1) \mathcal{D}'^2 < 0, & \Delta\phi = 0 \end{cases} \quad (5.2.57)$$

which makes the prospective loss of concavity around $\Delta\phi \simeq \pi$ particularly transparent.

Given this potential failure of concavity, we must scrutinize the full entropy functional S_{EE} and constrain the mixing $N = N_A/N_G$ to retain static SSA. Differentiating twice yields

$$\frac{d^2 S_{EE}}{(d\Delta\phi)^2} = \frac{N^2 e^{2S_G} S_A'' + e^{2S_A} S_G'' + N e^{S_A+S_G} (S_A'' - (S_A' - S_G')^2 + S_G'')}{(N e^{S_G} + e^{S_A})^2}. \quad (5.2.58)$$

The most stringent static constraint is obtained at $\Delta\phi = \pi$, where $\mathcal{D}'(\Delta\phi = \pi) = 0$, reducing the expression to

$$\left. \frac{d^2 S_{EE}}{(d\Delta\phi)^2} \right|_{\Delta\phi=\pi} = \mathcal{D}'' \left(\frac{\mu(\cosh((\pi - \mathcal{D})\mu) - N \cosh(\mathcal{D}\mu))}{\sinh((\pi - \mathcal{D})\mu) + N \sinh(\mathcal{D}\mu)} + \cot(\mathcal{D}) \right). \quad (5.2.59)$$

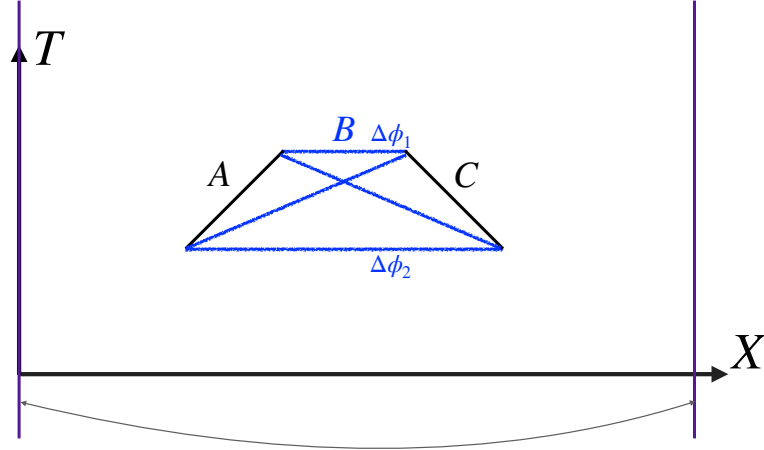


Figure 5.2.4: Setup for boosted strong subadditivity. The subsystems B and $A \cup B \cup C$ are placed on different time slices of the stretched horizon (represented as a cylinder). This figure is from [2].

This translates into the necessary inequality

$$\cot(2\alpha) + \frac{\mu(\cosh((\pi - 2\alpha)\mu) - N \cosh(2\alpha\mu))}{\sinh((\pi - 2\alpha)\mu) + N \sinh(2\alpha\mu)} \geq 0.$$

Hence static SSA imposes the explicit bounds

$$\text{Static SSA: } -\frac{\sinh((\pi - 2\alpha)\mu)}{\sinh(2\alpha\mu)} \leq N \leq \frac{\sinh((\pi - 2\alpha)\mu) + \mu \tan(2\alpha) \cosh((\pi - 2\alpha)\mu)}{\mu \tan(2\alpha) \cosh(2\alpha\mu) - \sinh(2\alpha\mu)}. \quad (5.2.60)$$

A numerical scan confirms that $\Delta\phi = \pi$ yields the tightest condition, so (5.2.60) is both necessary and sufficient to guarantee concavity and thus static SSA for our entropy ansatz. Denoting the upper edge by N_{crt} (see Fig. 5.2.3), one observes

$$N_{\text{crt}}(\alpha, \mu) \rightarrow \begin{cases} > 1, & \alpha < \frac{\pi}{4}, \\ = 1, & \alpha = \frac{\pi}{4}, \\ < 1, & \alpha > \frac{\pi}{4}, \end{cases} \quad (5.2.61)$$

namely, the admissible region consistent with static SSA shrinks as the stretched horizon approaches the cosmological horizon.

Boosted SSA

We now turn to strong subadditivity in a Lorentz-boosted arrangement, in which the relevant intervals are anchored on distinct time slices of the stretched horizon. In this setting, the entanglement entropy is a function of the bulk geodesic separation $\mathcal{D}$, and the SSA condition

takes the form

$$\text{SSA} = 2S(\mathcal{D}_{AB}) - S(\mathcal{D}_B) - S(\mathcal{D}_{ABC}) \geq 0. \quad (5.2.62)$$

To extract its infinitesimal content, consider two nearby configurations whose separations differ by a small parameter $\delta = \Delta\phi_1 - \Delta\phi_2 \ll 1$.⁶ Expanding the three geodesic lengths in δ yields

$$\begin{aligned} \mathcal{D}_{AB} &\approx \mathcal{D}_B + \mathcal{D}'_{AB} \delta + \frac{\delta^2}{2} \mathcal{D}''_{AB} + O(\delta^3), \\ \mathcal{D}_{ABC} &\approx \mathcal{D}_B + \mathcal{D}'_{ABC} \delta + \frac{\delta^2}{2} \mathcal{D}''_{ABC} + O(\delta^3). \end{aligned} \quad (5.2.63)$$

Substituting these series into the SSA expression gives

$$\begin{aligned} \text{SSA} &= (2\mathcal{D}'_{AB} - \mathcal{D}'_{ABC}) S' \delta \\ &+ \left((\mathcal{D}''_{AB} - \frac{1}{2} \mathcal{D}''_{ABC}) S' + S'' \left((\mathcal{D}'_{AB})^2 - \frac{1}{2} (\mathcal{D}'_{ABC})^2 \right) \right) \delta^2 + O(\delta^3). \end{aligned} \quad (5.2.64)$$

Here S' and S'' denote derivatives with respect to $\mathcal{D}$:

$$S' = \frac{dS(\mathcal{D})}{d\mathcal{D}}, \quad S'' = \frac{d^2S(\mathcal{D})}{d\mathcal{D}^2}. \quad (5.2.65)$$

The kinematics of the setup imply

$$\mathcal{D}'_{ABC} = 2\mathcal{D}'_{AB}, \quad (5.2.66)$$

so the inequality simplifies to

$$\text{SSA} = \left((\mathcal{D}''_{AB} - \frac{1}{2} \mathcal{D}''_{ABC}) S' - (\mathcal{D}'_{AB})^2 S'' \right) \delta^2 + O(\delta^3). \quad (5.2.67)$$

Thus the infinitesimal boosted SSA requirement can be written as

Infinitesimal SSA Condition: $\left(\mathcal{D}''_{AB} - \frac{1}{2} \mathcal{D}''_{ABC} \right) S' - (\mathcal{D}'_{AB})^2 S'' \geq 0.$

(5.2.68)

As a consistency check, specialize to the static case, for which $\delta_{ABC} = 2\delta_{AB}$ implies $\mathcal{D}'_{ABC} = 2\mathcal{D}'_{AB}$ and $\mathcal{D}''_{ABC} = 4\mathcal{D}''_{AB}$. In this case

$$\text{Static SSA} = - \left(\mathcal{D}'' \frac{dS}{d\mathcal{D}} + (\mathcal{D}')^2 \frac{d^2S}{d\mathcal{D}^2} \right) \delta^2 + O(\delta^3), \quad (5.2.69)$$

⁶The interval size must still respect the non-negativity constraint in eq. (5.2.37), i.e. $\delta \gtrsim \mathcal{D}_\epsilon / \cos \alpha$.

Using the chain rule between $\Delta\phi$ and $\mathcal{D}$,

$$\frac{d}{d\Delta\phi} = \mathcal{D}' \frac{d}{d\mathcal{D}}, \quad (5.2.70)$$

one recovers the concavity condition

$$\boxed{\text{Static SSA} = \frac{d^2 S}{d(\Delta\phi)^2} = \mathcal{D}'' S' + (\mathcal{D}')^2 S'' \leq 0}, \quad (5.2.71)$$

in agreement with eq. (5.2.49). In terms of the bulk geodesic length, the infinitesimal static SSA may be rewritten as

$$\text{Static SSA} = \mathcal{D}'' S' + (\mathcal{D}')^2 S'' \propto S'' \sin(\mathcal{D}) + \left(\frac{\cos(2\alpha) \cos(\mathcal{D}) - 1}{\cos(\mathcal{D}) - \cos(2\alpha)} \right) S' \leq 0. \quad (5.2.72)$$

The coefficient multiplying S' satisfies

$$\left(\frac{\cos(2\alpha) \cos(\mathcal{D}) - 1}{\cos(\mathcal{D}) - \cos(2\alpha)} \right) \in (-\infty, -1], \quad (5.2.73)$$

and diverges to $-\infty$ as $\mathcal{D} \rightarrow 2\alpha$. Therefore a necessary requirement is

$$S'|_{\mathcal{D}=2\alpha} \geq 0, \quad (5.2.74)$$

which is equivalent to

$$S'|_{\mathcal{D}=2\alpha} = \cot(2\alpha) + \frac{\mu(\cosh((\pi - 2\alpha)\mu) - N \cosh(2\alpha\mu))}{\sinh((\pi - 2\alpha)\mu) + N \sinh(2\alpha\mu)} \geq 0. \quad (5.2.75)$$

This reproduces the bound in eq. (5.2.60).

Moreover, this necessary condition is also sufficient. The argument proceeds by expressing the static SSA inequality entirely as a function of $\mathcal{D}$, combining terms over a common positive denominator, and observing that the numerator is a quadratic polynomial in N ,

$$\text{Static SSA} = \frac{aN^2 + bN + c}{\text{Positive Denominator}} \leq 0, \quad (5.2.76)$$

with $a > 0$ for $\mathcal{D} \in (0, 2\alpha)$. For each fixed $\mathcal{D}$, the allowed window is

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} \leq N \leq \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \quad (5.2.77)$$

A direct analysis shows that the lower (upper) edge is monotonically increasing (decreasing) in $\mathcal{D}$. Since $\mathcal{D}$ is bounded above by 2α , both extrema occur at $\mathcal{D} = 2\alpha$, and they coincide

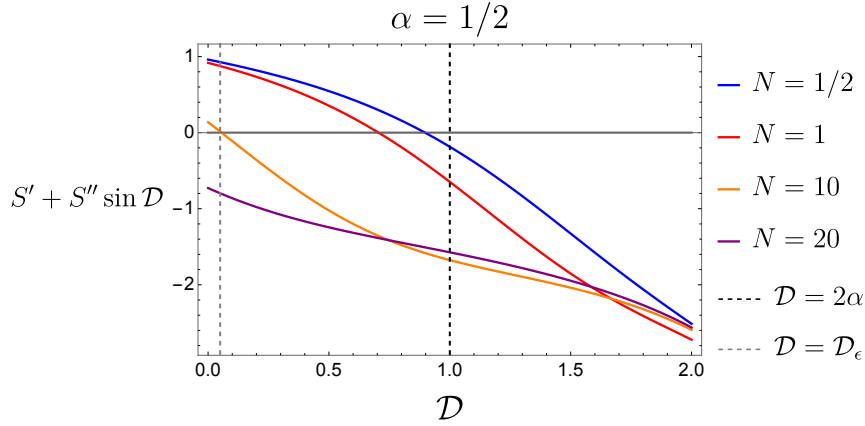


Figure 5.2.5: Representative behavior of $S' + S'' \sin \mathcal{D}$ as a function of the geodesic length $\mathcal{D}$ for several N , with $\alpha = 1/2$ and $\mathcal{D}_\epsilon = 1/20$. This figure is from [2].

precisely with the endpoints of the interval in eq. (5.2.60). Hence the necessary and sufficient conditions match, completing the proof.

Boundary null interval limit

We next analyze a limiting configuration in which the boundary intervals A and C become null with respect to the induced metric on the stretched horizon, while B and $A \cup B \cup C$ are anchored on distinct time slices. Because nullness is defined using the boundary metric, we refer to this regime as the *boundary null limit*. The induced line element is

$$ds^2 = -\cos^2 \alpha dt^2 + L^2 \sin^2 \alpha d\phi^2. \quad (5.2.78)$$

Imposing the null condition in this metric fixes the composite sizes to

$$\Delta\phi_{AB} = \frac{\Delta\phi_1 + \Delta\phi_2}{2}, \quad \Delta t_{AB} = \frac{\Delta\phi_2 - \Delta\phi_1}{2} L \tan \alpha. \quad (5.2.79)$$

In this limit, the relevant derivatives of the geodesic lengths are

$$\begin{aligned} \mathcal{D}'_{AB} &= \frac{\sin \alpha \cos \left(\frac{\Delta\phi_1}{2} \right)}{2 \sqrt{1 - \sin^2 \alpha \sin^2 \left(\frac{\Delta\phi_1}{2} \right)}}, \\ \mathcal{D}''_{AB} &= -\frac{\sin \alpha \csc \left(\frac{\Delta\phi_1}{2} \right) \left(\cos(2\alpha) \sin^2 \left(\frac{\Delta\phi_1}{2} \right) + 1 \right)}{8 \left(1 - \sin^2 \alpha \sin^2 \left(\frac{\Delta\phi_1}{2} \right) \right)^{3/2}}, \\ \mathcal{D}''_{ABC} &= -\frac{\sin \alpha \cos^2 \alpha \sin \left(\frac{\Delta\phi_1}{2} \right)}{2 \left(1 - \sin^2 \alpha \sin^2 \left(\frac{\Delta\phi_1}{2} \right) \right)^{3/2}}, \end{aligned} \quad (5.2.80)$$

Plugging these into the infinitesimal boosted SSA criterion (5.2.68) and taking $\Delta\phi < \pi$, the inequality reduces to⁷

$$\boxed{S' + 2S'' \sin(\alpha) \sin\left(\frac{\Delta\phi_1}{2}\right) \sqrt{1 - \sin^2 \alpha \sin^2\left(\frac{\Delta\phi_1}{2}\right)} \leq 0,} \quad (5.2.83)$$

with

$$\begin{aligned} S' &= \frac{dS(\mathcal{D}_B)}{d\mathcal{D}_B} = \cot(\mathcal{D}_B) + \frac{\mu(\cosh((\pi - \mathcal{D}_B)\mu) - N \cosh(\mathcal{D}_B\mu))}{\sinh((\pi - \mathcal{D}_B)\mu) + N \sinh(\mathcal{D}_B\mu)} \\ S'' &= \frac{d^2 S(\mathcal{D}_B)}{d\mathcal{D}_B^2} = -\frac{1}{\sin^2(\mathcal{D}_B)} + \frac{\mu^2 (N^2 - 2N \cosh(\pi\mu) + 1)}{(\sinh((\pi - \mathcal{D}_B)\mu) + N \sinh(\mathcal{D}_B\mu))^2}. \\ \mathcal{D}_B &= \arccos\left(\sin^2 \alpha \cos \Delta\phi_1 + \cos^2 \alpha\right). \end{aligned} \quad (5.2.84)$$

For notational convenience, let us write $\mathcal{D}_B \equiv \mathcal{D}$ from now on. In terms of $\mathcal{D}$, the boosted SSA condition tidies up to

$$\boxed{S' + S'' \sin(\mathcal{D}) \leq 0.} \quad (5.2.85)$$

Using the entropy proposal (5.2.1), this becomes

$$\frac{\mu^2 \sin \mathcal{D} (N^2 - 2N \cosh(\pi\mu) + 1)}{(\sinh((\pi - \mathcal{D})\mu) + N \sinh(\mathcal{D}\mu))^2} + \frac{\mu(\cosh((\pi - \mathcal{D})\mu) - N \cosh(\mathcal{D}\mu))}{\sinh((\pi - \mathcal{D})\mu) + N \sinh(\mathcal{D}\mu)} - \tan\left(\frac{\mathcal{D}}{2}\right) \leq 0. \quad (5.2.86)$$

Expanding near the cutoff $\mathcal{D} \sim \mathcal{D}_\epsilon$ yields

$$\frac{\mu (N - \cosh(\pi\mu))}{\sinh(\pi\mu)} + \mathcal{O}(\mathcal{D}_\epsilon) \geq 0, \quad (5.2.87)$$

so at leading order one obtains

$$\boxed{\text{boosted SSA condition: } N \geq \cosh(\pi\mu).} \quad (5.2.88)$$

This bound guarantees boosted SSA near $\Delta\phi \rightarrow 0$, but on its own it is merely *necessary*.

⁷Starting from eq. (5.2.68), one finds

$$\text{boosted SSA} = -\delta^2 \left(\frac{\sin \alpha \cos^2(\Delta\phi/2)}{8 \sin(\Delta\phi/2) (1 - \sin^2 \alpha \sin^2(\Delta\phi/2))^{3/2}} \right) \times (S' + S'' \sin(\mathcal{D})). \quad (5.2.81)$$

The prefactor is non-negative and therefore does not affect the sign of the inequality. It also clarifies the divergence $\text{SSA} \sim \frac{1}{\Delta\phi}$ as $\mathcal{D} \rightarrow 0$, and the vanishing at

$$\text{boosted SSA}|_{\Delta\phi=\pi} = 0. \quad (5.2.82)$$

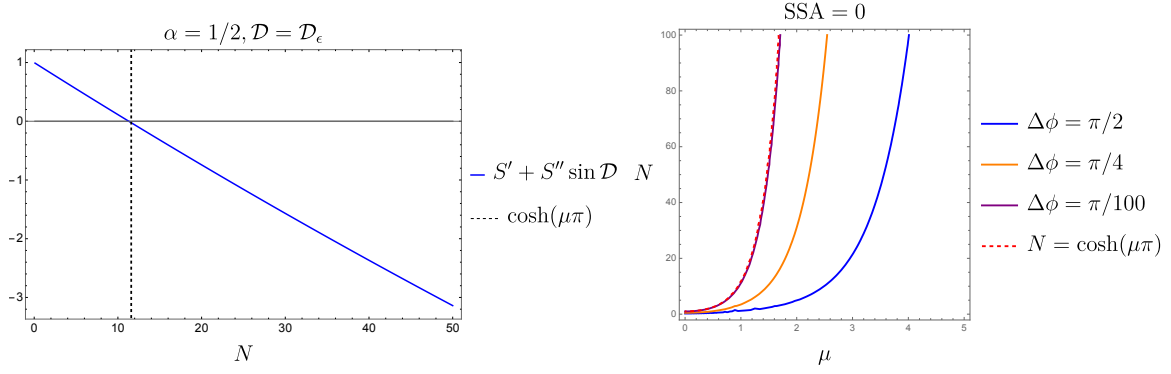


Figure 5.2.6: Left: $S' + S'' \sin \mathcal{D}$ versus N at $\mathcal{D} = \mathcal{D}_\epsilon = 1/100$ with $\alpha = 1/2$. Right: curves $N(\mu)$ for which $\text{SSA} = 0$ in (5.2.68); the tightest constraint arises in the $\mathcal{D} = \Delta\phi = 0$ (null) configuration. The plot is from [2].

To promote it to a *sufficient* condition, we must ensure the inequality (5.2.85) holds for all $\mathcal{D}$ in its allowed range. For fixed $\mathcal{D}$, boosted SSA requires

$$\frac{\mu^2 \sin \mathcal{D} (N^2 - 2N \cosh(\pi\mu) + 1)}{(\sinh((\pi - \mathcal{D})\mu) + N \sinh(\mathcal{D}\mu))^2} + \frac{\mu(\cosh((\pi - \mathcal{D})\mu) - N \cosh(\mathcal{D}\mu))}{\sinh((\pi - \mathcal{D})\mu) + N \sinh(\mathcal{D}\mu)} - \tan\left(\frac{\mathcal{D}}{2}\right) \leq 0. \quad (5.2.89)$$

A brief manipulation⁸ shows that the sharpest constraint is obtained by maximizing over $\mathcal{D}$:

$$N \geq \max_{\mathcal{D}} \left(\frac{\sinh(\pi\mu) \sqrt{-2\mu^2 \cos(2\mathcal{D}) + 2\mu^2 + 1} + \sinh((2\mathcal{D} - \pi)\mu) - 2\mu \sin \mathcal{D} \cosh(\pi\mu)}{\sinh(2\mathcal{D}\mu) - 2\mu \sin \mathcal{D}} \right) \\ = \cosh(\mu\pi), \quad (5.2.90)$$

consistent with the necessary bound above. Hence the condition in eq. (5.2.88) is both necessary and sufficient. Figure 5.2.6 illustrates this numerically.

In summary, the boundary-null limit cleanly separates the boosted SSA constraint from the purely static case. By expanding the SSA functional for nearby intervals, we obtained the compact inequality (5.2.85) involving S' and S'' , which both reproduces the static result as a special case and furnishes a sharp lower bound on the antipodal weight N . The condition (5.2.88) thus serves as a stringent test of the viability of the entropy proposal (5.2.40) in dynamical (boosted) configurations.⁹

⁸After putting terms over a common denominator, the numerator becomes a quadratic in N : $aN^2 + bN + C < 0$ with $a = 2\mu \sin \mathcal{D} - \sinh(2\mathcal{D}\mu) \leq 0$.

⁹Numerics indicate that the null configuration yields the strongest bound on N , though other boost angles can be analyzed analogously.

Constraints on holographic entanglement entropy

We conclude by collecting the constraints implied by our analysis of holographic entanglement entropy in the static patch of de Sitter.

Our proposal starts from the most general two-point function of a massive scalar in dS, and uses the bulk geodesic separation $\mathcal{D}$ in dS₃ to define

$$S_{\text{EE}} = -C \cdot \log \left(\frac{\sin \mathcal{D}_\epsilon \sinh \mu(\pi - \mathcal{D}) + N \sinh(\mu \mathcal{D})}{\sin \mathcal{D} \sinh \mu(\pi - \mathcal{D}_\epsilon) + N \sinh(\mu \mathcal{D}_\epsilon)} \right), \quad (5.2.91)$$

where the dimensionless parameter N encodes the relative weight of the global dS branch versus its antipodal branch and thereby captures the choice of boundary conditions (and state) in static patch holography. To tame the universal short-distance divergence and ensure non-negativity of the entropy, a bulk cutoff scale $\mathcal{D}_\epsilon$ is introduced.

With this candidate functional in hand, we subjected it to basic entropic tests. On a constant-time slice (the “static” setup), strong subadditivity furnishes the necessary and sufficient bounds

$$\text{Static SSA: } -\frac{\sinh((\pi - 2\alpha)\mu)}{\sinh(2\alpha\mu)} \leq N \leq \frac{\sinh((\pi - 2\alpha)\mu) + \mu \tan(2\alpha) \cosh((\pi - 2\alpha)\mu)}{\mu \tan(2\alpha) \cosh(2\alpha\mu) - \sinh(2\alpha\mu)}. \quad (5.2.92)$$

To probe time-dependent configurations, we analyzed a boosted arrangement in which the intervals live on different time slices. This “boosted SSA” yields a complementary necessary-and-sufficient lower bound,

$$\text{Boosted SSA condition: } N \geq \cosh(\pi\mu) \geq 1. \quad (5.2.93)$$

Notably, this constraint is independent of the stretch parameter α and is generically stricter than the static lower bound.

Taken together, these inequalities carve out a nontrivial admissible domain in the *bulk* parameter space of the static patch proposal. Only within this domain does the functional behave like the von Neumann entropy of a legitimate density matrix. Imposing (5.2.92) and (5.2.93) simultaneously further restricts where the stretched horizon may sit: since the static upper bound obeys $N_{\text{crt}}(\mu, \alpha) < 1$ once $\alpha > \pi/4$, compatibility requires $\alpha \leq \pi/4$; cf. Fig. 5.2.7. Our final constraint can thus be summarized as

$$\boxed{\text{SSA conditions: } \cosh(\mu\pi) \leq N \leq N_{\text{crt}}(\mu, \alpha) \quad \text{and} \quad \alpha \leq \frac{\pi}{4}}, \quad (5.2.94)$$

with saturation at $\alpha = \pi/4$ and $\mu = 0$.

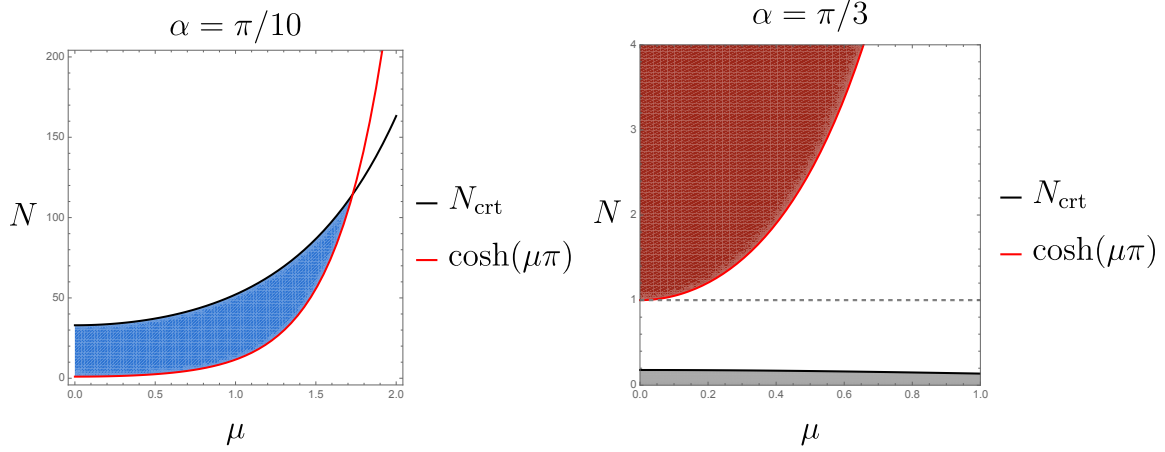


Figure 5.2.7: Allowed regions satisfying both static and boosted SSA. Left: for $\alpha \leq \pi/4$, the shaded blue area delineates the viable band in N -space. Right: for $\alpha > \pi/4$, the two requirements are incompatible and no overlap remains. The plots are from [2].

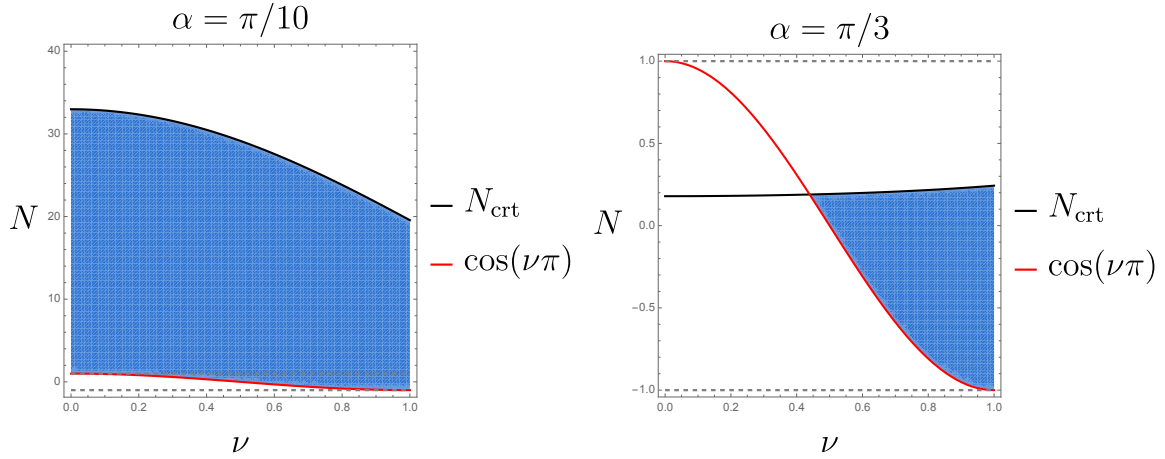


Figure 5.2.8: Allowed parameter space (blue) for the light-mass twist operator, i.e. the region in which both the static and boosted SSA constraints are satisfied. The figures are from [2].

5.2.4 Twist operators with light mass

Up to now we have worked with a twist operator whose conformal dimension is encoded by the real parameter

$$\mu = \sqrt{m^2 L^2 - 1}, \quad (5.2.95)$$

appropriate to a heavy scalar. We now turn to the complementary regime of a light twist operator.¹⁰ It is convenient to introduce

$$0 \leq \nu = \sqrt{1 - m^2} \leq 1, \quad (5.2.96)$$

¹⁰Strictly speaking, the twist operator mass should be fixed by the (as-yet unspecified) boundary QFT on the stretched horizon. In what follows we treat it as a free parameter to be constrained by entropic inequalities.

so that $\mu = i\nu$.

Analytically continuing (5.2.91) to imaginary μ gives the corresponding entropy functional,

$$S_{\text{EE}} = -C \cdot \log \left(\frac{\sin \mathcal{D}_\epsilon}{\sin \mathcal{D}} \frac{\sin \nu(\pi - \mathcal{D}) + N \sin(\nu \mathcal{D})}{\sin \nu(\pi - \mathcal{D}_\epsilon) + N \sin(\nu \mathcal{D}_\epsilon)} \right). \quad (5.2.97)$$

By the same continuation, the static and boosted SSA bounds become

$$\boxed{\text{SSA Conditions: } \cos(\pi\nu) \leq N \leq \frac{\sin((\pi - 2\alpha)\nu) + \nu \tan(2\alpha) \cos((\pi - 2\alpha)\nu)}{\nu \tan(2\alpha) \cos(2\alpha\nu) - \sin(2\alpha\nu)},} \quad (5.2.98)$$

and we have verified numerically that the replacement $\mu \rightarrow i\nu$ reproduces the correct constraints on N .

Unlike the heavy-mass case, here the two inequalities always admit an overlap. This is already clear at the edge $\nu = 1$, where the upper and lower bounds obey

$$N_{\text{crt}}(\nu = 1, \alpha) = -1 + \frac{2\pi}{4\alpha - \sin(4\alpha)} > -1 = \cos(\pi\nu)|_{\nu=1}. \quad (5.2.99)$$

Figure 5.2.8 illustrates the resulting allowed region in the (N, α) plane for representative choices of parameters.

5.2.5 Finite entropy inequalities: bulk causality vs boundary causality

In the preceding subsections, we restricted attention to the *infinitesimal* regime, where adjacent intervals differ by parametrically small separations. In that limit, boosted strong subadditivity reduces to local differential constraints on S , which allowed us to extract sharp bounds on the parameter $N = N_A/N_G$ controlling antipodal contributions. Such tests, however, interrogate only the short-distance behavior of the entropy functional; they need not ensure consistency for *finite* intervals, where intrinsically nonlocal features of the bulk geometry become relevant. To address this gap, we now analyze the finite (global) forms of the entropy inequalities, placing the relevant subsystems at macroscopic angular and temporal separations on the stretched horizon.

A central ingredient in the finite-interval analysis is the misalignment between bulk and boundary causal structures in de Sitter holography, a feature that distinguishes it from standard AdS/CFT. In asymptotically AdS spacetimes obeying an averaged null energy condition, Gao-Wald time delay ensures that bulk null curves cannot “outrun” boundary light rays: bulk causal curves never beat boundary null geodesics, and boundary spacelike separation remains noncausal through the bulk.

By contrast, on the stretched horizon the induced metric is flat, defining a natural boundary lightcone for the putative dual QFT, while the relevant bulk curves are de Sitter geodesics

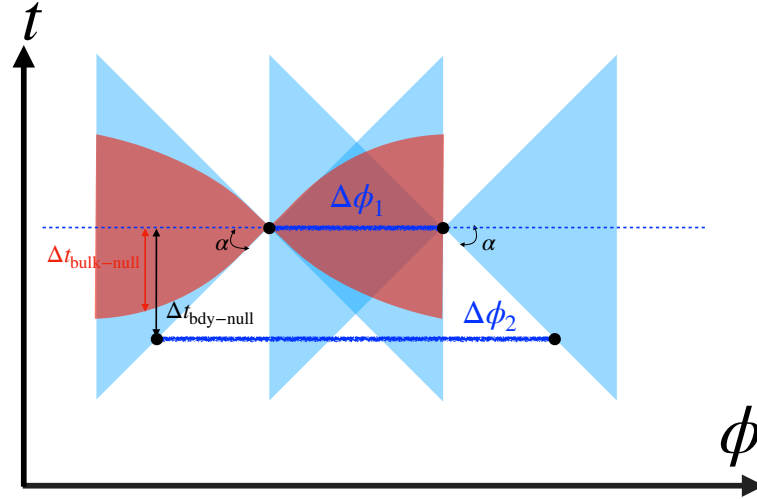


Figure 5.2.9: Bulk-vs-boundary causality on the stretched horizon. The shaded region indicates points spacelike to a given boundary point according to the boundary metric. The blue (boundary) domain of dependence follows the induced Lorentz symmetry, while the red one is the causal domain inherited from bulk dS. The illustration is from [2].

whose null boundary is set by the condition $\mathcal{D} = 0$. Their intersection with the stretched horizon induces a *different* causal structure on the boundary. Although one might wish the boundary theory to respect the induced Lorentz symmetry, we will see that the causality inherited from the bulk is the one compatible with our holographic entropy functional. Concretely, for an interval with spatial size $\Delta\phi$ and time separation Δt we have

$$\text{Boundary length :} \quad \Delta L = \sqrt{\Delta X^2 - \Delta T^2} = L \cos \alpha \sqrt{\tan^2 \alpha (\Delta\phi)^2 - (\Delta\tilde{t})^2}, \quad (5.2.100)$$

while the holographic entropy is governed by the bulk geodesic distance between the same endpoints,

$$\text{Bulk distance :} \quad \mathcal{D} = \arccos \left(\sin^2 \alpha \cos(\Delta\phi) + \cos^2 \alpha \cosh(\Delta\tilde{t}) \right), \quad (5.2.101)$$

which is not boost invariant with respect to the boundary metric. These two notions define distinct null separations,

$$\text{Boundary null interval :} \quad \Delta t_{\text{bdy null}} = L \cdot \Delta\phi \tan \alpha, \quad (5.2.102)$$

$$\text{Bulk null interval :} \quad \Delta t_{\text{bulk null}} = L \cdot \text{arccosh} \left[\frac{1 - \sin^2 \alpha \cos \Delta\phi}{\cos^2 \alpha} \right]. \quad (5.2.103)$$

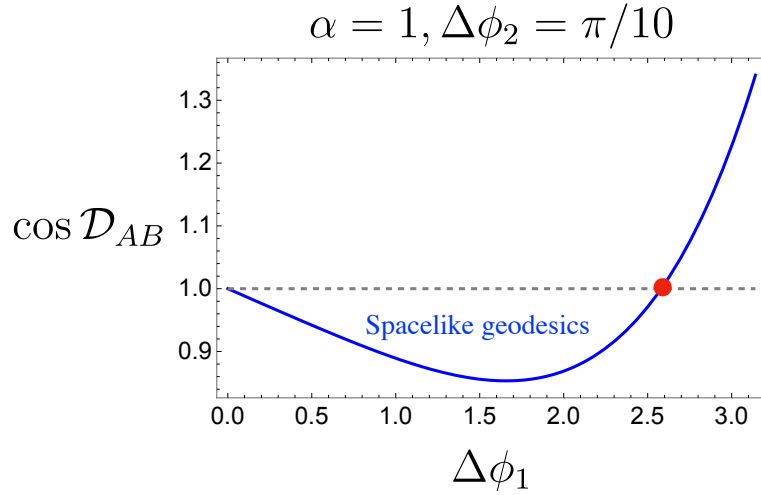


Figure 5.2.10: Geodesic length for the combined interval $A \cup B$ when A is chosen null with respect to the *boundary* lightcone. In this setting a bulk spacelike geodesic of length $\mathcal{D}_{AB}$ may cease to exist. The plot is from [2].

¹¹ For any fixed $\Delta\phi$ one finds¹²

$$\Delta t_{\text{bulk null}} \leq \Delta t_{\text{bdy null}} , \quad (5.2.104)$$

with equality only at $\Delta\phi = 0$. Thus the boundary causal domain is always *larger* than the one induced from the bulk. Equivalently, bulk causality in dS is more restrictive: bulk causal curves can reach earlier than purely boundary curves and can connect boundary points that are spacelike with respect to the boundary metric.

For infinitesimal intervals, $\Delta\phi \rightarrow 0$, the two null times coincide:

$$\Delta t_{\text{bulk null}} \approx L\Delta\phi \tan \alpha + \mathcal{O}((\Delta\phi)^2) \approx \Delta t_{\text{bdy null}} , \quad (5.2.105)$$

which explains why our earlier infinitesimal boosted SSA analysis did not need to distinguish them. For finite intervals, however, the two notions differ appreciably, and this mismatch affects SSA. In the configuration of figure 5.2.4, push the auxiliary intervals A and C to extremality—either null with respect to the *boundary* lightcone or null with respect to the *bulk* geodesics. If A and C are boundary-null, the joint geodesic for $A \cup B$ obeys

$$\cos \mathcal{D}_{AB} = \cos^2 \alpha \cosh \left[\frac{\Delta\phi_2 - \Delta\phi_1}{2} \tan \alpha \right] + \sin^2 \alpha \cos \left(\frac{\Delta\phi_2 + \Delta\phi_1}{2} \right) , \quad (5.2.106)$$

with $\Delta t_{AB} = \Delta t_A = \Delta t_C = \Delta t_{\text{bdy null}}(\Delta\phi_A)$. This extremal choice is problematic for two

¹¹A similar comparison in AdS/CFT is subtle: in AdS_3 a spacelike geodesic does not, in general, limit to a bulk null geodesic as $\mathcal{D} \rightarrow 0$ because the boundary is timelike. In fact, boundary-anchored null geodesics in AdS_3 reconverge at the antipodal boundary point.

¹²Since $\partial_{\Delta\phi} (\Delta t_{\text{bdy null}} - \Delta t_{\text{bulk null}}) > 0$, the minimum is attained at $\Delta\phi = 0$ and equals zero.

reasons. First, a bulk spacelike geodesic $\mathcal{D}_{AB}$ fails to exist when

$$\Delta t_{\text{bdy null}}(\Delta\phi_A) > \Delta t_{\text{bulk null}}(\Delta\phi_{AB}), \quad (5.2.107)$$

as sketched in figure 5.2.10. Second, even when a geodesic exists, the *finite* boosted SSA can be violated despite satisfying the infinitesimal bound. In contrast, when A and C are chosen to be bulk-null, all cases we tested preserve the finite boosted SSA. A complete characterization of finite boosted SSA within static patch holography would be worthwhile to develop.

Irrespective of the detailed form of the entropy functional, a replica-inspired holographic prescription in de Sitter should tie the entropy to the length of a bulk geodesic anchored at the boundary endpoints. The causality discussion above therefore applies quite generally. The upshot is that the causal structure inherited from the de Sitter bulk is the one compatible with holographic entanglement entropy in the static patch, whereas the intrinsic boundary lightcone overestimates the true causal domain of the dual theory.

5.2.6 Brief conclusion

In this work, we propose a novel holographic entanglement entropy in de Sitter space. We consider a stretched horizon in the static patch holography and take a subsystem on that surface. We exploit the twist operator formalism in the replica trick and compute it through holographic dictionary. In general, the scalar two-point function in de Sitter space admits the non-local α -vacuum contribution and this makes us possible to satisfy the strong subadditivity. We derive constraint on parameters such as the position of the stretched horizon r_* , mixing of Euclidean and antipodal Green's function N and the mass of the twist operator. For heavy mass case, we obtain the upper bound $r_* \leq \frac{1}{\sqrt{2}}$, but for the heavy mass case, we can place the stretched horizon at any position. This is a novel constraint on the static patch holography. We also discuss the lightcone on the stretched horizon. We have two choices; one is the null defined by the induced metric and the other is defined by the geodesic distance. Finally, we find that the first choice correctly reproduces the boosted version of SSA even at the finite subsystem and thus it is the correct result.

Chapter 6

Half de Sitter holography

Rather than placing a timelike boundary at the cosmological horizon, one may pursue the “half de Sitter” holographic program introduced in [3]. In this construction, a timelike hypersurface is inserted directly in global de Sitter spacetime, and Dirichlet boundary conditions are imposed so that the boundary dynamics decouple from gravity—mirroring the familiar logic of AdS/CFT.

A key ingredient is a finite radial (or global) cutoff, in close spirit to the finite-cutoff AdS/CFT implementations inspired by the $T\bar{T}$ deformation [105]. In the de Sitter context, this cutoff carves out only a portion of the global geometry bounded by the chosen timelike surface. The resulting setup yields a controlled arena for holography in de Sitter: it retains many technical conveniences of finite-cutoff AdS/CFT while also making explicit the structural similarities—and the important differences—between those constructions and prospective holographic duals of de Sitter space.

6.0.1 CFT on de Sitter Space from AdS/CFT

Before turning to de Sitter holography, it is helpful to first revisit AdS/CFT in a setting where the boundary CFT resides on de Sitter space. As a warm-up, we take a CFT_2 placed on dS_2 , whose metric is

$$ds^2_{\text{dS}_2} = -dt^2 + \cosh^2 t d\phi^2 = \frac{1}{\cos^2 T} \left(-dT^2 + d\phi^2 \right), \quad (6.0.1)$$

Throughout, the de Sitter radius is set to unity, and the global and conformal times are related by $\cosh t = 1/\cos T$. Note that T is compact, taking values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, whereas the global time t ranges over $(-\infty, \infty)$. For concreteness, we focus on $\text{AdS}_3/\text{CFT}_2$ with the CFT living on dS_2 . The bulk dual is global AdS_3 ,

$$ds^2(g) = -\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\phi^2, \quad (6.0.2)$$

Here ρ is the AdS radial coordinate and τ denotes global time. In AdS₃, codimension-two extremal surfaces reduce to geodesics, so their “area” is computed by the geodesic distance $D^{(g)}_{12}$ in the metric (6.0.2) between points (ρ_1, τ_1, ϕ_1) and (ρ_2, τ_2, ϕ_2) :

$$\cosh D^{(g)}_{12} = \cos(\tau_1 - \tau_2) \cosh \rho_1 \cosh \rho_2 - \cos(\phi_1 - \phi_2) \sinh \rho_1 \sinh \rho_2, . \quad (6.0.3)$$

6.0.2 de Sitter and Hyperbolic Slicing of AdS₃

To place a CFT on dS₂, we adopt the de Sitter sliced form of AdS₃ :

$$ds^2_{(d)} = d\eta^2 + \sinh^2 \eta (-dt^2 + \cosh^2 t d\phi^2), \quad (6.0.4)$$

with the coordinates (t, η) related to global AdS variables (τ, ρ) by

$$\sinh \rho = \cosh t \sinh |\eta|, \quad \tan \tau = \tanh \eta \sinh t \quad (6.0.5)$$

From (6.0.5), the geodesic distance between points (t_1, η_1, ϕ_1) and (t_2, η_2, ϕ_2) in this slicing follows as

$$\cosh D^{(d)}_{12} = \cosh \eta_1 \cosh \eta_2 - \sinh \eta_1 \sinh \eta_2 (\cosh t_1 \cosh t_2 \cos(\phi_2 - \phi_1) - \sinh t_1 \sinh t_2) . \quad (6.0.6)$$

The holographic CFT₂ with physical metric (6.0.1) resides at the conformal boundary $\eta \rightarrow \infty$ of AdS₃. Introducing a UV cutoff in the field theory corresponds to regulating the bulk with a finite surface $\eta = \eta_\infty$, which fixes the CFT cutoff ϵ via

$$e^{\eta_\infty} = \frac{1}{\tanh \frac{\xi}{2}} \approx \frac{2}{\epsilon} . \quad (6.0.7)$$

For later convenience, we also make use of the hyperbolic slicing of AdS₃, whose metric is

$$ds^2_{(h)} = -d\mu^2 + \cosh^2 \mu (d\xi^2 + \sinh^2 \xi d\phi^2) . \quad (6.0.8)$$

In terms of global coordinates (τ, ρ) , the map is

$$\sinh \rho = \sinh \xi \cosh \mu, \quad \tan \tau = \cot \mu \cosh \xi, \quad (6.0.9)$$

and the geodesic distance between (μ_1, ξ_1, ϕ_1) and (μ_2, ξ_2, ϕ_2) is

$$\cosh D^{(h)}_{12} = \cos \mu_1 \cos \mu_2 (\cosh \xi_1 \cosh \xi_2 - \cos(\phi_1 - \phi_2) \sinh \xi_1 \sinh \xi_2) + \sin \mu_1 \sin \mu_2 . \quad (6.0.10)$$

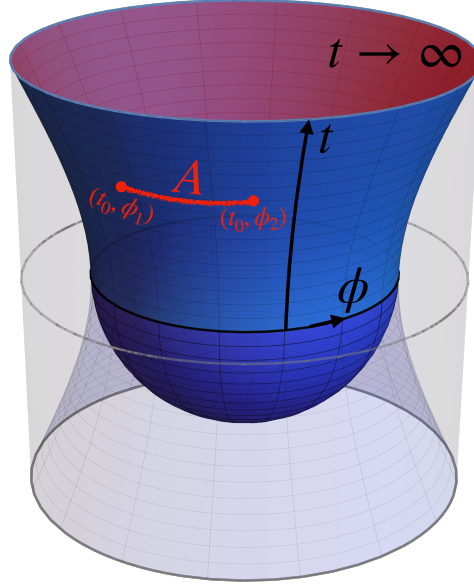


Figure 6.0.1: The setup for computing the entanglement entropy of a CFT₂ on dS₂ via AdS₃/CFT₂. The region $t < 0$ is replaced by a Euclidean cap (half-sphere) following the Hartle–Hawking prescription (blue), while in the Schwinger–Keldysh approach one restricts the Lorentzian path integral to $0 \leq t \leq t_0$ and glues a copy at $t = t_0$. The illustration is from [3].

The two slicings are related by the analytic continuation

$$\eta = i \left(\mu - \frac{\pi}{2} \right), \quad \xi = i \frac{\pi}{2} - t. \quad (6.0.11)$$

6.0.3 Case 1: Entanglement entropy in Schwinger-Keldysh prescription

In this section, we compute the holographic entanglement entropy [11, 42, 54] for a CFT living on dS₂, adopting the Schwinger-Keldysh prescription (case 1) for time-dependent backgrounds. See [106] for related analyses in higher dimensions.

We assume that the quantum state at $t = 0$ is prepared by an Euclidean path integral on a hemisphere via the Hartle-Hawking construction, as sketched in figure 6.0.1. To define the entanglement entropy S_A , we take the subsystem A to be an interval with endpoints (t_0, ϕ_1) and (t_0, ϕ_2) , also shown in figure 6.0.1.

On the gravity side, we glue two copies of AdS₃ truncated at $t = t_0$, following the holographic Schwinger-Keldysh contour [107, 108]. This avoids subtleties associated with the conformal boundary of dS₂ at $t = \infty$. Using the de Sitter-sliced geodesic length (6.0.6),

the holographic entanglement entropy evaluates to

$$S_A^{\text{con}} = \frac{c}{3} \log \left[\frac{2 \cosh t_0 \sin\left(\frac{|\phi_1 - \phi_2|}{2}\right)}{\epsilon} \right], \quad (6.0.12)$$

with the cutoff surface fixed by (6.0.7). At late times this grows linearly, $S_A \simeq \frac{c}{3} t_0$, reflecting the contribution of the connected geodesic anchoring the two boundary points. Within the Schwinger-Keldysh setup, (6.0.12) is the final result for the holographic entanglement entropy.

Remark on (strong) subadditivity. This will be relevant later for half-de Sitter holography. In the $\epsilon \rightarrow 0$ limit, (strong) subadditivity is satisfied as in standard AdS/CFT [55, 59]. Moreover, both subadditivity $S_A + S_B \geq S_{AB}$ and strong subadditivity $S_{AB} + S_{BC} \geq S_A + S_{ABC}$ continue to hold on a constant time slice even with a finite radial cutoff.

To make this explicit, place the dS_2 boundary at $\eta = \eta_0$ in AdS_3 , keeping η_0 finite. For a subregion A on $t = t_0$, the geodesic distance between its endpoints is

$$D_{12}^{(d)} = \text{arccosh} \left(\cosh^2 \eta_0 - \sinh^2 \eta_0 \left[\cosh^2 t_0 \cos(\phi_2 - \phi_1) - \sinh^2 t_0 \right] \right), \quad (6.0.13)$$

which is proportional to S_A^{con} . For interval size $\Delta\phi = |\phi_1 - \phi_2| < \pi$, the second derivative obeys

$$\frac{d^2 D_{12}^{(d)}}{d(\Delta\phi)^2} = - \frac{\sqrt{2} \cosh t_0 \sinh \eta_0 \sin\left(\frac{\Delta\phi}{2}\right) \left(1 + \cosh^2 t_0 \sinh^2 \eta_0\right)}{\left(1 + \cosh D_{12}^{(d)}\right)^{3/2}} \leq 0, \quad (6.0.14)$$

implying concavity and hence the (static) SSA on constant time slices.

By contrast, for boosted subsystems the strong subadditivity that underlies the entropic c -theorem [109] is known to be violated in AdS/CFT with a finite cutoff [98]. This breakdown is attributed to the nonlocality induced by the $T\bar{T}$ deformation in the cutoff theory, which also obstructs a standard Hamiltonian (quantum-mechanical) interpretation. As we will see later, in half-de Sitter holography both subadditivity and strong subadditivity can already fail on constant (unboosted) time slices, signaling spatial nonlocality of the dual theory on the half-dS boundary.

6.0.4 Case 2: Holographic pseudo entropy with a final state projection

We now analyze a setup in which the future boundary is projected onto a boundary state $|B\rangle$ at $t = \infty$, and interpret the corresponding extremal area as a holographic pseudo entropy [31]. In the bulk description, this projection is implemented by inserting an end-of-the-world (EOW)

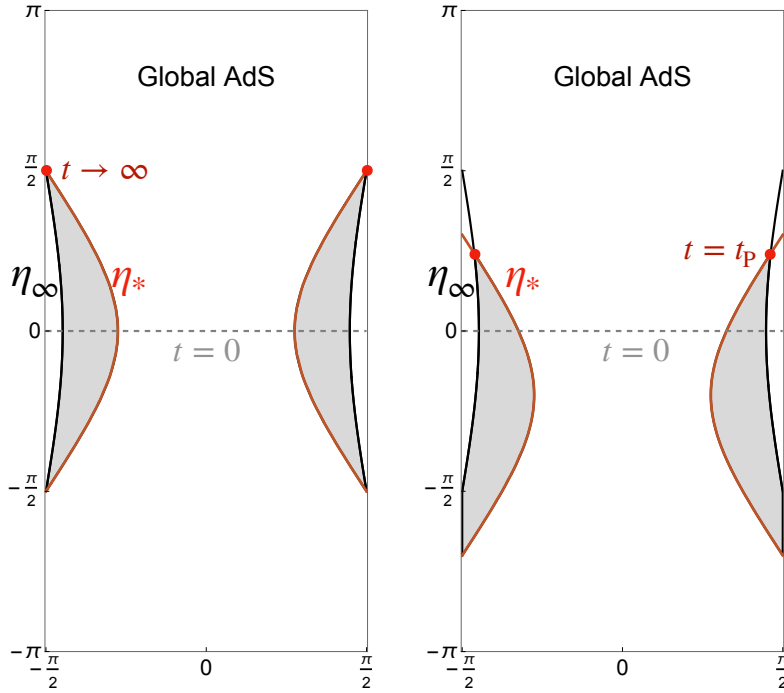


Figure 6.0.2: Geometry dual to a CFT_2 on dS_2 with a final-state projection (gray region). The angular direction of AdS_3 is suppressed. The solid black curve denotes the dS_2 boundary supporting the CFT. *Left*: projection imposed at $t_p \rightarrow \infty$ via a dS_2 end-of-the-world (EOW) brane (red) at fixed η_* . *Right*: projection carried out at a finite time t_p . The figure is from [3].

brane and imposing Neumann boundary conditions

$$K_{ij} - K h_{ij} + T h_{ij} = 0, \quad \text{with} \quad K = 2T, \quad (6.0.15)$$

where K_{ij} is the extrinsic curvature of the brane, K its trace, h_{ij} the induced metric, and T the brane tension. Since the ambient spacetime is AdS_3 , solutions to (6.0.15) are themselves maximally symmetric. Concretely (see, e.g., [110]),

$$R_{ij}[h] = h_{ij}(\varepsilon T^2 - 1), \quad R[h] = 2(\varepsilon T^2 - 1), \quad (6.0.16)$$

with $\varepsilon = +1$ for timelike branes and $\varepsilon = -1$ for spacelike branes. Thus the intrinsic worldvolume is either two-dimensional Minkowski ($|T| = 1$), de Sitter ($|T| > 1$), AdS ($|T| < 1$), or hyperbolic space ($\varepsilon = -1$). De Sitter EOW branes within $AdS/BCFT$ were introduced in [111] and applied to pseudo entropy in [112] in the context of black-hole final states [113].

Requiring the brane to intersect the dS_2 boundary as $t \rightarrow \infty$ admits two classes of embeddings: a timelike dS brane and a spacelike hyperbolic brane. Using the dS and

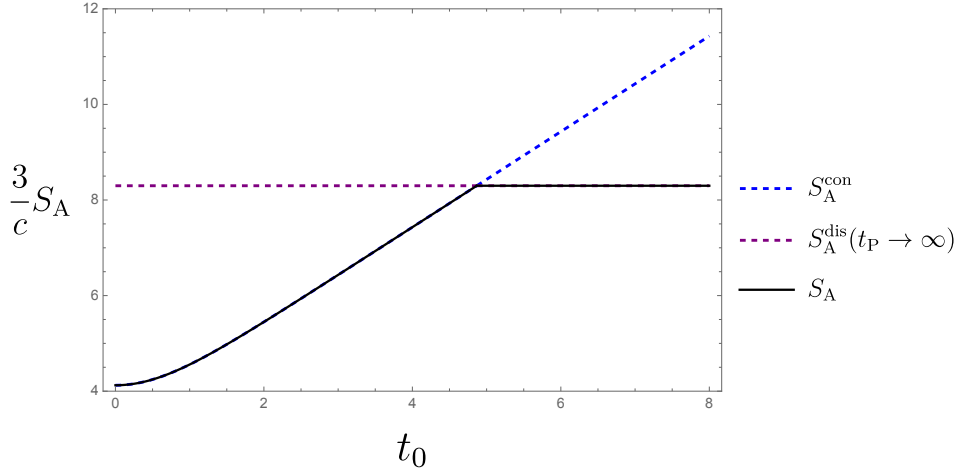


Figure 6.0.3: Time dependence of the connected (blue) and disconnected (purple) contributions, S_A^{con} and S_A^{dis} . For illustration we set $\eta_* = 3$, $\phi_2 - \phi_1 = \pi/5$, and $\epsilon = 10^{-2}$.

hyperbolic slicings of AdS_3 , their profiles in global AdS_3 take the form

$$\begin{aligned} \text{dS}_2 \text{ brane:} \quad & \cosh \rho \cos \tau = \pm \cosh \eta_*, \\ \text{H}_2 \text{ brane:} \quad & \cosh \rho \cos \tau = \sin \mu_*, \end{aligned} \quad (6.0.17)$$

where the constants η_* and μ_* are fixed by the brane tension and, equivalently, by the boundary condition encoded in the boundary state at $t = \infty$.

We first focus on the dS_2 EOW brane. Its mean curvature is

$$K|_{\eta=\eta_*} = -\frac{\partial_\eta \sqrt{-\gamma}}{\sqrt{-\gamma}} \Big|_{\eta=\eta_*} = -\frac{2 \cosh \eta_*}{\sinh \eta_*}, \quad (6.0.18)$$

implying $T = -\coth \eta_* < 0$ and intrinsic scalar curvature $R = 2/\sinh^2 \eta_*$. The boundary entropy associated with the projected state $|B\rangle$ is then [112]

$$S_{\text{bdy}} = \frac{c}{6} \log \sqrt{\frac{1-\mathcal{T}}{1+\mathcal{T}}} = \frac{c}{6} \log \sqrt{\frac{|\mathcal{T}|-1}{|\mathcal{T}|+1}} - i \frac{\pi c}{12} = -\frac{c}{6} \eta_* - i \frac{\pi c}{12}, \quad (6.0.19)$$

which reflects the choice of final-state boundary condition at $t \rightarrow \infty$.

The EOW brane introduces a disconnected extremal surface: a geodesic from a boundary endpoint $(t_0, \eta_\infty, \phi_i)$ of A to a point (t_*, η_*, ϕ_*) on the brane. Rotational symmetry fixes $\phi_* = \phi_i$, and t_* is determined by extremizing the length. For the dS EOW brane, using (6.0.6) one finds the disconnected contribution

$$S_A^{\text{dis}} = \frac{c}{3}(\eta_\infty - \eta_*) = \frac{c}{3} \log \frac{2}{\epsilon} - \frac{c}{3} \eta_*, \quad (6.0.20)$$

whose constant term coincides with the real part of (6.0.19).

A direct comparison between the connected contribution S_A^{con} and the disconnected one S_A^{dis} shows the following behavior: at early times, the connected geodesic dominates, so the holographic pseudo entropy for the interval A grows linearly with time. At late times, however, S_A^{dis} always becomes favorable and the pseudo entropy approaches a constant value. This phase transition is illustrated in figure 6.0.3. For $\eta_* = 0$, the transition occurs when

$$\cosh t_0 \cdot \sin \frac{|\phi_1 - \phi_2|}{2} = 1. \quad (6.0.21)$$

This condition captures the null geodesic in dS_2 (see figure 6.1.1) and coincides with (6.1.10).

For a hyperbolic EOW brane, the disconnected pseudo entropy (PE) follows from (6.0.6) as

$$S_A^{\text{dis}} = \frac{c}{3} \log \frac{2}{\epsilon} + i \frac{c}{3} \left(\frac{\pi}{2} - \mu_* \right), \quad (6.0.22)$$

where the parameter range is $-\frac{\pi}{2} \leq \mu_* \leq \frac{\pi}{2}$.

Before proceeding, let us note a potential interpretation: this holographic computation of pseudo entropy might in fact coincide with a genuine entanglement entropy. The reason is the periodicity of global AdS_3 in the time coordinate τ , which could imply a π -periodic evolution of the boundary state, $e^{-i\pi H} |B\rangle \propto |B\rangle$. While this is suggested by the classical geometry, establishing it at the quantum level—i.e., within the full CFT dynamics—remains nontrivial, and we leave a careful analysis to future work.

6.0.5 Holographic pseudo entropy under final state projection at $t = t_p$

For later use, we consider a CFT on dS_2 with a final-state projection imposed at a finite time $t = t_p$. In the $AdS_3/BCFT_2$ description, this corresponds to inserting the EOW brane at an earlier time, as shown in the right panel of figure 6.0.2. Evidently, S_A vanishes at $t = t_p$, so the time evolution of S_A displays a Page-curve-like profile, as sketched in the right panel of figure 6.0.4.

Because the dS slicing does not globally cover the brane, it is convenient to work in global coordinates (6.0.2). We label a boundary point on dS by $(\tau_0, \rho_\infty, \phi_0)$, for which the dS boundary obeys

$$\cosh \rho_\infty \cos \tau_0 = \text{constant} = \cosh \eta_\infty, \quad (6.0.23)$$

with η_∞ fixing the boundary location. Likewise, a point on the brane will be denoted (τ_*, ρ_*, ϕ_*) . Since the brane itself is a two-dimensional dS space, its profile can be parametrized as

$$\cosh \rho_* \cos (\tau_* + \tau_{\text{shift}}) = \text{constant} = \cosh \eta_*, \quad (6.0.24)$$

where η_* is determined by the brane tension. Fixing the projection time to $t = t_p$ (or $T = T_p$

on the boundary) then determines the shift parameter as

$$\tau_{\text{shift}} = \arccos\left(\frac{\cosh \eta_* \cos \tau_0}{\cosh \eta_\infty}\right) - \tau_p = \arccos\left(\frac{\cosh \eta_* \cos \tau_0}{\cosh \eta_\infty}\right) - \arctan(\tanh \eta_\infty \tan T_p), \quad (6.0.25)$$

using the boundary coordinate relations

$$\begin{cases} \cosh \rho = \sqrt{\cosh^2 \eta + \sinh^2 \eta \tan^2 T}, \\ \tan \tau = \tanh \eta \tan T. \end{cases} \quad (6.0.26)$$

Next, the geodesic distance D_{0b} between the dS boundary and the dS brane is

$$\cosh D_{0b} = \cos(\tau_0 - \tau_*) \cosh \rho_\infty \cosh \rho_* - \cos(\phi_0 - \phi_*) \sinh \rho_\infty \sinh \rho_*. \quad (6.0.27)$$

Extremizing with respect to the angular coordinate immediately yields $\phi_0 = \phi_*$ by symmetry. We then maximize along the timelike direction by solving

$$\frac{\partial D_{0b}}{\partial \tau_*} = 0, \quad \text{with} \quad \rho_* = \rho_*(\tau_*), \quad (6.0.28)$$

which reduces to

$$\cosh \eta_* \sinh \rho_\infty \tan(\tau_* + \tau_{\text{shift}}) - \frac{\sin(\tau_* + \tau_{\text{shift}})}{\cos(\tau_0 + \tau_{\text{shift}})} \cosh \rho_\infty \sqrt{\cosh^2 \eta_* - \cos^2(\tau_0 + \tau_{\text{shift}})} = 0. \quad (6.0.29)$$

This extremization is particularly transparent in global coordinates. A short algebra gives

$$\begin{aligned} \cos(\tau_* + \tau_{\text{shift}}) &= \cosh \eta_* \sqrt{\frac{\cosh^2 \rho_\infty \sin^2(\tau_0 + \tau_{\text{shift}}) - \sinh^2 \rho_\infty}{\cosh^2 \rho_\infty \sin^2(\tau_0 + \tau_{\text{shift}}) - \sinh^2 \rho_\infty \cosh^2 \eta_*}}, \\ \cosh \rho_* &= \sqrt{\frac{\cosh^2 \rho_\infty \sin^2(\tau_0 + \tau_{\text{shift}}) - \sinh^2 \rho_\infty \cosh^2 \eta_*}{\cosh^2 \rho_\infty \sin^2(\tau_0 + \tau_{\text{shift}}) - \sinh^2 \rho_\infty}}. \end{aligned} \quad (6.0.30)$$

Inserting these expressions into the geodesic distance yields the extremal separation between the boundary and the brane, i.e. the area of the disconnected HRT surface.

We now specialize to the cutoff boundary at $\eta_\infty \sim \frac{2}{\epsilon}$. In this regime, the boundary coordinates simplify to

$$\tau_0 = T_0, \quad \rho_\infty \approx \log\left(\frac{2}{\epsilon \cos T_0}\right), \quad \tau_{\text{shift}} = \frac{\pi}{2} - T_p. \quad (6.0.31)$$

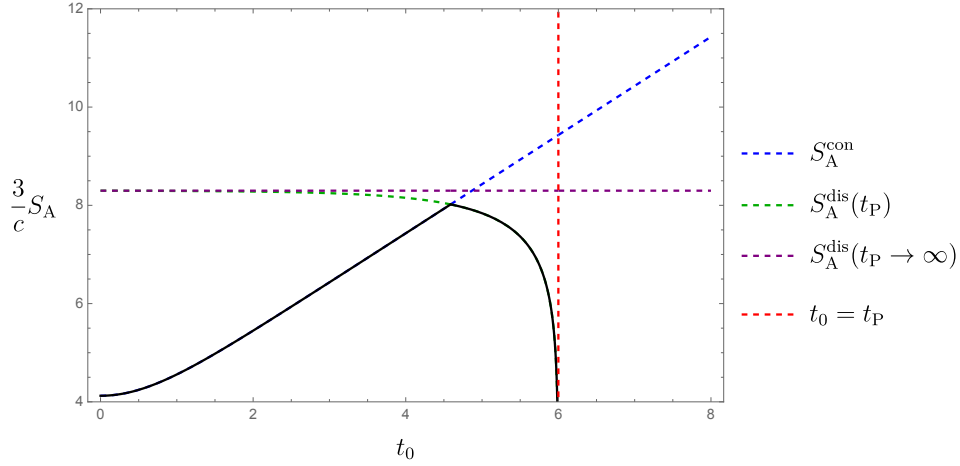


Figure 6.0.4: Time evolution of holographic entanglement entropy (denoted by the black curve) with respect to the global time t_0 . The projection time is chosen as $t_P = 6$ with $\eta_* = 3$. We set $\phi_2 - \phi_1 = \frac{\pi}{5}$, $\epsilon = \frac{1}{100}$ for this numerical plot. The plot is from [3].

Accordingly, the extremal point on the brane satisfies

$$\begin{aligned} \sin(T_P - \tau_*) &= \cosh \eta_* \sqrt{\frac{\sin^2(T_P - T_0)}{\cosh^2 \eta_* - \cos^2(T_P - T_0)}}, \\ \cosh \rho_* &= \sqrt{1 + \frac{\sinh^2 \eta_*}{\sin^2(T_P - T_0)}}, \end{aligned} \quad (6.0.32)$$

and the corresponding geodesic distance becomes

$$D_{0b} \approx \log \left(\frac{2 \cos(T_0 + \tau_{\text{shift}})}{\epsilon \cos T_0} e^{-\eta_*} \right). \quad (6.0.33)$$

Consequently, the contribution from the disconnected HRT surface reads

$$S_A^{\text{dis}} = \frac{2D_{0b}}{4G_N} = \frac{c}{3} \log \left(\frac{2 \sin(T_P - T_0)}{\epsilon \cos T_0} \right) - \frac{c}{3} \eta_*. \quad (6.0.34)$$

Rewriting in terms of the de Sitter time t , we finally obtain

$$S_A^{\text{dis}} = \frac{c}{3} \log \left[\frac{2}{\epsilon} \cdot \frac{\sinh t_P - \sinh t_0}{\cosh t_P} \right] - \frac{c}{3} \eta_*, \quad (6.0.35)$$

where we used $\sinh t_P = \tan T_P$. In the presence of a final-state projection, the holographic pseudo entropy is the minimum of the two candidates: $S_A = \min[S_A^{\text{con}}, S_A^{\text{dis}}]$.

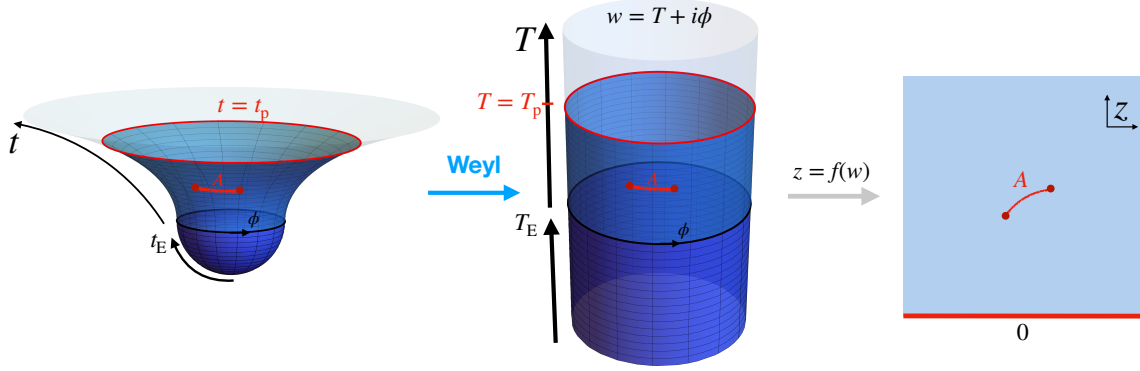


Figure 6.0.5: Conformal maps used in the computation of pseudo entropy. The projection at fixed time t_p is indicated by the red circle. The illustration is from [3].

6.0.6 Entanglement/Pseudo entropy of CFT_2 on dS_2

It is instructive to juxtapose the holographic results for entanglement entropy and pseudo entropy with a direct computation in two-dimensional CFT. In particular, following the standard replica construction for 2d CFTs [39, 40], the two-point function of twist operators on dS_2 is obtained via a Weyl rescaling:

$$\langle \sigma(t_0, \phi_1) \bar{\sigma}(t_0, \phi_2) \rangle_{\text{dS}_2} = (\cos T_0)^{2\Delta_n} \langle \sigma(T_0, \phi_1) \bar{\sigma}(T_0, \phi_2) \rangle_{\text{cylinder}}. \quad (6.0.36)$$

Here we introduce conformal coordinates (T, ϕ) through

$$ds_{\text{dS}_2}^2 = -dt^2 + \cosh^2 t d\phi^2 = \frac{1}{\cosh^2 T} (-dT^2 + d\phi^2), \quad (6.0.37)$$

with the change of variables between global and conformal time given by $\cosh t = \frac{1}{\cos T}$.

The Euclidean preparation of the Hartle-Hawking state corresponds to a half-sphere, whose metric is

$$ds_{\text{HH}}^2 = dt_E^2 + \cos^2 t_E d\phi^2 = \frac{1}{\cosh^2 T_E} (dT_E^2 + d\phi^2), \quad (6.0.38)$$

where the Euclidean conformal time T_E is defined via $\cos t_E = \frac{1}{\cosh T_E}$ and ranges $-\frac{\pi}{2} \leq t_E \leq 0$, $-\infty \leq T_E \leq 0$.

Case 1: Schwinger-Keldysh description without EOW

To begin, recall that for a single interval on the spatial cylinder (circumference 2π) the entanglement entropy is

$$S_A^{\text{cylinder}} = \frac{c}{3} \log \left(\frac{2}{\epsilon} \sin \frac{|\phi_1 - \phi_2|}{2} \right), \quad (6.0.39)$$

where ϵ is the UV cutoff and the 2π periodicity of ϕ has been used. Incorporating the Weyl factor that relates dS_2 to the cylinder contributes an additive term $\frac{\epsilon}{3} \log \cosh t_0$, yielding

$$S_A^{dS_2} = \frac{c}{3} \log \cosh t_0 + \frac{c}{3} \log \left(\frac{2}{\epsilon} \sin \frac{|\phi_1 - \phi_2|}{2} \right), \quad (6.0.40)$$

which coincides with the holographic result in eq. (6.0.12).

Case 2: Final projection at $t = t_p$

To evaluate the pseudo entropy of a CFT_2 on dS_2 , we follow the procedure sketched in figure 6.0.5. We start from the two-point function together with its Weyl rescaling:

$$\begin{aligned} & \langle B | e^{iH(T_p - T)} \sigma(T_0, \phi_1) \bar{\sigma}(T_0, \phi_2) | HH \rangle_{dS_2} \\ &= (\cos T_0)^{2\Delta_n} \langle B | e^{-iH(T_p - T)} \sigma(T_0, \phi_1) \bar{\sigma}(T_0, \phi_2) | 0 \rangle_{\text{cylinder}}, \end{aligned} \quad (6.0.41)$$

where $|B\rangle$ denotes a boundary (Cardy) state in BCFT [114], and $|HH\rangle$ is the Hartle–Hawking state on dS_2 . We first isolate the transition amplitude

$$\langle B | e^{-iH(T_p - T)} \sigma(T_0, \phi_1) \bar{\sigma}(T_0, \phi_2) | 0 \rangle_{\text{cylinder}}, \quad (6.0.42)$$

momentarily setting aside the Weyl factor.

The cylinder relevant for (6.0.42) consists of an infinite Euclidean segment $-\infty \leq T_E \leq 0$ glued to a finite Lorentzian piece $0 \leq T \leq T_p$. After Wick rotation, we equivalently work with the Euclidean cylinder $-\infty \leq T_E \leq T_p^E = iT_p$. Introducing the complex coordinate

$$w = T_E + i\phi, \quad (6.0.43)$$

we map the Euclidean cylinder to the upper half-plane via

$$\frac{e^w}{e^{T_p^E}} = \frac{z - i}{z + i} \quad \Leftrightarrow \quad z = f(w) = -i \frac{e^{w - T_p^E} + 1}{e^{w - T_p^E} - 1}, \quad (6.0.44)$$

as depicted in figure 6.0.5.

Using the method of images together with factorization, the transition amplitude splits into two pieces:

1. Connected contribution:

$$S_A^{\text{con, cyl}} = \frac{c}{6} \log \frac{|f(w_1) - f(w_2)|^2}{\epsilon^2 |f'(w_1)| |f'(w_2)|}. \quad (6.0.45)$$

With the insertions

$$w_1 = i(T_0 + \phi_1), \quad w_1^* = i(T_0 - \phi_1), \quad w_2 = i(T_0 + \phi_2), \quad w_2^* = i(T_0 - \phi_2), \quad (6.0.46)$$

this reproduces eq. (6.0.40), matching the holographic result for Case 1.

2. Disconnected contribution:

$$S_A^{\text{dis,cyl}} = \frac{c}{6} \log \frac{|f(w_1) - \bar{f}(\bar{w}_1)| |f(w_2) - \bar{f}(\bar{w}_2)|}{\epsilon^2 |f'(w_1)| |f'(w_2)|} + 2S_{\text{bdy}}. \quad (6.0.47)$$

Substituting (6.0.46) yields

$$S_A^{\text{dis,cyl}} = \frac{c}{6} \log \frac{-4}{\epsilon^2} \sin^2 (T_p - T_0) + 2S_{\text{bdy}} = \frac{c}{3} \log \left(\frac{2}{\epsilon} \sin (T_p - T_0) \right) + \frac{\pi i}{12} c + 2S_{\text{bdy}}. \quad (6.0.48)$$

Including the Weyl factor finally gives

$$S_A^{\text{dis,dS}} = \frac{c}{3} \log \left(\frac{2 \sin (T_p - T_0)}{\epsilon \cos T_0} \right) + \frac{\pi i}{12} c + 2S_{\text{bdy}}. \quad (6.0.49)$$

Combining the two branches reproduces the time dependence shown in figure 6.0.4. In particular, for post-selection at future infinity, $T_p = \frac{\pi}{2}$, the disconnected piece simplifies to

$$S_A^{\text{dis,dS}} = \frac{c}{3} \log \left(\frac{2}{\epsilon} \right) + \frac{\pi i}{12} c + 2S_{\text{bdy}}. \quad (6.0.50)$$

In summary, the boundary CFT₂ computation of pseudo entropy agrees with the holographic expressions for S_A^{dis} in (6.0.20) and (6.0.35), once the boundary entropy is identified via (6.0.19).

6.0.7 Holographic entanglement in higher dimensions

In this section we analyze holographic entanglement (pseudo) entropy in higher dimensions, obtained from the areas of extremal surfaces in $(d+1)$ -dimensional AdS. The de Sitter-sliced metric reads

$$ds^2 = d\eta^2 + \sinh^2 \eta (-dt^2 + \cosh^2 t (d\phi^2 + \sin^2 \phi d\Omega_{d-2}^2)). \quad (6.0.51)$$

Case 1: Schwinger–Keldysh without EOW

We begin with the connected contribution S_A^{con} . Working in global AdS coordinates,

$$ds^2 = d\rho^2 - \cosh^2 \rho d\tau^2 + \sinh^2 \rho (d\phi^2 + \sin^2 \phi d\Omega_{d-2}^2), \quad (6.0.52)$$

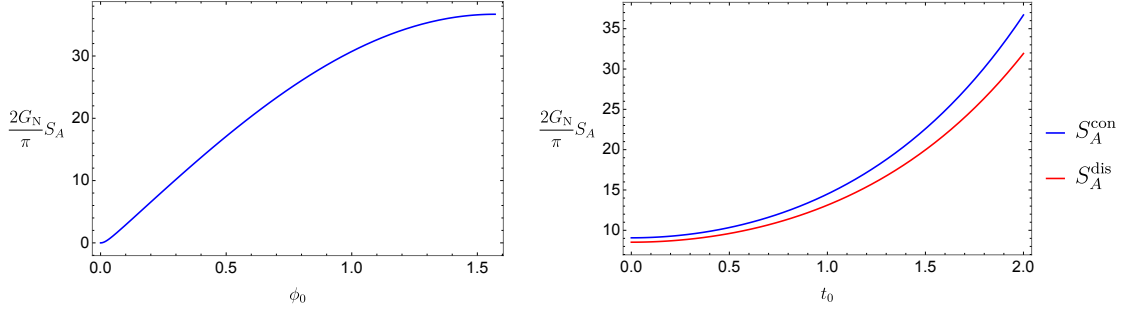


Figure 6.0.6: Holographic entanglement entropies S_A^{con} (6.0.58) and S_A^{dis} (6.0.63) from the Schwinger–Keldysh construction without an EOW brane. For both panels we set $d = 3$, $\phi_0 = \frac{\pi}{2}$, $\eta_\infty = 3$, and $\eta_* = 1$. The plots are from [3].

we take the boundary region A to be

$$0 \leq \phi \leq \phi_0, \quad \tau = \tau_0, \quad \rho = \rho_\infty. \quad (6.0.53)$$

In these coordinates an extremal surface may be parameterized as $\phi = \phi(\rho)$. Its area, divided by $4G_N$, gives

$$S_A^{\text{con}} = \frac{\text{Vol}(\text{S}^{d-2})}{4G_N} \int d\rho \sqrt{1 + \sinh^2 \rho \phi'^2} (\sinh \rho \sin \phi)^{d-2}, \quad (6.0.54)$$

where $\phi' \equiv \partial_\rho \phi$. Varying the functional yields the Euler-Lagrange equation

$$\phi'' - \frac{d-2}{\tan \phi \sinh^2 \rho} + \frac{d \phi'}{\tanh \rho} - \frac{(d-2)\phi'^2}{\tan \phi} + (d-1) \sinh \rho \cosh \rho \phi'^3 = 0. \quad (6.0.55)$$

A convenient first integral solving this equation is $\tanh \rho \cos \phi = C$, with C an integration constant. Enforcing the boundary data (6.0.53) fixes

$$\tanh \rho \cos \phi = \tanh \rho_\infty \cos \phi_0. \quad (6.0.56)$$

The connected entropy then becomes

$$S_A^{\text{con}} = \frac{\text{Vol}(\text{S}^{d-2})}{4G_N} \int_1^{\frac{\cosh \rho_\infty}{\cosh \rho_{\min}}} du (u^2 - 1)^{\frac{d-3}{2}}, \quad (6.0.57)$$

with $\rho_{\min}$ set by $\tanh \rho_{\min} = \tanh \rho_\infty \cos \phi_0$. Specializing to $d = 3$ gives

$$\begin{aligned} S_A^{\text{con}} &= \frac{\pi}{2G_N} \left(\frac{\cosh \rho_\infty}{\cosh \rho_{\min}} - 1 \right) = \frac{\pi}{2G_N} \left(\sqrt{\cosh^2 t_0 \sin^2 \phi_0 \sinh^2 \eta_\infty + 1} - 1 \right) \\ &\approx \frac{\pi}{2G_N} \left(\frac{\cosh t_0 \sin \phi_0}{\epsilon} - 1 \right), \end{aligned} \quad (6.0.58)$$

whose dependence on t_0 and ϕ_0 is illustrated in figure 6.0.6.

Case 2: Final projection at $t = \infty$

Next, we analyze the contribution from disconnected extremal surfaces that run between the dS boundary and a dS end-of-the-world (EOW) brane, which enters the holographic pseudo entropy. Concretely, we consider an extremal surface anchored at $(\eta_\infty, t_0, \phi_0, \Omega_0)$ on the boundary and at $(\eta_*, t_*, \phi_*, \Omega_0)$ on the brane, and then maximize its area by varying t_* . In AdS_3 the extremal surface obeys $t = \text{const}$ and $\phi = \text{const}$, but this simplification does not persist in higher dimensions. Motivated by the residual symmetry, we take $\phi_* = \frac{\pi}{2}$ along the surface¹. With this choice, the area functional becomes

$$S_A^{\text{dis}} = \frac{\text{Vol}(\text{S}^{d-2})}{4G_N} \int d\eta \sqrt{1 - \sinh^2 \eta t'^2} (\sinh \eta \cosh t)^{d-2}. \quad (6.0.59)$$

The corresponding Euler–Lagrange equation takes the form

$$\frac{(d-2) \tanh t}{\sinh^3 \eta} + \frac{d t'}{\tanh \eta \sinh \eta} - \frac{(d-2) \tanh t t'^2}{\sinh \eta} - (d-1) \cosh \eta t'^3 + \frac{t''}{\sinh \eta} = 0. \quad (6.0.60)$$

A useful particular solution—equivalent to $\tau = \text{const}$ in global AdS—is

$$\tanh \eta \sinh t = \tanh \eta_\infty \sinh t_0 = \tanh \eta_* \sinh t_*. \quad (6.0.61)$$

While more general solutions should exist (and, in principle, the area should be maximized over the endpoint on the EOW brane), this optimization is technically involved. In what follows we focus on the above tractable branch, which yields the minimal value of S_A^{dis} .

Substituting (6.0.61) back into the functional gives

$$\begin{aligned} S_A^{\text{dis}} &= \frac{\text{Vol}(\text{S}^{d-2})}{4G_N} \int d\eta \sqrt{1 - \sinh^2 \eta t'^2} (\sinh \eta \cosh t)^{d-2} \\ &= \frac{\text{Vol}(\text{S}^{d-2})}{4G_N} \int_{\sqrt{1+\cosh^2 t_* \sinh^2 \eta_*}}^{\sqrt{1+\cosh^2 t_0 \sinh^2 \eta_\infty}} dv (v^2 - 1)^{\frac{d-3}{2}}. \end{aligned} \quad (6.0.62)$$

In particular, for $d = 3$ we obtain

$$S_A^{\text{dis}} = \frac{\pi \sqrt{\tanh^2 \eta_\infty \sinh^2 t_0 + 1}}{2G_N} (\cosh \eta_\infty - \cosh \eta_*) \simeq \frac{\pi}{2G_N} \cosh t_0 \left(\frac{1}{\epsilon} - \cosh \eta_* \right), \quad (6.0.63)$$

and the behavior is displayed in figure 6.0.6. Unlike the AdS_3 setup, both S_A^{con} and S_A^{dis} grow

¹This restriction is justified at $\phi = \frac{\pi}{2}$ due to the $\mathbf{Z}_2$ symmetry $\theta \rightarrow \frac{\pi}{2} - \phi$. In general, the extremal surface can depend on (ϕ, t, η) .

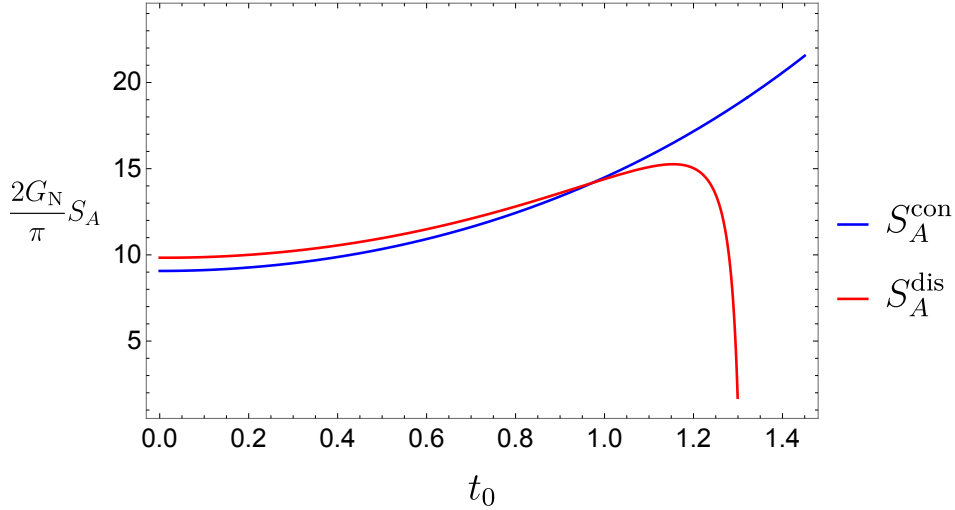


Figure 6.0.7: Time dependence of S_A^{con} (6.0.58) and S_A^{dis} (6.0.66). We set $\eta_\infty = 3$, $\phi_0 = \frac{\pi}{2}$, $\tau_{\text{shift}} = \frac{\pi}{6}$, and $\cosh \eta_c = 0.2$. At early times the connected saddle dominates. After a phase transition the disconnected branch becomes dominant and tends to zero at the projection point $t_0 = t_p$, where the two branes meet. The plot is from [3].

exponentially with t_0 at late times. Comparing (6.0.58) with (6.0.63), we always find

$$S_A^{\text{dis}} \leq S_A^{\text{con}}, \quad (6.0.64)$$

which follows from the elementary bound $\cosh X \geq 1$.

More on Case 2: Final projection at $t = t_p$

Next, we analyze a finite-time final-state projection. Concretely, we translate the projection brane along the global time direction as in Sec. 6.0.5. A backward shift by τ_{shift} places the brane at $\cosh \rho \cos(\tau + \tau_{\text{shift}}) = \cosh \eta_c$, where η_c is fixed by the tension of the de Sitter brane. The particular branch of extremal surfaces relevant here is the one already obtained in the previous subsection, eq. (6.0.61), specialized to $\phi_0 = \frac{\pi}{2}$. Consequently, the endpoint on the brane is determined by the intersection conditions $\tanh \eta_* \sinh t_* = \tanh \eta_\infty \sinh t_0$ and $\cosh \rho_* \cos(\tau_* + \tau_{\text{shift}}) = \cosh \eta_c$.

A short calculation gives

$$\cosh \eta_* = \frac{\cosh \eta_c}{\cos \tau_{\text{shift}} - \tanh \eta_\infty \sinh t_0 \sin \tau_{\text{shift}}}, \quad (6.0.65)$$

which, for $d = 3$, leads to

$$S_A^{\text{dis}} = \frac{\pi \sqrt{\tanh^2 \eta_\infty \sinh^2 t_0 + 1}}{2G_N} \left(\cosh \eta_\infty - \frac{\cosh \eta_c}{\cos \tau_{\text{shift}} - \tanh \eta_\infty \sinh t_0 \sin \tau_{\text{shift}}} \right). \quad (6.0.66)$$

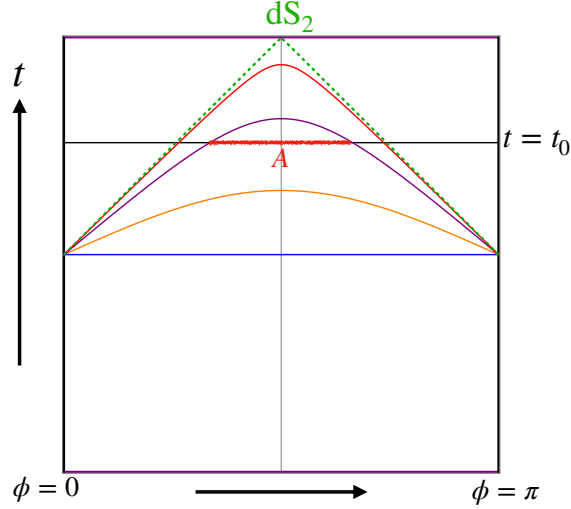


Figure 6.1.1: Penrose diagram of the dS_2 boundary. Colored curves indicate representative spacelike geodesics; the boundary null geodesics are shown as green dashed lines. The diagram is from [3].

Figure 6.0.7 illustrates the result: as in AdS_3 , the disconnected contribution vanishes at the projection time t_p .

6.1 Holography for a half dS without EOW brane (Case 1)

We now turn to our central objective: developing holography with a positive cosmological constant. To cast de Sitter holography in the familiar setting where bulk gravity is dual to a quantum system on a timelike boundary, we restrict the global dS_{d+1} geometry to the subset

$$0 \leq \theta \leq \theta_0. \quad (6.1.1)$$

In line with the holographic principle, gravity on this half of dS_{d+1} is expected to be dual to a boundary theory residing on the slice at $\theta = \theta_0$. We will assume $0 \leq \theta_0 \leq \frac{\pi}{2}$; the choice $\theta_0 = \frac{\pi}{2}$ corresponds to an exact half of the full de Sitter spacetime.

Our strategy is to diagnose this proposed duality via holographic entanglement entropy. For simplicity we will mainly analyze the three-dimensional case, dS_3 . There, the entropy S_A for an interval A is obtained from the length of the bulk geodesic anchored at the endpoints of A . This requirement already favors the range $0 \leq \theta_0 \leq \frac{\pi}{2}$: if $\theta_0 > \frac{\pi}{2}$, the geodesics that compute S_A no longer lie entirely within the truncated region, even for arbitrarily small subsystems.

6.1.1 Geodesic Length in dS_3

To set the stage for the computation of holographic entanglement entropy S_A , we first examine geodesics in dS_3 with metric

$$ds^2 = -dt^2 + \cosh^2 t (d\theta^2 + \sin^2 \theta d\phi^2). \quad (6.1.2)$$

Consider two bulk points

$$P_1 = (t_1, \theta_1, \phi_1), \quad P_2 = (t_2, \theta_2, \phi_2), \quad (6.1.3)$$

whose invariant separation is

$$D_{12} = (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2)) \cosh t_1 \cosh t_2 - \sinh t_1 \sinh t_2. \quad (6.1.4)$$

Specializing to two boundary points at $\theta = \theta_0$ and equal time $t = t_0$ with angular positions $\phi = \phi_1, \phi_2$, we obtain

$$\cos D_{12} = \left(\cos^2 \theta_0 + \sin^2 \theta_0 \cos(\phi_1 - \phi_2) \right) \cosh^2 t_0 - \sinh^2 t_0. \quad (6.1.5)$$

A spacelike geodesic between the endpoints exists precisely when

$$\cos^2 \theta_0 + \sin^2 \theta_0 \cos(\phi_1 - \phi_2) \geq 1 - \frac{2}{\cosh^2 t_0}. \quad (6.1.6)$$

The saturation locus traces the past light cone of future infinity on the boundary, as in figure 6.1.2. Denoting by $\Delta\phi_{\max}$ the value of $\Delta\phi = \phi_1 - \phi_2$ at which the bound is saturated, spacelike geodesics appear only for $\Delta\phi \leq \Delta\phi_{\max}$.

For generic distinct endpoints $\phi_1 \neq \phi_2$, the inequality is eventually violated at late times. Conversely, when

$$\cos(2\theta_0) \geq 1 - \frac{2}{\cosh^2 t_0}, \quad (6.1.7)$$

a spacelike connector exists for any choice of ϕ_1, ϕ_2 . Representative profiles of $D_{12}(\Delta\phi)$ at fixed t_0 are shown in figure 6.1.3.

If condition (6.1.6) fails, the geodesic length analytically continues to a complex value with fixed real part,

$$D_{12} = \pi + i \operatorname{arccosh} \left[\sinh^2 t - \left(\cos^2 \theta_0 + \sin^2 \theta_0 \cos(\phi_1 - \phi_2) \right) \cosh^2 t_0 \right]. \quad (6.1.8)$$

The imaginary contribution originates from a timelike segment, while the real part π

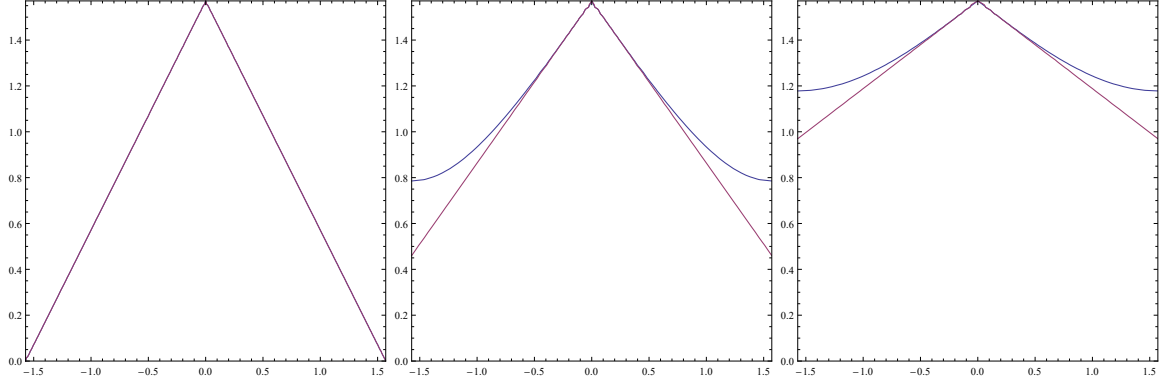


Figure 6.1.2: Boundaries beyond which no spacelike geodesic exists for $\theta = \frac{\pi}{2}$ (left), $\theta_0 = \frac{\pi}{4}$ (middle), and $\theta_0 = \frac{\pi}{8}$ (right). The horizontal and vertical axes are ϕ and T with ranges $-\pi/2 \leq \phi \leq \pi/2$ and $0 \leq T \leq \pi/2$. We consider geodesics joining $\phi_1 = -\phi_2$ at each time. Blue curves arise from the dS_3 geodesic condition via D_{12} , while red curves come from the dS_2 boundary metric; they coincide only when $\theta = \frac{\pi}{2}$. These plots are from [3].

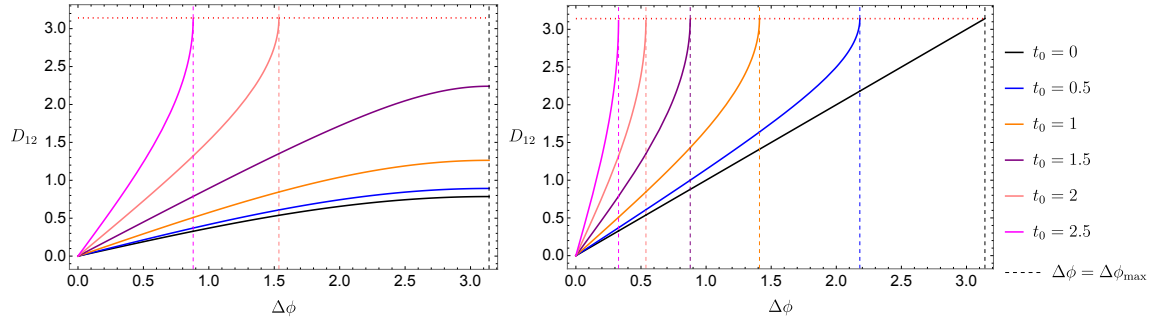


Figure 6.1.3: Spacelike geodesic length D_{12} . Left: D_{12} versus $\Delta\phi = |\phi_1 - \phi_2|$ at fixed times t_0 for $\theta_0 = \frac{\pi}{8}$. For sufficiently late times, spacelike geodesics disappear beyond a critical $\Delta\phi$. Right: D_{12} as a function of $\Delta\phi$ in the maximal half, $\theta_0 = \frac{\pi}{2}$. The horizontal red dotted line indicates $D_{12} = \pi$; vertical dashed lines mark the critical size $\Delta\phi_{\max}$. These plots are from [3].

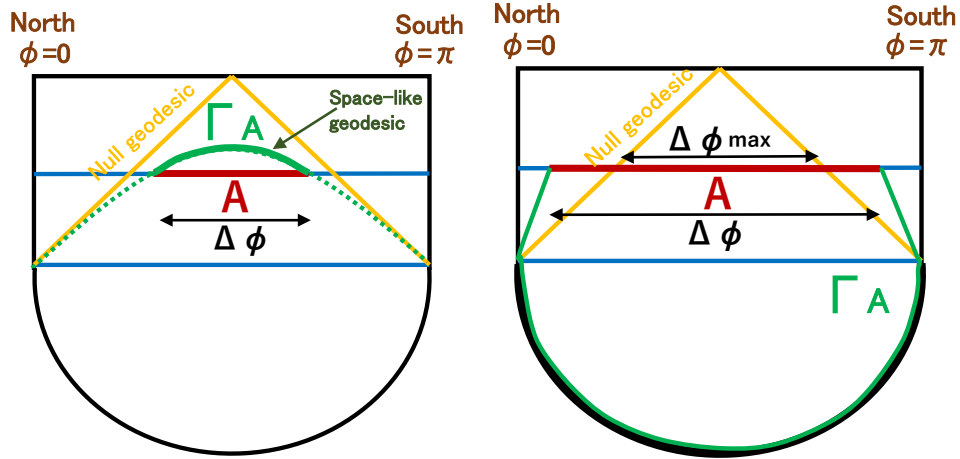


Figure 6.1.4: Schematic geodesics Γ_A on the boundary dS_2 for $\theta = \frac{\pi}{2}$. Left: spacelike connector for $\Delta\phi \leq \Delta\phi_{\max}$. Right: for $\Delta\phi > \Delta\phi_{\max}$, the relevant saddle combines a timelike and a spacelike segment, consistent with (6.1.8). The figure is from [3].

corresponds to a Euclidean (hemispherical) piece (see, e.g., [30]).

In the maximal case $\theta_0 = \frac{\pi}{2}$, the $\mathbb{Z}_2$ symmetry $\theta \rightarrow \pi - \theta$ ensures that geodesics connecting boundary points at $\theta = \theta_0$ lie entirely within the boundary dS_2 (figure 6.1.1). At $t_0 = 0$ one finds

$$D_{12} = \Delta\phi, \quad (6.1.9)$$

whereas for $t_0 \neq 0$ there is a critical range $\Delta\phi > \Delta\phi_{\max}$ beyond which no spacelike connector exists. The threshold is

$$\Delta\phi_{\max} = 2 \left(\frac{\pi}{2} - \arccos \left(\frac{1}{\cosh t_0} \right) \right) = \pi - 2 \arctan [\sinh t_0]. \quad (6.1.10)$$

This curve reproduces the boundary of the past light cone of a point at $t = \infty$ and matches the condition (6.0.21) encountered in the AdS/CFT analysis (see figure 6.1.2, left). The corresponding behavior of D_{12} is depicted in figure 6.1.3 (right).

For $\Delta\phi \leq \Delta\phi_{\max}$, the spacelike connector in the boundary dS_2 is sketched in figure 6.1.4 (left). When $\Delta\phi > \Delta\phi_{\max}$, the relevant saddle effectively decomposes into a timelike segment and a spacelike one, yielding the complex length in (6.1.8); this is illustrated in figure 6.1.4 (right). This transition at $\Delta\phi = \Delta\phi_{\max}$ is a distinctive feature of the de Sitter setup and has no direct analogue in the AdS case.

6.1.2 Holographic Entanglement Entropy and Violation of Subadditivity

We now evaluate the holographic entanglement entropy in Case 1. In this setting, the interval entropy S_A is computed from the connected geodesic according to

$$S_A = \frac{D_{12}}{4G_N}, \quad (6.1.11)$$

with D_{12} as derived in the previous subsection.

Fix a time slice $t = t_0$ and take the boundary interval A to be $\phi_1 \leq \phi \leq \phi_2$. For a generic boundary location θ_0 , the geodesic formula yields a real, positive S_A only when $\Delta\phi \leq \Delta\phi_{\max}$, as illustrated in figure 6.1.3. At the endpoint $\Delta\phi = \Delta\phi_{\max}$, one finds $D_{12} = \pi$, i.e. half the de Sitter horizon length, so that $S_A = \frac{1}{2}S_{\text{ds}}$.

Viewed as a function of time, S_A is well-defined for any $\Delta\phi$ during the interval $0 \leq t \leq t_{\max}$, where $t_{\max}$ is determined by saturation of (6.1.7):

$$t_{\max} = \text{arccosh} \left(\frac{1}{\sin \theta_0} \right) = \log \left[\cot \left(\frac{\theta_0}{2} \right) \right]. \quad (6.1.12)$$

Beyond this time, $t > t_{\max}$, two issues arise: for $\Delta\phi > \Delta\phi_{\max}$ the geodesic length becomes complex, and even within the allowed range $\Delta\phi \leq \Delta\phi_{\max}$ the function $S_A(\Delta\phi)$ turns convex, signaling a breakdown of (strong) subadditivity. Indeed, the second derivative of the geodesic length is

$$\frac{d^2 D_{12}}{d\Delta\phi d\Delta\phi} = \frac{\sqrt{2} \cosh t \sin \theta_0 \sin \left(\frac{\theta_0}{2} \right) (\cosh^2 t \sin^2 \theta_0 - 1)}{(1 + \cos D_{12})^{3/2}}, \quad (6.1.13)$$

which becomes positive precisely for $t > t_{\max}$. In terms of the conformal time T (defined by $\cosh t \equiv \frac{1}{\cos T}$), the corresponding threshold is

$$T_{\text{crt}} = \frac{\pi}{2} - \theta_0, \quad (6.1.14)$$

namely the intersection of the boundary $\theta = \theta_0$ with the cosmological horizon at $T = \frac{\pi}{2} \pm \theta$.

The convexity directly contradicts subadditivity. Since translational invariance implies $S_A = S(L(A)) \equiv S(y)$, convexity $\partial_y^2 S(y) > 0$ yields $S(2y) > 2S(y)$, violating $S_A + S_B \geq S_{AB}$. Consequently, strong subadditivity $S_{AB} + S_{BC} \geq S_{ABC} + S_B$ also fails. Figure 6.1.5 displays the regions where subadditivity holds; we will return to their interpretation below. This behavior stands in sharp contrast with AdS/CFT, in which strong subadditivity is guaranteed [55, 59].

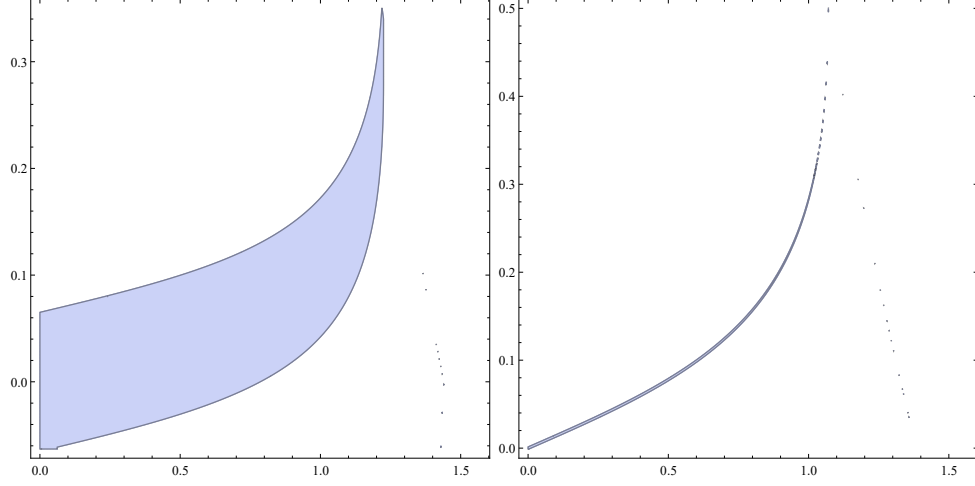


Figure 6.1.5: Regions of subadditivity for $\theta_0 = \frac{\pi}{4}$ (left) and $\theta_0 = \frac{\pi}{2} - 0.01$ (right). We take two adjacent intervals A and B with endpoints $(\phi, T) = (0, T_0) \rightarrow (1/2, T_0 + \Delta T)$ and $(1/2, T_0 + \Delta T) \rightarrow (1, T_0)$, respectively, where T is defined via $\cos T = \frac{1}{\cosh t}$. Shaded domains indicate $S_A + S_B - S_{AB} \geq 0$. The plot is from [3].

6.1.3 Static Time Slices

To clarify the origin of the subadditivity violation, we first examine the maximally extended case $\theta_0 = \frac{\pi}{2}$. In this setting it is convenient to introduce static coordinates (χ, s) on dS_2 via

$$\cos \chi = \cos \phi \cosh t, \quad \cosh s = \frac{\sin \phi}{\sqrt{\frac{1}{\cosh^2 t} - \cos^2 \phi}}, \quad (6.1.15)$$

which bring the metric to

$$ds^2 = d\chi^2 - \sin^2 \chi ds^2. \quad (6.1.16)$$

For two boundary points (χ_1, s_0) and (χ_2, s_0) at equal static time s_0 , the corresponding bulk dS_3 geodesic has length $D_{12} = |\chi_1 - \chi_2|$. Consequently, on a constant- s slice the holographic entanglement entropy is linear in the interval size $|A|$ and exactly saturates subadditivity. By contrast, any other choice of time slicing—including constant t —breaks strong subadditivity, since the effective interval size shrinks as

$$|A| = \int_{\chi_1}^{\chi_2} d\chi \sqrt{1 - \sin^2 \chi \left(\frac{ds}{d\chi} \right)^2} \leq \chi_2 - \chi_1 = D_{12}. \quad (6.1.17)$$

Thus, for $\theta_0 = \frac{\pi}{2}$, the static slice $s = \text{const.}$ is the unique time foliation compatible with a boundary Hilbert space interpretation for a unitary theory.

A complementary viewpoint comes from the bulk causal structure. Consider the Wheeler-DeWitt (WDW) patch in dS_3 associated with a generic boundary slice on dS_2 . Because all

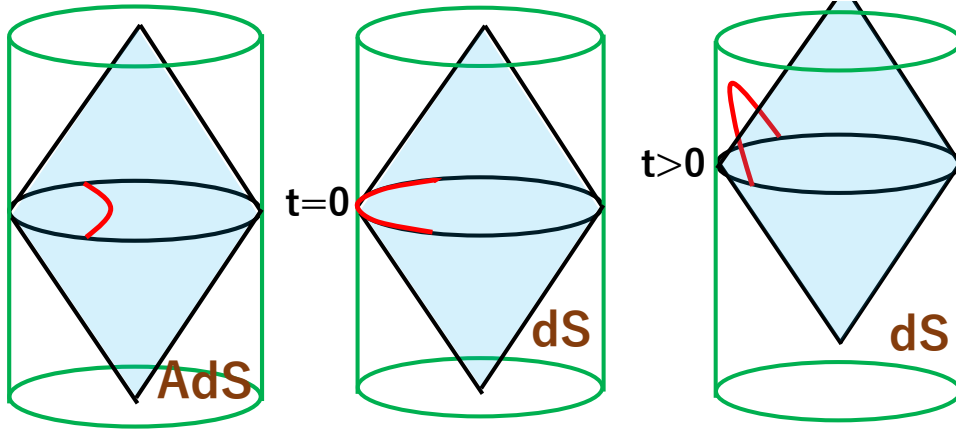


Figure 6.1.6: Schematic Wheeler-DeWitt patches (blue) and geodesics (red) in AdS_3 (left), dS_3 at $t = 0$ (middle), and dS_3 at $t > 0$ (right). The figures are from [3].

spacelike geodesics between boundary points lie entirely on the boundary dS_2 , they fall outside the WDW patch unless the slice is $s = \text{const.}$ (see figure 6.1.6, in contrast with the AdS case). In a sensible holographic dual, bulk representatives of boundary data at a given time should lie within the corresponding WDW patch, reinforcing the special role of the static slicing.

For general θ_0 , the conclusion is less sharp but remains qualitatively the same. This is illustrated in the left panel of figure 6.1.5: along constant- t (equivalently constant- T) slices, the region where subadditivity holds is progressively pushed towards the future and eventually disappears. As $\theta_0 \rightarrow \frac{\pi}{2}$, the allowed region narrows significantly, as shown in the right panel of figure 6.1.5.

6.1.4 Time slices and dual Hilbert space

Guided by the preceding analysis of holographic entanglement entropy, we now turn to its dual interpretation. For definiteness, we specialize to the maximal boundary choice $\theta_0 = \frac{\pi}{2}$. From the perspective of a putative unitary boundary theory, the discussion in the previous subsection selects the static foliation by constant s (on the dS_2 boundary) as the admissible notion of time. Exploiting the $SO(2, 1)$ isometries of dS_2 , one can obtain a family of equivalent time slices by boosting and rotating the canonical slice at $t = 0$; these are illustrated in the left panel of figure 6.1.7. We will refer to any slice obtained from $t = 0$ by an $SO(2, 1)$ transformation as a *nice slice*. By construction, this family includes the constant- s slices defined in (6.1.16) but excludes the constant- t slices with $t = t_0 > 0$. It is natural to identify all nice slices with a single Hilbert space $\mathcal{H}_{\text{dS}}$ describing the dual to the half dS_3 region. Since the extremal curve Γ_A that computes boundary entanglement lies entirely within the boundary dS_2 , the dual state appears maximally entangled, and the size of $\mathcal{H}_{\text{dS}}$ should be characterized by the

de Sitter entropy,

$$S_{dS} = \frac{2\pi}{4G_N}. \quad (6.1.18)$$

This viewpoint parallels the surface/state duality of [83], which associates to each codimension-two spacelike surface Σ a state $|\Phi_\Sigma\rangle$. The present setup differs, however, because we introduce an actual time-like boundary by restricting to half of dS_3 , with boundary dS_2 located at $\theta = \theta_0$. Even so, one expects a meaningful dual description already prior to adding such a boundary. Our analysis suggests a refinement of the surface/state proposal: for the duality to behave consistently, the extremal surface Γ_A should lie within the Wheeler-DeWitt patch of Σ .

A useful corollary of this picture concerns constant- t slices with $t_0 > 0$. As sketched in the right panel of figure 6.1.7, any such slice can be covered by multiple nice slices. Because the proper length of a constant- t slice grows exponentially, $\sim e^{t_0}$, the number of nice slices required to cover it also grows, and their associated Hilbert spaces overlap extensively (each repeatedly covering the $t = 0$ slice). Consequently, a constant- t slice—other than $t = 0$ —overcounts degrees of freedom and cannot be regarded as defining a single well-posed Hilbert space. This aligns with the earlier observation that subadditivity is violated on constant- t slices: partitioning such a slice into segments does not correspond to a factorization of a boundary Hilbert space.

This stands in contrast with local quantum field theory, where any two Cauchy slices spanning the same causal diamond yield equivalent Hilbert-space factorizations. In the present de Sitter holographic setting, even if an interval A on a constant- t slice and the corresponding Γ_A on a nice slice occupy the same causal diamond (cf. the left panel of figure 6.1.4), only the latter provides a faithful subsystem of $\mathcal{H}_{dS}$ with a consistent entropic interpretation.

6.1.5 Non-locality and Holographic Entanglement Entropy

It is natural to ask why only the nice slices (constant s slices) furnish a well-defined Hilbert space. While we do not claim a decisive resolution, the evidence points to the nonlocal character of the dual quantum system as the essential ingredient. First, introducing a time-like boundary to realize half dS_3 implements a finite cutoff for the boundary theory. Second, the volume-law entanglement seen on nice slices coincides with what is typical for vacuum states of strongly nonlocal field theories [115, 116]. In fact, within replica-based definitions one can even encounter growth that outstrips a volume law, $S_A \sim |A|^p$ with $p > 1$, signaling a breakdown of subadditivity. Concretely, [115] analyzed a d -dimensional nonlocal free scalar whose action is

$$S = \int dx^d \phi(x) e^{(-\partial_x^2)^q} \phi(x), \quad (6.1.19)$$

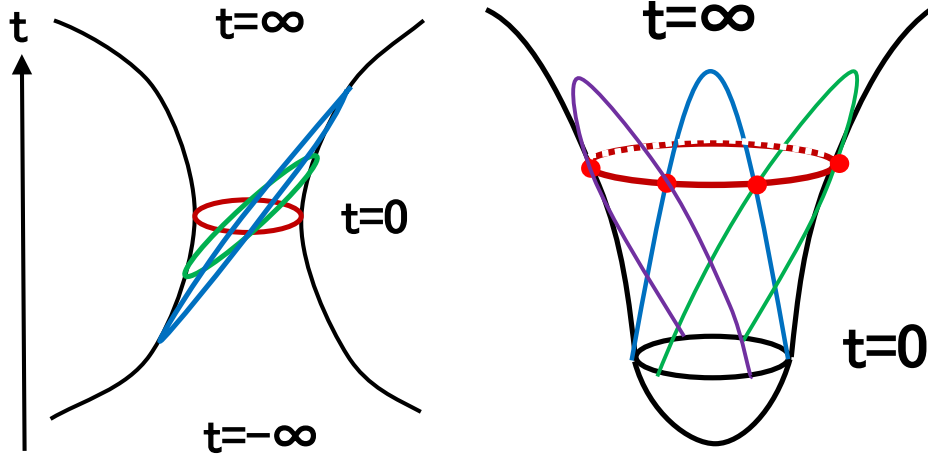


Figure 6.1.7: Schematic “nice” time slices (constant s) on the boundary dS_2 that support a consistent Hilbert-space interpretation. Left: nice slices generated from $t = 0$ by $SO(2, 1)$ transformations. Right: a constant- t slice (red) decomposed into multiple constant- s slices (purple, blue, green); the overlaps indicate overcounting and the absence of a clean factorization. The figures are from [3].

and found a UV divergence of the form

$$S_A \propto \left(\frac{L_A}{\epsilon} \right)^{d-2+2q}, \quad (6.1.20)$$

with L_A the linear scale of A . For $q > \frac{1}{2}$ this exceeds a volume law. Note that replica/path-integral-based “entropies” are not guaranteed to satisfy subadditivity, unlike the von Neumann entropy of a density matrix on a Hilbert space, which does.

In our half dS_3 setup, it is therefore natural to expect an effectively nonlocal boundary dynamics, which in turn explains the failure of subadditivity on generic choices of time slice. As θ_0 decreases, such nonlocal effects are softened, and the region of slices obeying subadditivity broadens, consistent with figure 6.1.5.

From this perspective, even though a clean Hilbert-space interpretation breaks down for generic slices (including constant- t ones), one can still define an “entropy” via the replica method—at the price of allowing violations of strong subadditivity. In such nonlocal theories, a standard Hamiltonian formulation is unavailable because the Euclidean action contains higher-than-second-order time derivatives. This pattern echoes finite-cutoff AdS/CFT: integrating out UV modes generically yields a highly nonlocal dual, and boosted strong subadditivity is indeed known to fail (see, e.g., [64]). A notable novelty here is that, in the half de Sitter construction, both subadditivity and strong subadditivity already fail for unboosted configurations.

If this nonlocal interpretation is correct, S_A should remain definable even when A is so

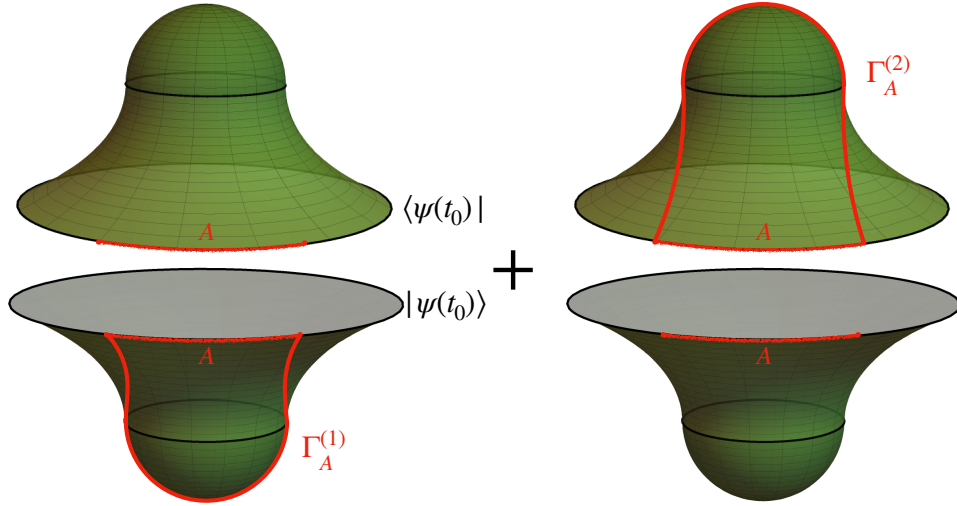


Figure 6.1.8: Sketches of calculation of holographic entanglement entropy in the Schwinger-Keldysh geometry. The sum of the contributions associated with two extremal surfaces $\Gamma_A^{(1)}, \Gamma_A^{(2)}$ gives rise to the real entanglement entropy. The illustrations are from [3].

large that no spacelike geodesic connects its endpoints. Applying the geodesic prescription (6.1.11) in that regime produces a complex length,

$$D_{12} = \pi + i \operatorname{arccosh} [\sinh^2 t - \cos(2\theta_0) \cosh^2 t_0] . \quad (6.1.21)$$

(See [30, 93] for discussions of complex geodesics in de Sitter.) As illustrated in the right panel of figure 6.1.4, the real part arises from an Euclidean (semispherical) segment, whereas the imaginary part comes from a timelike geodesic in Lorentzian dS. In a replica computation of holographic entanglement (for a covariant derivation in AdS/CFT, see [108]), one finds two candidate extremal surfaces—one in the bra geometry and one in the ket—shown in figure 6.1.8, paralleling the construction in [117].

The combined contribution to the replica partition function is

$$Z_{\text{tot}}^{(n)} = e^{(1-n)S} + e^{(1-n)S^*}, \quad (6.1.22)$$

with the entropy extracted via

$$S_A = \lim_{n \rightarrow 1} \frac{1}{1-n} \log Z_{\text{tot}}^{(n)} \simeq \frac{S + S^*}{2} = \operatorname{Re}[S]. \quad (6.1.23)$$

Accordingly, the physical answer is the real part,

$$S_A = \frac{\pi}{4G_N} = \frac{1}{2} S_{\text{dS}}, \quad (6.1.24)$$

where $S_{\text{dS}} = \frac{2\pi}{4G_N}$ denotes the de Sitter entropy.

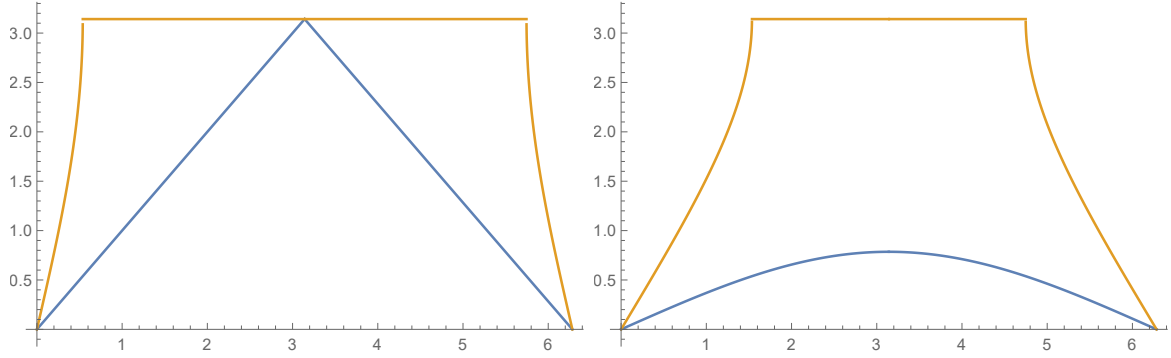


Figure 6.1.9: Plots of holographic entanglement entropy S_A on a constant t time slice based on the prescription (6.1.23) for $d = 2$. We plotted S_A as a function of $\Delta\phi$ in the range $0 \leq \Delta\phi \leq 2\pi$ for $\theta_0 = \frac{\pi}{2}$ (left) and $\theta_0 = \frac{\pi}{8}$ (right). The blue and orange curves correspond to $t = 0$ and $t = 2$. Note that we have the critical time $t_{\max} = 0$ for $\theta_0 = \frac{\pi}{2}$ and $t_{\max} \simeq 1.61$ for $\theta_0 = \frac{\pi}{8}$. These plots are from [3].

Putting this together, S_A increases monotonically for $0 \leq \Delta\phi \leq \Delta\phi_{\max}$ and then saturates at $\frac{1}{2}S_{\text{dS}}$ when $\Delta\phi \geq \Delta\phi_{\max}$. For $\Delta\phi > \pi$, the usual holographic identity $S_A = S_{A^c}$ holds, with A^c the complement of A . Representative curves are shown in figure 6.1.9.

6.1.6 Overcounting of de Sitter Hilbert Space

In earlier studies of holographic entanglement entropy for a single interval, it was found that the entropy is bounded above by one half of the dS entropy, *i.e.*,

$$S_A \leq \frac{1}{2}S_{\text{dS}}, \quad (6.1.25)$$

as illustrated in figure 6.1.9. This is natural because the associated bulk dual occupies only half of the de Sitter spacetime. One can extend the same method to a subsystem A consisting of multiple disconnected intervals. Since the computation follows the standard prescription [118], we omit the details. The key point, however, is that for a union of several intervals the holographic entanglement entropy is not constrained by S_{dS} at late times, because the sum of lengths of disconnected geodesics can grow without bound.

Specializing to the boundary placed at $\theta = \theta_0 = \frac{\pi}{2}$, eq. (6.1.25) shows that a single interval attains the maximal value $\frac{1}{2}S_{\text{dS}}$ when its length equals the critical size $\Delta\phi_{\max}$. Note that the maximum size $\Delta\phi_{\max}(t_0)$ decreases to zero as the boundary time t_0 increases. Consider then a slice at $t = t_0$ partitioned into $2N$ identical intervals with $\frac{\pi}{N} \geq \Delta\phi_{\max}(t_0)$. An explicit example with $N = 3$ is depicted in figure 6.1.10. For a subsystem made of N such intervals, the holographic entanglement entropy becomes

$$S_{A_1 \cup A_2 \cup \dots \cup A_N} = N \times S_{A_i} = \frac{N}{2}S_{\text{dS}} \geq S_{\text{dS}}. \quad (6.1.26)$$

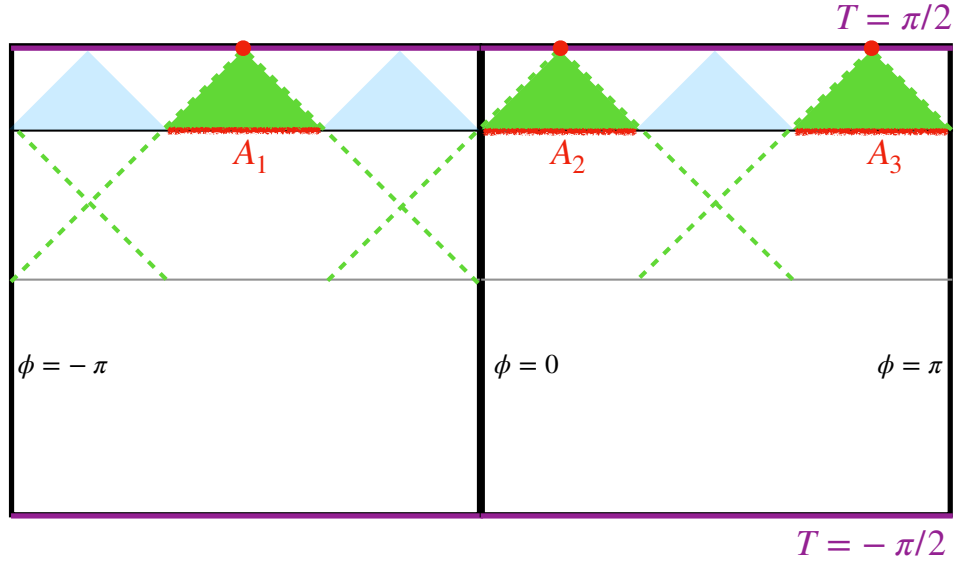


Figure 6.1.10: Penrose diagram of dS_2 , representing the dS boundary located at $\theta = \theta_0 = \frac{\pi}{2}$ in the dS_3 bulk. Green dashed segments indicate the null limit of geodesics. On the time slice $T = T_0$ (equivalently $t = t_0$), the system is partitioned into six segments. The diagram is cited from [3].

As $N \rightarrow \infty$ while $\lim_{t_0 \rightarrow \infty} \Delta\phi_{\max} \rightarrow 0$, this quantity can diverge.

At first sight, this seems to contradict the de Sitter entropy bound, given the expectation of a finite-dimensional Hilbert space. However, as suggested by the breakdown of (strong) subadditivity, a generic time slice need not correspond to a pure state in a single de Sitter Hilbert space. The unbounded growth in eq. (6.1.26) is better interpreted as an overcounting of the de Sitter Hilbert-space dimension. As indicated in figure 6.1.10, each critical interval A_i at $t = t_0$ may be viewed as a maximally entangled state within an individual de Sitter Hilbert space. Evolving back to $t = 0$, these single Hilbert spaces overlap along the slice. A related tensor-network picture with overlapping qubits was discussed in [119], though their analysis concerns bulk dS rather than our boundary perspective in figure 6.1.10. Moreover, [103] argued that the static patch of de Sitter is described by a type II_1 algebra; such an algebraic structure may be relevant to understanding the overcounting phenomenon.

6.1.7 Higher dimensional half de Sitter space

It is straightforward to generalize our half dS_3 analysis to a half dS_{d+1} with $d \geq 3$. Below we focus on $\theta_0 = \frac{\pi}{2}$ in the global dS_{d+1} . The time-like boundary is dS_d with global metric

$$ds^2 = -dt^2 + \cosh^2 t (d\phi^2 + \sin^2 \phi d\Omega_{d-2}^2), \quad (6.1.27)$$

where $d\Omega_{d-1}^2$ denotes the metric on the unit sphere S^{d-2} . As in the $d = 2$ case, we argue that gravity on a half dS_{d+1} is dual to a non-local field theory on dS_d .

To probe this duality we compute the holographic entanglement entropy, defined by the area of the extremal surface Γ_A anchored on the boundary of the subsystem A .

We take A to be a disk on the dS_d boundary:

$$t = t_0, \quad 0 \leq \phi \leq \phi_1. \quad (6.1.28)$$

For this choice, the profile of the extremal surface Γ_A ending on $\phi = \phi_1$ of (6.1.28) is

$$\frac{\cos \phi}{\tanh t} = \frac{1 + L^2}{1 - L^2}. \quad (6.1.29)$$

This follows from mapping the extremal surface

$$t^2 - \sum_{i=1}^{d-1} (x_i)^2 = L^2, \quad (6.1.30)$$

in Poincaré dS_d with metric

$$ds^2 = \frac{-dt^2 + \sum_{i=1}^{d-1} dx_i^2}{t^2}. \quad (6.1.31)$$

The constant L is fixed by the choice of A through

$$\frac{1 + L^2}{1 - L^2} = \frac{\cos \phi_1}{\tanh t_0}. \quad (6.1.32)$$

Figure 6.1.11 shows the profile of (6.1.29).

The $(d - 1)$ -dimensional surface Γ_A wraps S^{d-2} . Its induced metric is

$$ds^2 = \frac{4L^2(1 + L^2)^2}{((1 + L^2)^2 - (1 - L^2)^2 \cos^2 \phi)^2} d\phi^2 + \frac{(1 + L^2)^2 \sin^2 \phi}{(1 + L^2)^2 - (1 - L^2)^2 \cos^2 \phi} d\Omega_{d-2}^2. \quad (6.1.33)$$

The surface is space-like for $0 < L^2 \leq 1$, becomes light-like at $L^2 = 0$, and time-like when $L^2 < 0$.

The area is

$$A(\Gamma_A) = \text{Vol}(S^{d-2}) \int_0^{\phi_1} d\phi \frac{2L(1 + L^2)^{d-1} (\sin \phi)^{d-2}}{((1 + L^2)^2 - (1 - L^2)^2 \cos^2 \phi)^{\frac{d}{2}}}. \quad (6.1.34)$$

Viewing this as a function of ϕ_1 and t_0 , we write $A(\phi_1, t_0)$. At $t_0 = 0$ and $\phi_1 = \frac{\pi}{2}$,

$$A\left(\frac{\pi}{2}, 0\right) = \frac{1}{2} \text{Vol}(S^{d-1}). \quad (6.1.35)$$

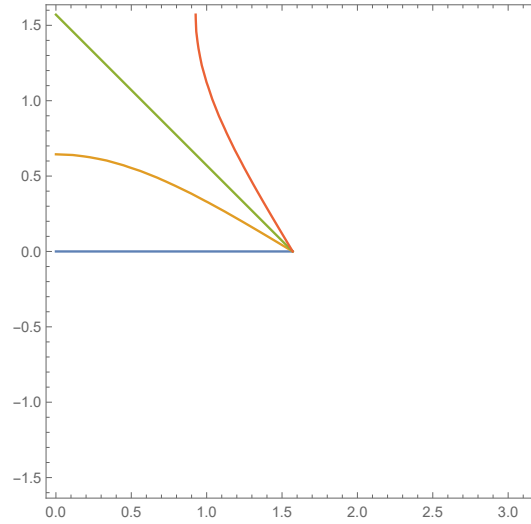


Figure 6.1.11: Profiles of the extremal surfaces (6.1.29) in global dS. The horizontal and vertical axes are ϕ and T , respectively. Blue, yellow, green, and red curves correspond to $L^2 = 1, 1/4, 0$, and $-1/4$, respectively. The figure is from [3].

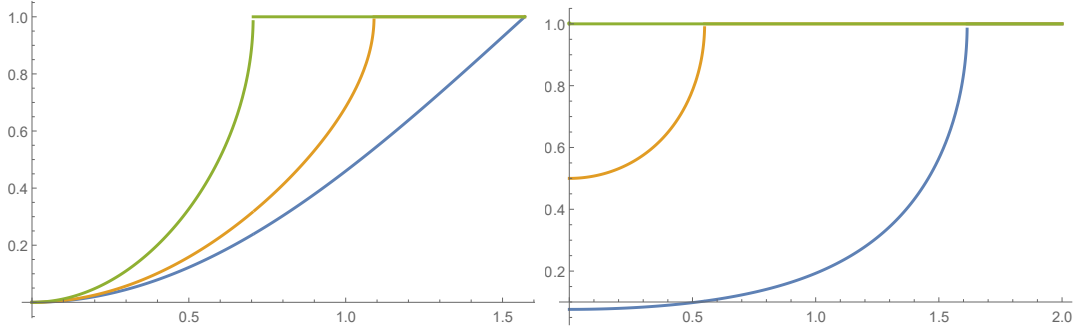


Figure 6.1.12: Holographic entanglement entropy for a half dS₄ ($d = 3$) in case 1. Left: $\frac{1}{2\pi}A(\Gamma_A)$ as a function of ϕ_1 at $t = 0$ (blue), $t = 1/2$ (orange), and $t = 1$ (green). Right: $\frac{1}{2\pi}A(\phi_1, t_0)$ versus t_0 for $\phi_1 = \pi/8$ (blue), $\phi_1 = \pi/3$ (orange), and $\phi_1 = \pi/2$ (green). The plots are from [3].

Hence the corresponding holographic entanglement entropy equals $\frac{1}{2}S_{dS}$. The behavior of $A(\phi_1, t_0)$ is shown in figure 6.1.12: generally it increases monotonically in t_0 and ϕ_1 and saturates at the maximal value $\frac{1}{2}S_{dS}$. When $L^2 < 0$ (so Γ_A is time-like), we instead pick the alternative extremal surface that passes through and wraps a hemisphere in the Euclidean region, as in the right panel of figure 6.1.4. By the same reasoning as in (6.1.24), the holographic entanglement entropy is given by the real part of $A(\Gamma_A)$, namely $\frac{1}{2}S_{dS}$.

6.2 Holography for a half dS with EOW brane (Case 2)

Now we consider a second holographic setup for a half de Sitter space that includes an EOW brane (case 2). We treat the future boundary $t = \infty$ as an EOW brane and impose Neumann boundary conditions for gravity there. On the dual field-theory side, this corresponds to a

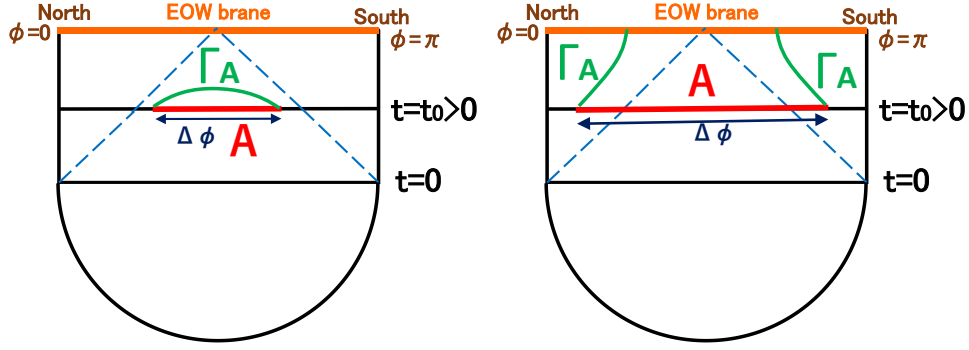


Figure 6.2.1: Geodesic configurations in dS_2 used to compute the holographic pseudo entropy in a half dS_3 . For a small subsystem A with $\Delta\phi \leq \Delta\phi_{\max}$, the relevant geodesic Γ_A is space-like and connected (left). When A is larger, $\Delta\phi > \Delta\phi_{\max}$, Γ_A consists of two disconnected time-like geodesics that end on the EOW brane (right). The illustrations are from [3].

final-state projection, in parallel with the AdS/CFT discussion in section 6.0.4.

To extract properties of the dual geometry, we evaluate the holographic entanglement entropy from the area of an extremal surface. Because the field-theory description involves a final-state projection, the appropriate quantity is the *holographic pseudo entropy* [31]. For clarity, we focus on the maximal case $\theta_0 = \frac{\pi}{2}$ in dS_{d+1} . We argue that gravity on half dS_{d+1} with an EOW brane at $t = t_p$ is dual to a non-local field theory on the dS_d boundary equipped with a final-state projection at $t = t_p$. A new feature here is that a space-like extremal surface may end on the EOW brane.

We first examine the situation in which the EOW brane sits at future infinity, $t_p \rightarrow \infty$, which corresponds in the dual to a final-state projection at $t = \infty$.

Let us analyze the pseudo entropy for $d = 2$. If the interval A is small, $\Delta\phi < \Delta\phi_{\max}$, the pseudo entropy S_A equals the length of the connected space-like geodesic D_{12}^{con} , as in case 1. For $\Delta\phi > \Delta\phi_{\max}$, when no connected space-like geodesic exists, S_A is computed from two disconnected geodesics connecting each endpoint of A to the EOW brane, as in figure 6.2.1.

Note that for $\Delta\phi > \Delta\phi_{\max}$ those disconnected geodesics are time-like, implying that the holographic pseudo entropy is purely imaginary. This is the key difference from case 1. In case 2, because Γ_A can terminate on the EOW brane, the real part of S_A from this contribution (right panel of figure 6.2.1) is smaller than that of the connected configuration in case 1 (right panel of figure 6.1.4), whose real part is positive.

In summary, for $d = 2$ the behavior of S_A agrees with section 6.1.5 when $\Delta\phi < \Delta\phi_{\max}$, whereas for $\Delta\phi > \Delta\phi_{\max}$ we have $\text{Re}[S_A] = 0$:

$$\begin{aligned} \text{For } 0 \leq \Delta\phi < \Delta\phi_{\max} : \quad S_A &= \frac{D_{12}^{\text{con}}}{4G_N}, \\ \text{For } \Delta\phi > \Delta\phi_{\max} : \quad \text{Re}[S_A] &= 0. \end{aligned} \tag{6.2.1}$$

The extension to higher dimensions follows directly by using $A(\phi_1, t_0)$ from case 1 in

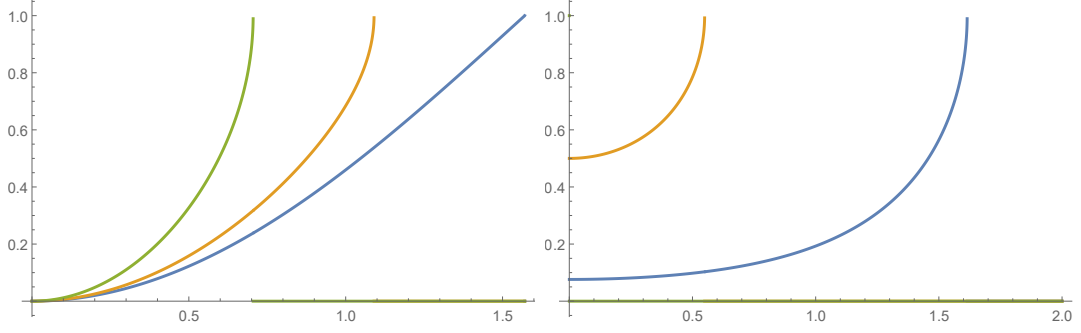


Figure 6.2.2: Holographic pseudo entropy for a half dS_4 ($d = 3$) in case 2. Left: $\frac{1}{2\pi} A(\Gamma_A)$ as a function of ϕ_1 at $t = 0$ (blue), $t = 1/2$ (orange), and $t = 1$ (green). Right: $\frac{1}{2\pi} A(\Gamma_A)$ as a function of t_0 for $\phi_1 = \pi/8$ (blue), $\phi_1 = \pi/3$ (orange), $\phi_1 = \pi/2$ (green). The plots are from [3].

section 6.1.7. When ϕ_1 is smaller than the saturation value of case 1, S_A matches the case-1 result; for larger ϕ_1 we obtain $\text{Re}[S_A] = 0$. An explicit plot for $d = 3$ is shown in figure 6.2.2.

We now interpret these results in the dual field theory on dS_d . For $\Delta\phi < \Delta\phi_{\text{max}}$, S_A coincides with case 1 and thus exhibits the same violation of subadditivity, attributable to non-locality. For $\Delta\phi > \Delta\phi_{\text{max}}$, a case-2 specific phenomenon emerges: the real part of S_A vanishes.

This transition at $\Delta\phi = \Delta\phi_{\text{max}}$ is natural from the field-theory viewpoint. Recall that the pseudo entropy is defined from the transition matrix, which depends on two states $|\psi_1\rangle$ and $|\psi_2\rangle$. In our setup, $|\psi_1\rangle$ is obtained by forward time evolution from the Hartle-Hawking state, whereas $|\psi_2\rangle$ is prepared by backward evolution from the final state. Phenomenological studies [112, 120, 121] indicate that the real part of the pseudo entropy typically measures entanglement in an intermediate state between the two, while the meaning of the imaginary part is less clear. Here the final state is implemented by a boundary condition at $t = t_p = \infty$, and analyses of space-like EOW branes [111] suggest that such a state has no real-space entanglement, as for standard boundary states [74]. On a slice $t = t_0$, a disentangling effect can thus arise from $t = \infty$: two points separated by more than $2(t_p - t_0)$ (assuming lightlike propagation on dS_d) do not exchange signals. This threshold coincides with $\Delta\phi_{\text{max}}$, explaining why $\text{Re}[S_A]$ vanishes for $\Delta\phi > \Delta\phi_{\text{max}}$. For $\Delta\phi < \Delta\phi_{\text{max}}$, the EOW brane is causally irrelevant and the result matches case 1.

A similar analysis applies when the final projection is inserted at finite time $t = t_p$. Evidently $S_A = 0$ at $t = t_p$. Moreover, the real part of S_A vanishes once Γ_A becomes light-like. This can occur already for $t < t_p$ if A is large, after which an imaginary part turns on (since Γ_A becomes time-like) and persists until $t = t_p$, as illustrated in figure 6.2.3. It is noteworthy that this Page-curve-like evolution of $\text{Re} S_A$ is qualitatively similar to the AdS/CFT behavior in eq. (6.0.35).

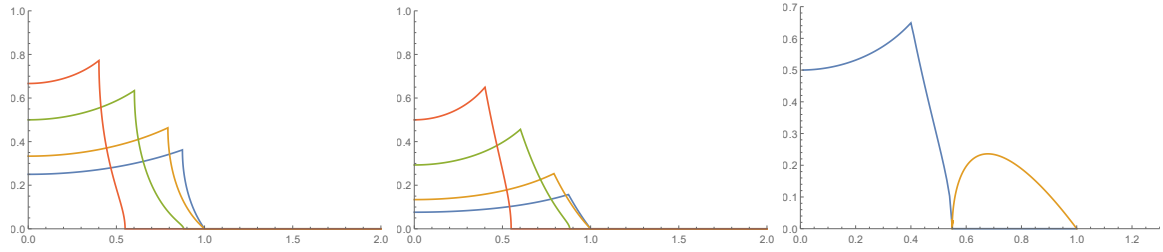


Figure 6.2.3: Holographic pseudo entropy for half dS_3 ($d = 2$, left) and half dS_4 ($d = 3$, middle and right) in case 2 with an EOW brane at $t_p = 1$. We plot $\frac{1}{\pi}A(\Gamma_A)$ for $d = 2$ (left) and $\frac{1}{2\pi}A(\Gamma_A)$ for $d = 3$ (middle) versus t_0 , choosing $\phi_1 = \frac{\pi}{8}$ (blue), $\phi_1 = \frac{\pi}{6}$ (orange), $\phi_1 = \frac{\pi}{4}$ (green), and $\phi_1 = \frac{\pi}{3}$ (red). Right: $\text{Re}\left[\frac{1}{2\pi}A(\Gamma_A)\right]$ (blue) and $\text{Im}\left[\frac{1}{2\pi}A(\Gamma_A)\right]$ (orange) versus t_0 for $\phi_1 = \frac{\pi}{3}$ at $d = 3$. The real part drops to zero when the geodesic becomes null, and both parts vanish for $t > t_p (= 1)$. The plots are from [3].

Chapter 7

Holography in FLRW spacetime

7.1 Holography in FLRW spacetime

7.1.1 Basics of the flat FLRW universe

In this subsection we collect basic facts about the flat FLRW universe.

Geometry. Friedmann–Lemaître–Robertson–Walker (FLRW) space is a solution of the Einstein equation under the condition of isotropy and homogeneity of matter distribution. Consider a flat FLRW spacetime in $(d+1)$ dimensions:

$$ds^2 = a^2(\eta) \left[-d\eta^2 + dr^2 + r^2 d\Omega_{d-1}^2 \right] , \quad (7.1.1)$$

where η denotes conformal time, $a(\eta)$ is the scale factor, and $d\Omega_{d-1}^2$ is the line element on the $(d-1)$ -dimensional unit sphere. Although the analysis below mainly employs η , many physical quantities are more transparently expressed in terms of the physical time t defined by $dt = a(\eta)d\eta$, in which case

$$ds^2 = -dt^2 + a^2(\eta(t)) \left[dr^2 + r^2 d\Omega_{d-1}^2 \right] . \quad (7.1.2)$$

For example, the Hubble parameter $H = \dot{a}/a$ (dot: t -derivative) characterizes the expansion rate, and the cosmic acceleration is measured by $\ddot{a}$.

Apparent horizon. In our holographic analysis the cosmological horizon is central. In a flat FLRW universe, the apparent horizon coincides with the sphere expanding at the speed of light. By the Hubble law, its physical radius R_H is

$$R_H = H^{-1} . \quad (7.1.3)$$

Equivalently, its comoving location is fixed by

$$r = r_H \quad \text{with} \quad r_H = \frac{1}{aH} = \frac{a}{a'}, \quad (7.1.4)$$

where a prime denotes differentiation with respect to η .

Benchmark examples. For illustration, consider a universe filled with an ideal fluid of constant equation-of-state parameter $w = p/\varepsilon$ with $\varepsilon > 0$. The Einstein equations imply the power-law evolution

$$a(\eta) = R \left(\frac{\eta}{\gamma} \right)^\gamma \quad \text{with} \quad \gamma = \frac{2}{d(1+w) - 2}, \quad (7.1.5)$$

where R sets the overall physical scale. We set $R = 1$ below. Choosing the expanding branch, we take $0 < \eta < \infty$ for $\gamma > 0$ and $-\infty < \eta < 0$ for $\gamma < 0$. Here $\gamma > 0$ corresponds to deceleration $\ddot{a} < 0$, while $\gamma < 0$ corresponds to acceleration $\ddot{a} > 0$. The flat chart of de Sitter space is recovered for $w = -1$, i.e. $\gamma = -1$, saturating the null energy condition $w \geq -1$.¹ For the power law (7.1.5), the comoving apparent-horizon radius is

$$r_H = \frac{\eta}{\gamma}. \quad (7.1.6)$$

Summary for three-dimensional universe. Specializing to a three-dimensional universe $d = 2$ (our holographic setting), the conformal-time metric is

$$ds^2 = a^2(\eta) \left[-d\eta^2 + dr^2 + r^2 d\phi^2 \right], \quad (7.1.7)$$

where ϕ is the S^1 angle, $0 \leq \phi < 2\pi$. For constant w , one simply has $\gamma = w^{-1}$, so the scale factor and the comoving apparent-horizon radius read

$$a(\eta) = (w\eta)^{\frac{1}{w}}, \quad r_H = \frac{a}{a'} = w\eta. \quad (7.1.8)$$

7.2 de Sitter holography

As the simplest FLRW background, we begin with de Sitter spacetime. A key advantage is that geodesic lengths admit a closed form by embedding the geometry into Minkowski space.

Three-dimensional de Sitter space with unit radius is defined by the embedding

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 = 1, \quad (7.2.1)$$

¹In our setup the null and weak energy conditions are equivalent because we assume $\varepsilon > 0$.

into four-dimensional flat spacetime,

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2. \quad (7.2.2)$$

Under this embedding, the geodesic distance $D > 0$ between two spatially separated points X_A and X_B is

$$\cos D = -X_A^0 X_B^0 + X_A^1 X_B^1 + X_A^2 X_B^2 + X_A^3 X_B^3, \quad (7.2.3)$$

which furnishes a convenient formula for holographic entanglement entropy. In what follows we work in the flat chart of de Sitter space and examine SSA in two holographic setups analogous to static patch holography [84] and half-de Sitter holography [3].

7.2.1 Holographic scenarios

Flat universe. The flat chart of de Sitter spacetime is given by

$$X^0 = \frac{\eta^2 - 1 - r^2}{2\eta}, \quad X^1 = \frac{\eta^2 + 1 - r^2}{2\eta}, \quad X^2 = \frac{r \cos \phi}{\eta}, \quad X^3 = \frac{r \sin \phi}{\eta} \quad (7.2.4)$$

with coordinate ranges

$$-\infty < \eta < 0, \quad 0 \leq r < \infty, \quad 0 \leq \phi < 2\pi. \quad (7.2.5)$$

The corresponding metric is

$$ds^2 = \frac{-d\eta^2 + dr^2 + r^2 d\phi^2}{\eta^2}. \quad (7.2.6)$$

For the observer at $r = 0$, the cosmological event horizon lies at $r = |\eta|$. The geodesic distance D between (η_A, r_A, ϕ_A) and (η_B, r_B, ϕ_B) reads

$$\cos D = 1 - \frac{-(\eta_A - \eta_B)^2 + r_A^2 + r_B^2 - 2r_A r_B \cos(\phi_A - \phi_B)}{2\eta_A \eta_B}. \quad (7.2.7)$$

Holographic setups. Holographic scenarios for expanding universes fall into two classes. With a spacelike boundary (e.g. dS/CFT [18]), bulk and boundary lack a common time, and the putative dual is non-unitary. With a timelike boundary (e.g. static patch [84] and half-de Sitter [3]), bulk and boundary share time, so the prospective dual is expected to be unitary. Since our aim is to identify setups compatible with a standard holographic dual, we focus primarily on the latter, while noting that our discussion can also be adapted to the former. See the final paragraph of this subsection as well.

With this motivation, we study holographic setups with a (1+1)-dimensional timelike

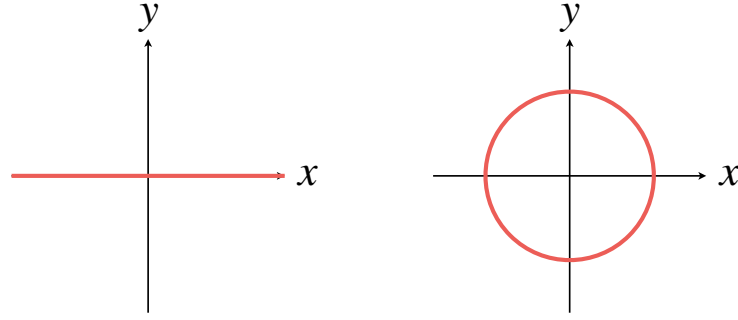


Figure 7.2.1: On a constant-time slice, the boundary location is shown in red for half holography (left) and horizon-type holography (right). The illustration is from [4].

boundary that runs along the conformal-time direction η and one spatial direction. On a constant-time slice $\eta = \eta_*$, the boundary is specified by embedding a one-dimensional curve into the (r, ϕ) plane. In this paper we focus on two symmetric embeddings (see also Fig. 7.2.1):

1. Half holography.

Motivated by half de Sitter holography [3], the boundary at $\eta = \eta_*$ is placed on the straight line

$$y = 0 \quad \text{with} \quad (x, y) := (r \cos \phi, r \sin \phi), \quad (7.2.8)$$

which bisects the constant- η slice. We refer to this as *half holography*. Owing to translational invariance of the flat universe, the boundary enjoys a shift symmetry in x , so the second derivative of the entanglement entropy provides a test of (strong) subadditivity.²

2. Horizon-type holography.

Inspired by static patch holography [84], the boundary at $\eta = \eta_*$ is taken to be the circle

$$r = \lambda |\eta_*|, \quad 0 \leq \phi < 2\pi. \quad (7.2.9)$$

Here λ is a convenient dimensionless parameter. The boundary coincides with the cosmological horizon at $\lambda = 1$, sits inside the horizon for $0 < \lambda < 1$, and outside for $\lambda > 1$. We call this the *horizon-type* setup. By rotational invariance, the boundary has a shift symmetry in ϕ , and again the second derivative of the entanglement entropy can be used to test (strong) subadditivity.

Remarks on holography with a spacelike boundary. The same analysis applies to scenarios with a spacelike boundary. For example, consider a boundary on the spacelike

²This is different from [3] since they work in global dS coordinates. Actually, we work in the flat slicing.

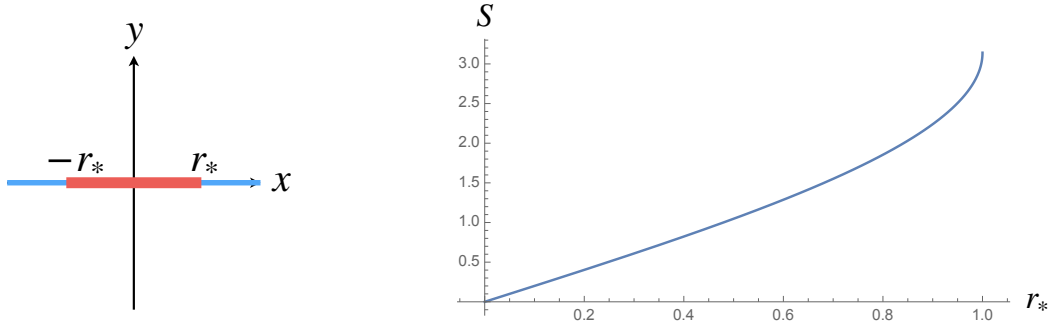


Figure 7.2.2: Left: Subsystem (red) and its complement (blue) for the half-holography setup. Right: Holographic entanglement entropy with the convention $4G = 1$. The figures are from [4].

surface $\eta = \eta_*$, as in dS/CFT [18]. If we identify y with Euclidean time, the constant (Euclidean) time slice reproduces the half-holography slice; identifying the radial coordinate r with Euclidean time reproduces the horizon-type slice. Since the holographic entanglement entropy depends only on the endpoints of the subsystem at a fixed time, the following computations carry over without modification.³ The same observation extends to our treatments of open/closed de Sitter holography and FLRW holography.

7.2.2 Half holography

We now evaluate the holographic entanglement entropy in the half-holography setup. Consider a subsystem defined on the interval

$$x \in (-r_*, r_*), \quad y = 0. \quad (7.2.10)$$

See Fig. 7.2.2. Without loss of generality, by exploiting the dilatation symmetry of de Sitter space we set $\eta_* = -1$ for clarity of presentation. Using Eq. (7.2.7) together with the RT prescription, we obtain

$$S = \frac{1}{2G} \arcsin r_*, \quad (7.2.11)$$

which remains real as long as the interval lies within the cosmological horizon, i.e. $r_* \leq 1$. Once the interval extends beyond the event horizon⁴ (i.e. $r_* > 1$ [3]), the entropy becomes complex. The second derivative with respect to the half-size parameter r_* is

$$\frac{\partial^2 S}{\partial r_*^2} = \frac{1}{2G} \frac{r_*}{(1 - r_*^2)^{\frac{3}{2}}}, \quad (7.2.12)$$

³While the RT prescription appears to yield the same value as in the timelike case, the notion of the state differs because the path-integral direction is different. It would be interesting to analyze this distinction in detail.

⁴In de Sitter space, the event horizon coincides with the apparent horizon.

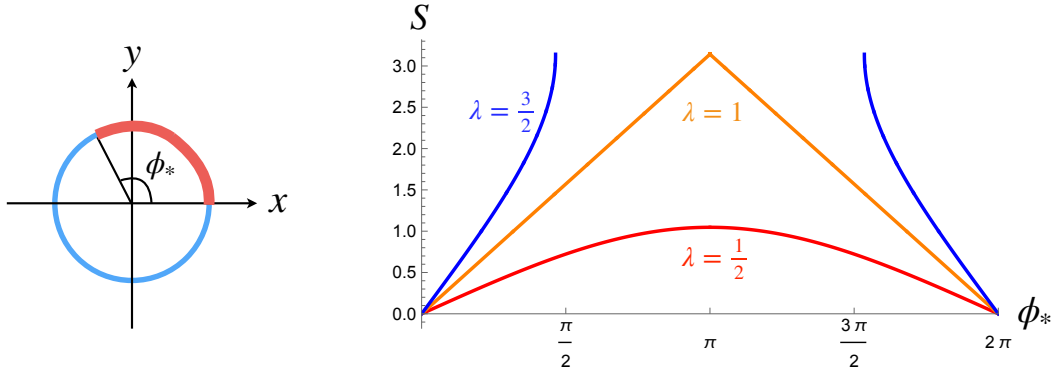


Figure 7.2.3: Left: Subsystem (red) and complement (blue) for the horizon-type holography. Right: Holographic entanglement entropy for $\lambda = \frac{1}{2}, 1, \frac{3}{2}$ with $4G = 1$. The figures are from [4].

which is strictly positive for intervals inside the horizon. Therefore, the half-holography scenario violates (strong) subadditivity. Plots of S and its second derivative are shown in Fig. 7.2.2.

7.2.3 Horizon-type holography

Next we analyze the horizon-type setup. Take the subsystem to be an arc

$$r = \lambda|\eta_*|, \quad \phi \in (0, \phi_*). \quad (7.2.13)$$

See Fig. 7.2.3. As before we choose $\eta_* = -1$. Recall that λ measures the boundary radius relative to the horizon. From Eq. (7.2.7) we find

$$S = \frac{1}{4G} \arccos \left[1 - 2\lambda^2 \sin^2 \frac{\phi_*}{2} \right], \quad (7.2.14)$$

which is real for all ϕ_* provided the boundary lies on or inside the horizon, $\lambda \leq 1$. If the boundary is outside the horizon, $\lambda > 1$, the entropy becomes complex whenever $\sin \frac{\phi_*}{2} > \frac{1}{\lambda}$. The second derivative with respect to ϕ_* is

$$\frac{\partial^2 S}{\partial \phi_*^2} = -\frac{(1 - \lambda^2)}{8G} \frac{\lambda \sin \frac{\phi_*}{2}}{(1 - \lambda^2 \sin^2 \frac{\phi_*}{2})^{\frac{3}{2}}}. \quad (7.2.15)$$

Thus, it is negative for all ϕ_* when the boundary sits inside the cosmological horizon, while it is purely imaginary outside the horizon, as first emphasized in [3]. We conclude that the horizon-type setup is compatible with (strong) subadditivity as long as the boundary is on or within the horizon; placing it outside leads to a violation. See also Fig. 7.2.3.

We now comment on the special case $\lambda = 1$, in which the boundary coincides with the

cosmological horizon. In this situation the holographic entanglement entropy is directly proportional to the size of the subsystem (or its complement):

$$S = \begin{cases} \frac{\phi_*}{4G} & (0 < \phi_* < \pi), \\ \frac{2\pi - \phi_*}{4G} & (\pi < \phi_* < 2\pi). \end{cases} \quad (7.2.16)$$

Consequently, the second derivative vanishes everywhere except at $\phi_* = \pi$. It is suggestive that when the boundary is identified with the cosmological horizon, de Sitter spacetime saturates the negativity bound on the second derivative, mirroring the fact that de Sitter also saturates the null energy condition. This observation motivates us to extend the analysis to more general FLRW universes and to investigate the role of the null energy condition.

7.3 FLRW holography

In this section we generalize the de Sitter analysis to FLRW backgrounds. In Secs. 7.3.1–7.3.3, we analytically evaluate the holographic entanglement entropy in a flat universe filled with an ideal fluid obeying a constant equation of state. In Sec. 7.3.4, we conclude by discussing where a holographic boundary consistent with SSA may be placed.

7.3.1 Cosmological setup

Generality. Consider a three-dimensional flat FLRW universe,

$$ds^2 = a(\eta)^2 [-d\eta^2 + dr^2 + r^2 d\phi^2] \quad (0 \leq r < \infty, \ 0 \leq \phi < 2\pi), \quad (7.3.1)$$

leaving the scale factor $a(\eta)$ arbitrary for the moment. In computing the holographic entanglement entropy, we will need the geodesic distance between two boundary points A, B on the same constant- η slice $\eta = \eta_*$:

$$(\eta, r, \phi) = (\eta_*, r_A, \phi_A), \ (\eta_*, r_B, \phi_B). \quad (7.3.2)$$

Exploiting translation and rotation invariance on the $\eta = \eta_*$ slice, these can be mapped to

$$(\eta, r, \phi) = (\eta_*, \tilde{r}, 0), \ (\eta_*, \tilde{r}, \pi) \quad \text{with} \quad \tilde{r} = \frac{1}{2} \sqrt{r_A^2 + r_B^2 - 2r_A r_B \cos(\phi_A - \phi_B)}, \quad (7.3.3)$$

or, in Euclidean coordinates $(x, y) = (r \cos \theta, r \sin \theta)$,

$$(\eta, x, y) = (\eta_*, \tilde{r}, 0), \ (\eta_*, -\tilde{r}, 0). \quad (7.3.4)$$

By symmetry the geodesic lies along $y = 0$, so it may be described by the function $\eta(x)$. The geodesic connecting (7.3.4) is obtained by minimizing

$$D(\eta_*, \tilde{r}) = \int_{-\tilde{r}}^{\tilde{r}} dx a(\eta(x)) \sqrt{1 - \eta'(x)^2} = 2 \int_0^{\tilde{r}} dx a(\eta(x)) \sqrt{1 - \eta'(x)^2}, \quad (7.3.5)$$

subject to $\eta'(0) = 0$ and $\eta(\tilde{r}) = \eta_*$, where the Z_2 symmetry $\eta(-x) = \eta(x)$ follows from $\eta(\pm\tilde{r}) = \eta_*$.

Translational invariance along x furnishes a conserved quantity:

$$\frac{a(\eta(x))}{\sqrt{1 - \eta'(x)^2}} = a(\eta_0), \quad (7.3.6)$$

with $\eta_0 := \eta(0)$ the turning-point time. Hence $a(\eta(x)) \leq a(\eta_0)$. In particular, for an expanding universe at $\eta = \eta_*$ (our standing assumption), one has $\eta_0 > \eta_*$. The geodesic profile then obeys

$$x(\eta) = \pm \int_{\eta}^{\eta_0} \frac{d\tilde{\eta}}{\sqrt{1 - \frac{a(\tilde{\eta})^2}{a(\eta_0)^2}}}, \quad \tilde{r} = |x(\eta_*)| = \int_{\eta_*}^{\eta_0} \frac{d\tilde{\eta}}{\sqrt{1 - \frac{a(\tilde{\eta})^2}{a(\eta_0)^2}}}, \quad (7.3.7)$$

where the $\pm$ sign labels the two halves of the curve (see Figs. 7.3.1–7.3.3). The corresponding geodesic length is

$$D(\eta_*, \tilde{r}) = 2 \int_{\eta_*}^{\eta_0} d\eta \frac{a(\eta)^2}{\sqrt{a(\eta_0)^2 - a(\eta)^2}}. \quad (7.3.8)$$

Dividing by $4G$ yields the holographic entanglement entropy. Our task reduces to evaluating (7.3.7)–(7.3.8).

Analytic results for constant w . Assume now an ideal fluid with constant equation-of-state parameter $w = p/\varepsilon$ ($\varepsilon > 0$). The scale factor then follows the power law

$$a(\eta) = (w\eta)^{\frac{1}{w}}. \quad (7.3.9)$$

Choosing the expanding branch, $\eta \in (0, \infty)$ for $w > 0$ (deceleration) and $\eta \in (-\infty, 0)$ for $w < 0$ (acceleration). For (7.3.9), the integrals (7.3.7)–(7.3.8) can be performed in closed

form:

$$\begin{aligned}
 x(\eta) &= \pm \left[\eta_0 \cdot \frac{\sqrt{\pi} \Gamma\left[\frac{w+2}{2}\right]}{\Gamma\left[\frac{w+1}{2}\right]} - \eta \cdot {}_2F_1\left[\frac{1}{2}, \frac{w}{2}; \frac{w+2}{2}; \left(\frac{\eta}{\eta_0}\right)^{\frac{2}{w}}\right] \right], \\
 \tilde{r} &= \eta_0 \cdot \frac{\sqrt{\pi} \Gamma\left[\frac{w+2}{2}\right]}{\Gamma\left[\frac{w+1}{2}\right]} - \eta_* \cdot {}_2F_1\left[\frac{1}{2}, \frac{w}{2}; \frac{w+2}{2}; \left(\frac{\eta_*}{\eta_0}\right)^{\frac{2}{w}}\right], \\
 D(\eta_*, \tilde{r}) &= (w\eta_0)^{1+\frac{1}{w}} \frac{\sqrt{\pi} \Gamma\left[\frac{w+2}{2}\right]}{\Gamma\left[\frac{w+3}{2}\right]} - \frac{2}{w+2} \frac{(w\eta_*)^{1+\frac{2}{w}}}{(w\eta_0)^{\frac{1}{w}}} {}_2F_1\left[\frac{1}{2}, \frac{w+2}{2}; \frac{w+4}{2}; \left(\frac{\eta_*}{\eta_0}\right)^{\frac{2}{w}}\right],
 \end{aligned} \tag{7.3.10}$$

where the $\pm$ again label the two branches. Although each term appears singular at $w = -2$, the total is finite due to mutual cancellation. For example, at $w = -2$ one finds

$$\tilde{r} = -\eta_* \sqrt{1 - \frac{\eta_0}{\eta_*}} - \frac{\eta_0}{2} \ln \frac{1 + \sqrt{1 - \frac{\eta_0}{\eta_*}}}{1 - \sqrt{1 - \frac{\eta_0}{\eta_*}}}, \quad D(\eta_*, \tilde{r}) = \sqrt{-2\eta_0} \ln \frac{1 + \sqrt{1 - \frac{\eta_0}{\eta_*}}}{1 - \sqrt{1 - \frac{\eta_0}{\eta_*}}}. \tag{7.3.11}$$

Finally, the power-law form (7.3.9) implies a simple scaling of the length with η_* ,

$$D(\eta_*, \tilde{r}) = |\eta_*|^{1+\frac{1}{w}} D\left(\pm 1, \frac{\tilde{r}}{|\eta_*|}\right), \tag{7.3.12}$$

with the plus (minus) sign for $w > 0$ ($w < 0$).

7.3.2 Half holography

First, we apply the analytic expressions in (7.3.10) for flat universes with constant w to the half-holography setup. As in the de Sitter example, we place the boundary at $\eta = \eta_*$ along the straight line $y = 0$ and choose a subsystem on the interval

$$x \in (-r_*, r_*), \quad y = 0. \tag{7.3.13}$$

The holographic entanglement entropy is then

$$S = \frac{D(\eta_*, r_*)}{4G}. \tag{7.3.14}$$

Its η_* -dependence follows a simple power law (see Eq. (7.3.12)) and can be absorbed by an overall rescaling of the scale factor. Accordingly, we set $\eta_* = \pm 1$ without loss of generality. We present results for $w > 0$, $-1 < w < 0$, and $w < -1$ in this order.

Decelerating universes ($w > 0$). Typical spacelike geodesics for $w > 0$ are shown on the left of Fig. 7.3.1. For any r_* there exists a unique spacelike geodesic connecting the boundary

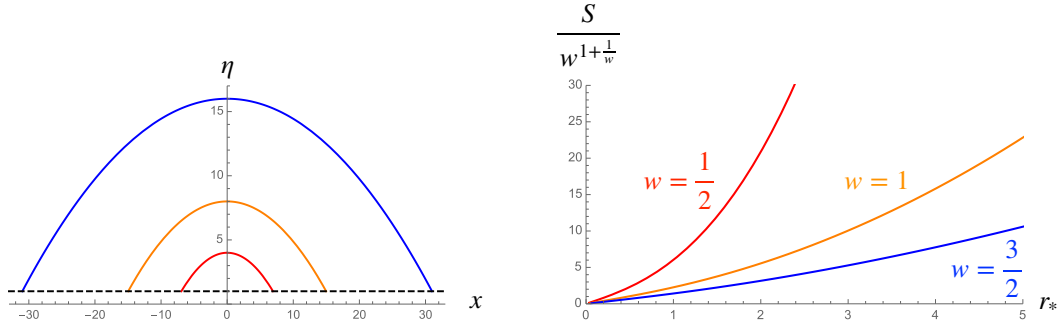


Figure 7.3.1: Left: Geodesic profiles for $w = \frac{1}{2}$. Red, orange, and blue curves correspond to $\eta_0 = 4, 8, 16$, respectively. The dashed line indicates the $\eta_* = 1$ slice on which the subsystem is defined. Right: Holographic entanglement entropy for $w = \frac{1}{2}, 1, \frac{3}{2}$ with $4G = 1$. For visibility, we plot $S/w^{1+\frac{1}{w}}$. The plots are from [4].

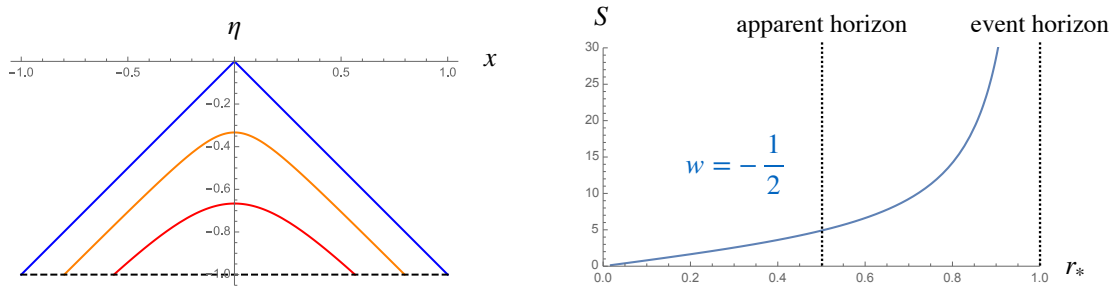


Figure 7.3.2: Left: Geodesic profiles for $w = -\frac{1}{2}$. Red, orange, and blue curves correspond to $\eta_0 = -\frac{3}{4}, -\frac{1}{2}, 0$, respectively. The dashed line indicates the $\eta_* = -1$ slice. Right: Holographic entanglement entropy for $w = -\frac{1}{2}$ with $4G = 1$. The entropy diverges as the subsystem approaches the event horizon, while nothing special occurs at the apparent horizon. The plots are from [4].

endpoints. Indeed, Eq. (7.3.10) shows that the subsystem size is a monotone function of the turning time η_0 , with the asymptotic behavior

$$r_* \simeq \eta_0 \cdot \frac{\sqrt{\pi} \Gamma\left[\frac{w+2}{2}\right]}{\Gamma\left[\frac{w+1}{2}\right]} \quad (\eta_0 \gg \eta_*). \quad (7.3.15)$$

Moreover, the right panel of Fig. 7.3.1 demonstrates that $S(r_*)$ is convex. Therefore, as in de Sitter, half holography conflicts with (strong) subadditivity.

Accelerating universes with $-1 < w < 0$. Consider now an accelerating universe satisfying the null energy condition, $-1 < w < 0$. For an observer at $r = 0$, the apparent horizon $r = w\eta$ lies inside the event horizon $r = -\eta$; at $\eta_* = -1$ they are at $r = |w|$ and $r = 1$, respectively. As shown on the left of Fig. 7.3.2, a spacelike geodesic connecting the two endpoints exists only if the subsystem is inside the *event* horizon, $r_* < 1$. Consequently, S ceases to be real when the interval extends beyond the event horizon—mirroring the de Sitter case. Furthermore, $S(r_*)$

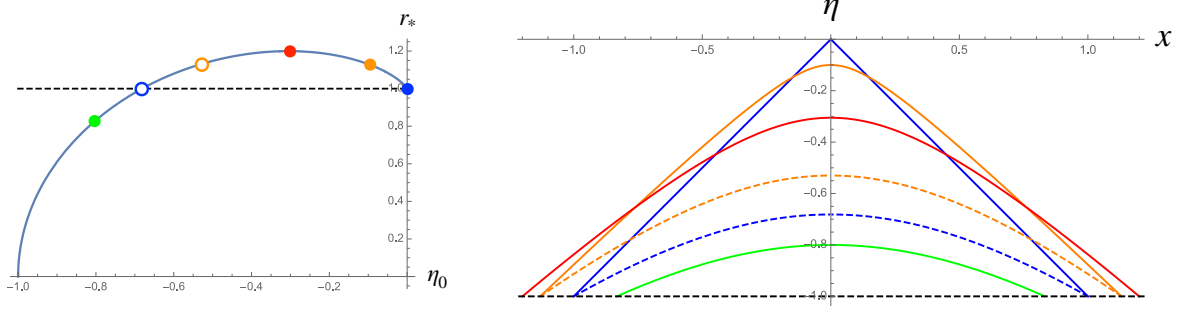


Figure 7.3.3: Left: Subsystem size r_* versus turning time η_0 for $w = -2$. The dashed line marks the event horizon $r_* = 1$. Colored and white circles correspond, respectively, to the solid and dashed curves of the same color in the right panel.

Right: Geodesic profiles for $w = -2$. Solid and dotted curves of the same color share the same r_* ; the solid curve has the shorter length. Red, orange, blue, and green correspond to $r_* \simeq 1.20, 1.13, 1.00, 0.83$. The plots are from [4].

is convex (right panel), so half holography again violates SSA. Unlike de Sitter, however, the entropy diverges as the interval approaches the event horizon; from (7.3.10) one finds the asymptotics $S \propto (1 - r_*)^{1+\frac{1}{w}}$ as $r_* \rightarrow 1$.

Accelerating universes with $w < -1$. Finally, for NEC-violating acceleration ($w < -1$), the apparent horizon $r = w\eta$ sits outside the event horizon $r = -\eta$ for an observer at $r = 0$. The left panel of Fig. 7.3.3 shows r_* as a function of η_0 and reveals that a spacelike connector persists even when the subsystem extends past the event horizon ($r_* > 1$). Remarkably, for $1 \leq r_* < r_{\max}$ there are *two* distinct spacelike geodesics; see the right panel for representative profiles. This behavior contrasts sharply with the $-1 \leq w < 0$ case.

Accordingly, the holographic entanglement entropy exhibits a sharp transition at $r_* = 1$; see Fig. 7.3.4. For $r_* < 1$, there is a single geodesic for each r_* and $S(r_*)$ is convex. At $r_* = 1$, a new branch (solid curve for $r_* \geq 1$) appears with a shorter length than the original branch (dotted curve), and thus it is selected as the Ryu–Takayanagi geodesic. In the range $1 \leq r_* \leq r_{\max}$ ($r_{\max} \simeq 1.20$ for $w = -2$) the entropy is *concave*. For $r_* > r_{\max}$ there is no spacelike connector, so S becomes complex. Evidently S fails SSA, rendering half holography incompatible, as in the previous cases. Since the null energy condition is violated, geodesics of real length can exist even beyond the event horizon.

7.3.3 Horizon-type holography

Next, we examine the horizon-type holography. In direct analogy with the de Sitter setup, we place the boundary at $\eta = \eta_*$ on the circle

$$r = \lambda r_H, \quad 0 \leq \phi < 2\pi, \quad (7.3.16)$$

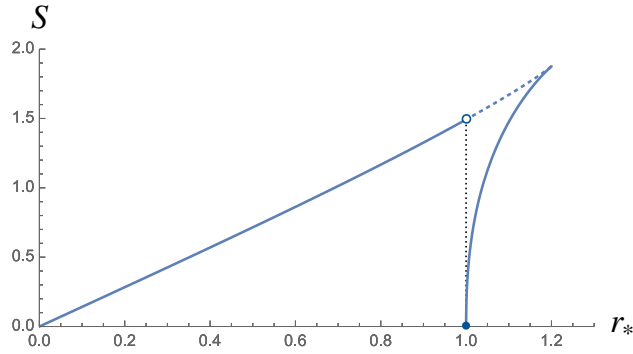


Figure 7.3.4: Holographic entanglement entropy for $w = -2$ as a function of the subsystem size r_* . The plot is from [4].

where $r_H := \frac{a(\eta_*)}{a'(\eta_*)}$ denotes the radius of the *apparent horizon*, and λ measures the boundary size relative to r_H . For the power-law scale factor (7.3.9) one has $r_H = w\eta_*$. We choose the subsystem to be the arc

$$r = \lambda r_H, \quad \phi \in (0, \phi_*) . \quad (7.3.17)$$

With this choice, $\tilde{r}$ in Eq. (7.3.3) becomes $\tilde{r} = \lambda r_H \sin \frac{\phi_*}{2}$, and the holographic entanglement entropy is

$$S = \frac{D(\eta_*, \lambda r_H \sin \frac{\phi_*}{2})}{4G} . \quad (7.3.18)$$

As before, we fix $\eta_* = \pm 1$ without loss of generality and present results for decelerating ($w > 0$) and accelerating ($w < 0$) universes in turn.

Decelerating universes. Figure 7.3.5 shows the entropy for $w = \frac{1}{2}$ as a function of ϕ_* . The qualitative features are: (i) for any $w > 0$, S is real for all ϕ_* ; (ii) S is concave when the interval is sufficiently large. These properties follow from the Z_2 symmetry $\phi_* \rightarrow 2\pi - \phi_*$ and the fact that the geodesic length is real for any $\tilde{r}$. However, if the boundary lies outside the apparent horizon ($\lambda > 1$), S becomes *convex* in the small- ϕ_* regime (see Sec. 7.3.4 for an analytic treatment). Thus, the horizon-type holography is compatible with SSA only when the boundary is on or inside the apparent horizon.

Accelerating universes. In accelerating backgrounds the event horizon is also relevant. For $-1 \leq w < 0$, the entropy is real only when the subsystem fits within the event horizon, so a sufficient condition for reality at all ϕ_* is $\lambda \leq |w|^{-1}$. Figure 7.3.6 displays $S(\phi_*)$ for $w = -\frac{1}{2}$; it is concave, and hence the horizon-type holography respects SSA only if the boundary is on or inside the apparent horizon (see also Sec. 7.3.4). In contrast, for $w < -1$ the entropy exhibits a phase transition when the boundary lies outside the event horizon,

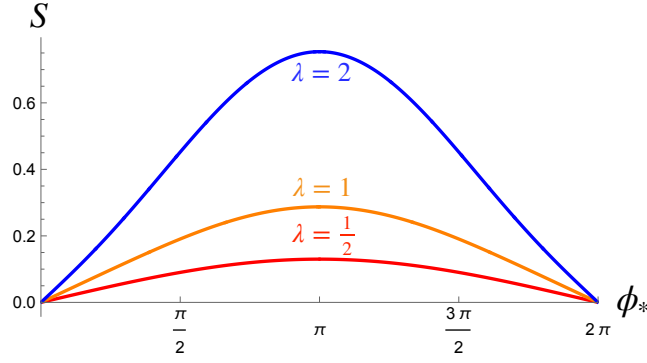


Figure 7.3.5: Holographic entanglement entropy for $w = \frac{1}{2}$ as a function of ϕ_* . The boundary sizes are $\lambda = \frac{1}{2}, 1, 2$. The plot is from [4].

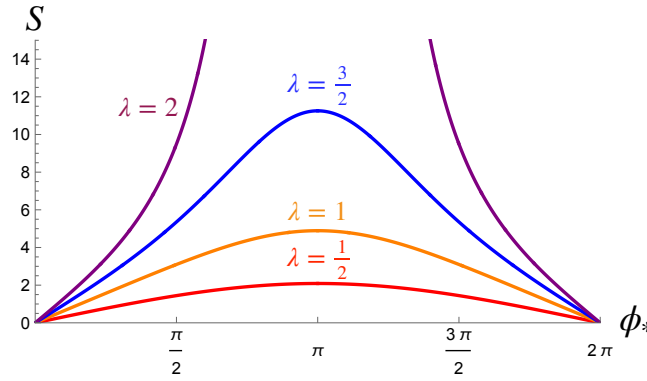


Figure 7.3.6: Holographic entanglement entropy for $w = -\frac{1}{2}$ versus ϕ_* . The boundary sizes are $\lambda = \frac{1}{2}, 1, \frac{3}{2}, 2$. The plot is from [4].

$\lambda > |w|^{-1}$ (see Fig. 7.3.7), a behavior tied to the violation of the null energy condition. As in the half-holography case, this renders the setup incompatible with SSA. If instead $\lambda \leq |w|^{-1}$, the entropy is concave and SSA-compatible.

7.3.4 Where can we place the holographic boundary?

Summarizing the flat FLRW analysis with a power-law scale factor: half holography fails SSA, while for the horizon-type construction compatibility depends on both λ and w . Necessary conditions for SSA can be phrased in terms of the subsystem size:

1. **Small subsystem.** SSA holds in the short-interval regime $\phi_* \ll 1$ *only if the boundary is on or inside the apparent horizon* ($\lambda \leq 1$). This statement is local and hence independent of w , applying to both accelerating and decelerating universes. Indeed, we prove below (Sec. 7.3.4) that it extends to general FLRW spacetimes beyond the flat constant- w family. The reliance on the apparent horizon reflects its local definition.
2. **Large subsystem.** At long distances, global causal structure matters, and the event horizon controls the behavior. This is especially significant when the accelerating

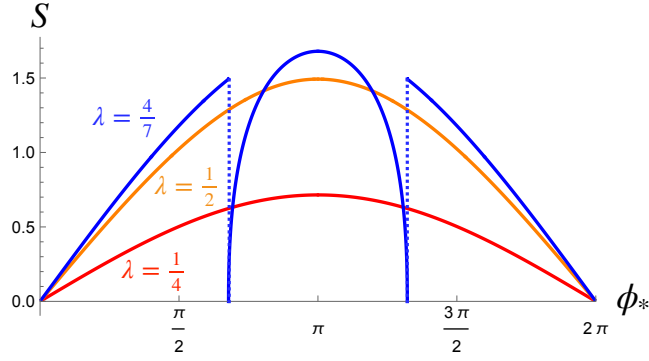


Figure 7.3.7: Holographic entanglement entropy for $w = -2$ as a function of ϕ_* . The boundary sizes are $\lambda = \frac{1}{4}, \frac{1}{2}, \frac{4}{7}$. The plot is from [4].

universe satisfies $w < -1$, so that the event horizon lies inside the apparent horizon. If the boundary sits outside the event horizon, S undergoes a phase transition and SSA is violated. Hence a necessary condition for SSA in accelerating universes is that the boundary be on or inside the event horizon.

Equivalently: if the boundary is placed *on* the apparent horizon, then SSA is satisfied *only when the null energy condition holds* ($w \geq -1$). This echoes earlier results [122–124] relating the null energy condition to consistency of holographic entanglement entropy in other backgrounds.

These observations suggest the following expectation for FLRW holography: demanding a conventional, unitary and local dual QFT implies that *the holographic boundary should lie inside both the apparent and event horizons of a geodesic observer*. In particular, for symmetric embeddings a natural choice is the horizon-type setup with $\lambda \leq 1$.⁵ To substantiate this, Sec. 7.3.4 analyzes the short-distance behavior of the entropy in general FLRW universes.

Finally, a remark on the original dS/CFT correspondence and FLRW generalizations. As noted in Sec. 7.2.1, our arguments carry over to a spacelike boundary at fixed η . Interpreting y as Euclidean time in the putative dual, the entanglement entropy reduces to that of half holography, which always violates SSA; the dual is therefore not a standard QFT and is expected to be non-unitary, since bulk and boundary do not share time. While such non-standard duals are interesting in their own right, our program is different: we seek criteria for holographic/cosmological setups under the working assumption that the dual QFT is well behaved.

⁵For $w > 0$, S can be concave even with $\lambda > 1$ when the subsystem is sufficiently large. A pragmatic—though somewhat artificial—way to maintain SSA would be to introduce a UV cutoff in the dual theory and restrict the range of ϕ_* .

Short-distance behavior in general FLRW universe.

To close this section, we analytically extract the short-distance behavior of the holographic entanglement entropy in general FLRW spacetimes. By contrast, the long-distance regime is sensitive to the full cosmic history, making a universal analysis difficult.

Consider a general flat FLRW background and recall the geodesic formulas (7.3.5)–(7.3.6). Combining them gives

$$S = \frac{1}{4G} \int_{-\tilde{r}}^{\tilde{r}} \frac{a^2(\eta(x))}{a(\eta(0))} dx, \quad (7.3.19)$$

where $\tilde{r} = r_*$ for half holography and $\tilde{r} = \lambda r_H \sin \frac{\phi_*}{2}$ for the horizon-type setup. Expanding at small $\tilde{r}$ yields

$$S = \frac{1}{4G} \left[2a(\eta_*)\tilde{r} - \frac{1}{3}a'(\eta_*)\eta''(\tilde{r})\tilde{r}^3 + O(\tilde{r}^5) \right], \quad (7.3.20)$$

using the boundary condition $\eta'(0) = 0$. The Euler-Lagrange equation from (7.3.5) implies

$$\eta''(\tilde{r}) = -\frac{a'(\eta_*)}{a(\eta_*)} = -\frac{1}{r_H}. \quad (7.3.21)$$

Substituting into (7.3.20) we obtain

$$S = \frac{a_*}{4G} \left[2\tilde{r} + \frac{1}{3} \frac{\tilde{r}^3}{r_H^2} + O(\tilde{r}^5) \right]. \quad (7.3.22)$$

In half holography $\tilde{r} = r_*$, with r_* equal to half the interval length, so S is convex at short distances for any flat FLRW universe. Thus, half holography is incompatible with SSA in any FLRW background.

For the horizon-type construction, $\tilde{r} = \lambda r_H \sin(\phi_*/2)$, where λ measures the boundary size relative to the apparent horizon and ϕ_* is the arc size. Plugging this into (7.3.22) leads to

$$S = \frac{\lambda a_* r_H}{4G} \left[\phi_* - \frac{1}{24} (1 - \lambda^2) \phi_*^3 + O(\phi_*^5) \right]. \quad (7.3.23)$$

This expansion also holds for closed and open universes, see [4] for details. We therefore conclude that the horizon-type setup satisfies SSA in the short-interval regime *only if* the boundary lies on or inside the apparent horizon, $\lambda \leq 1$. This supports the claim that a standard dual QFT requires placing the holographic boundary inside the apparent horizon.

Chapter 8

Conclusion

In this thesis, we have investigated several approaches to de Sitter holography through the lens of entanglement entropy. The current state of the art is that neither a definitive formulation of de Sitter holography nor a universally accepted notion of holographic entanglement entropy has been established. These difficulties are tightly connected to the non-unitarity of the putative dual CFT. In our discussion of holography, we have reviewed Strominger’s dS/CFT proposal [18], Susskind’s static patch holography [24], and our half–de Sitter holography [3], and we have presented our own work grounded in these distinct frameworks.

In the AdS/CFT correspondence, an extra spatial direction emerges; by analogy, one expects the emergence of time in the dS/CFT setting. In [26], we implement a Wick rotation from AdS/CFT and elucidate the emergence of time via the information metric. This provides a concrete hint as to how information about time evolution is encoded in CFT dynamics. The approach is in the same spirit as dS/CFT in that it exploits conformal symmetry to the fullest extent.

In [2], we propose a novel holographic entanglement entropy that satisfies strong subadditivity. The analysis is carried out within static patch holography, and our aim is to derive constraints on the position of the stretched horizon from strong subadditivity. Because de Sitter space admits a one-parameter family of vacua—the α -vacua—the corresponding two-point function is labeled by α . Under the assumption of the twist-operator computation in field theory and the holographic dictionary, the resulting entanglement entropy acquires a characteristic α -dependence, which is the essential new ingredient of our work. The nonlocality associated with the α -vacuum may help to explain the unconventional behavior of entanglement entropy in de Sitter space as compared to the AdS/CFT case.

In half–de Sitter holography [3], we introduce a timelike boundary and impose Dirichlet boundary conditions, an approach philosophically aligned with static patch holography. We find that a finite-cutoff geometry exhibits nonlocal behavior, akin to effects induced by higher-derivative terms. This work reveals a novel spreading of information beyond the lightcone in de Sitter space. It also bears on the following question: why does the volume of de Sitter

space grow exponentially even though the number of degrees of freedom appears finite, as inferred from the entropy? We expect this to be a central issue for future investigation.

In [4], we study holographic entanglement entropy in FLRW spacetime using the Ryu-Takayanagi prescription. We also introduce two types of holographic screens on which the dual QFT is placed and evaluate strong subadditivity. In particular, within the perfect-fluid approximation, the holographic screen should lie inside the apparent horizon for $w \geq -1$, whereas it should lie inside the event horizon for $w < -1$. Further pursuit of holographic duality in FLRW spacetime may deepen our understanding of the universe in holographic terms.

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