

HOMOTOPY TYPES OF THE COMPONENTS OF SPACES OF EMBEDDINGS OF COMPACT POLYHEDRA INTO 2-MANIFOLDS

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Homotopy types of the identity components of homeomorphism groups of 2-manifolds have been classified in [2, 7, 9]. In this article we classify the homotopy type of connected components of spaces of embeddings of compact connected polyhedra into 2-manifolds [11].

1. BACKGROUND

The homotopy type of the identity component $\mathcal{H}(M)_0$ of the group $\mathcal{H}(M)$ of $(C^0, \text{PL}, C^\infty)$ homeomorphisms on a surface M of finite type was studied in 1960's and its classification is now a classical result. In the C^0 -category, M.E. Hamstrom et al. [2, 7] studied the homotopy groups of $\mathcal{H}(M)_0$ and R. Luke - W.K. Mason [3] showed that $\mathcal{H}(M)$ is an ANR (absolute neighborhood retract). After the development of infinite-dimensional manifold theory in 1970's [4] it was shown that $\mathcal{H}(M)$ is a topological ℓ_2 -manifold, and the topological type of $\mathcal{H}(M)_0$ was determined based on its homotopy type.

The study of homeomorphism groups is closely related to the study of embedding spaces. For example, the following properties of embedding spaces played crucial roles in the works of Hamstrom and Luke - Mason: the triviality of the homotopy groups of the space of embeddings of a one point union of circles, ANR property of the space of embeddings of a circle, etc. However, these results on spaces of embeddings into 2-manifolds were restricted to partial cases.

In another viewpoint, the theory of conformal mappings in the complex plane [6] played an important role in 2-dimensional topology. Conformal mappings give canonical coordinates to domains in the complex plane. Those coordinates are used to extend homeomorphisms on the boundaries canonically to homeomorphisms on the domains.

Based upon these backgrounds, we have studied the remaining parts: a bundle theorem which connects homeomorphism groups of surfaces with spaces of embeddings into surfaces, homeomorphism groups of surfaces of infinite type and spaces of embeddings of compact polyhedra into surfaces, etc.

In the C^∞ -category, it is well known that the restriction maps from the homeomorphism group of a manifold N to the space of embeddings of a submanifold L into N is a principal bundle [5]. We have shown a similar result for any topological 2-manifold M and any compact subpolyhedron X of M [8]. Again the conformal mapping theorem is used to obtain canonical extension of embeddings of X to homeomorphisms of M . This bundle theorem has been used

to show that the space $\mathcal{E}(X, M)$ of embeddings of X into M is an ℓ^2 -manifold [8]. We have also provided a sufficient condition that the fiber of this bundle is connected [9]. Combining these results with the results on $\mathcal{H}(M)$ for M of finite type together, we have determined the homotopy type and the topological type of $\mathcal{H}(M)$ for M of infinite type [9].

Now we are in a position to answer the following problem:

Problem. For any 2-manifold M and any compact connected subpolyhedron X of M , determine the homotopy type and the topological type of the connected components of the space $\mathcal{E}(X, M)$ of embeddings of X into M .

2. MAIN RESULTS

2.1. Main Theorem.

Suppose M is a connected 2-manifold and X is a compact connected subpolyhedron of M with respect to some triangulation of M . Let $\mathcal{E}(X, M)$ denote the space of topological embeddings of X into M with the compact-open topology and let $\mathcal{E}(X, M)_0$ denote the connected component of the inclusion map $i_X : X \subset M$ in $\mathcal{E}(X, M)$.

If X is a point of M then $\mathcal{E}(X, M) \cong M$, and if X is a closed 2-manifold then $X = M$ and $\mathcal{E}(X, M)_0 = \mathcal{H}(M)_0$, whose homotopy type is already known [2, 7].

Assumption 1. Below we assume that X is neither a point nor a closed 2-manifold.

The inclusion map $i_X : X \subset M$ induces a homomorphism on the fundamental group $i_{X*} : \pi_1(X) \rightarrow \pi_1(M)$. Denote the image of i_{X*} by G . We have the following three cases:

- [1] G : not a cyclic group [2] G : a nontrivial cyclic group [3] $G = 1$

The homotopy type of $\mathcal{E}(X, M)_0$ can be classified in the term of this subgroup G . (The symbols $\mathbb{S}^1$, $\mathbb{T}^2$, $\mathbb{P}^2$, $\mathbb{K}^2$ denote the circle, torus, projective plane and Klein bottle respectively.)

Theorem 1. Suppose G is not a cyclic group.

- (1) $\mathcal{E}(X, M)_0 \simeq *$ if $M \not\cong \mathbb{T}^2, \mathbb{K}^2$.
- (2) $\mathcal{E}(X, M)_0 \simeq \mathbb{T}^2$ if $M \cong \mathbb{T}^2$.
- (3) $\mathcal{E}(X, M)_0 \simeq \mathbb{S}^1$ if $M \cong \mathbb{K}^2$.

Theorem 2. Suppose G is a nontrivial cyclic group.

- (1) $\mathcal{E}(X, M)_0 \simeq \mathbb{S}^1$ if $M \not\cong \mathbb{P}^2, \mathbb{T}^2, \mathbb{K}^2$.
- (2) $\mathcal{E}(X, M)_0 \simeq \mathbb{T}^2$ if $M \cong \mathbb{T}^2$.
- (3) Suppose $M \cong \mathbb{K}^2$.
 - (i) $\mathcal{E}(X, M)_0 \simeq \mathbb{T}^2$ if X is contained in an annulus which does not separate M .
 - (ii) $\mathcal{E}(X, M)_0 \simeq \mathbb{S}^1$ otherwise.
- (4) Suppose $M \cong \mathbb{P}^2$.
 - (i) $\mathcal{E}(X, M)_0 \simeq SO(3)/\mathbb{Z}_2$ if X is an orientation reversing circle in M .

(ii) $\mathcal{E}(X, M)_0 \simeq SO(3)$ otherwise.

When $G = 1$, under Assumption 1, X is contractible in M and X has a disk neighborhood. The 2-manifold M admits a smooth structure and has a Riemannian metric. By $S(TM)$ we denote the unit circle bundle of the tangent bundle TM . When M is nonorientable, let $\widetilde{M}$ denote the orientable double cover of M .

Theorem 3. *Suppose $G = 1$.*

- (1) $\mathcal{E}(X, M)_0 \simeq S(TM)$ if X is an arc or M is orientable.
- (2) $\mathcal{E}(X, M)_0 \simeq S(T\widetilde{M})$ otherwise.

Complement. If we choose a base point $x_0 \in X$ and consider the map $p : \mathcal{E}(X, M)_0 \rightarrow M$, $p(f) = f(x_0)$, then in Theorem 3 (1), (2) $\mathcal{E}(X, M)_0$ is fiber preserving (f.p.) homotopy equivalent over M to $S(TM)$ and $S(T\widetilde{M})$ respectively.

To determine the topological type of $\mathcal{E}(X, M)_0$ we can apply the theory of infinite-dimensional manifolds [4]. Since $\mathcal{E}(X, M)$ is a topological ℓ^2 -manifold [8], the topological type of $\mathcal{E}(X, M)_0$ is determined by its homotopy type [4]. If $\mathcal{E}(X, M)_0$ has the homotopy type of a compact polyhedron P , then $\mathcal{E}(X, M)_0 \cong P \times \ell^2$. In [10] we study the space of embedded images of X in M , $\mathcal{K}(X, M) = \{f(X) \mid f \in \mathcal{E}(X, M)\}$, equipped with the Frechet topology, and show that the natural map $\mathcal{E}(X, M) \rightarrow \mathcal{K}(X, M)$ is a principal $\mathcal{H}(X)$ -bundle.

2.2. Idea of Proof.

Theorems 1-3 are proved by the following considerations: First we take a regular neighborhood N of X and compare the homotopy types of $\mathcal{E}(X, M)_0$ and $\mathcal{E}(N, M)_0$ through the restriction map $\mathcal{E}(N, M)_0 \rightarrow \mathcal{E}(X, M)_0 : f \mapsto f|_X$. It is shown that, except two cases, this restriction map is a homotopy equivalence. The exceptional cases are treated separately. Below we consider the generic case. By Assumption 1 N has a boundary and admits a core Y which is a one point union of circles.

(1) If G is not a cyclic group, Y includes at least two *independent* essential circles. In this case it is shown that the restriction map $\mathcal{H}(M)_0 \rightarrow \mathcal{E}(N, M)_0$ is a homotopy equivalence and we have the conclusion follows from the homotopy type of $\mathcal{H}(M)_0$.

(2) If G is a nontrivial cyclic group, Y includes only one *independent* essential circle. One can eliminate *dependent* circles from Y without changing the homotopy type of $\mathcal{E}(Y, M)_0$. Thus the general case reduces to the case where X is an essential circle. In the latter case, we can deduce the conclusion by comparing with $\mathcal{H}(M)_0$ (cf. [7]). Generically, $\mathcal{H}(M)_0 \simeq *$ yields $\mathcal{E}(X, M)_0 \simeq \mathbb{S}^1$ (the circle of the rotations of X along itself).

(3) When $G = 1$, under Assumption 1, X has a disk neighborhood D . For simplicity we consider the case where M is orientable. The unit circle bundle $S(TM)$ can be embedded into $\mathcal{E}(X, M)_0$ in the following form: Fix a base point x_0 of X . We identify (D, x_0) with the unit

disk $(D(1), 0)$ in the plane $\mathbb{R}^2$. Thus we can regard as $X \subset D(1)$. If we choose a sufficiently small function $\varepsilon(x) : M \rightarrow (0, \infty)$, then at each point $x \in M$ the exponential map $\exp_x$ is defined on the $\varepsilon(x)$ -neighborhood of the origin in $T_x M$. For each $v \in S(T_x M)$ we take the unique orientation preserving (o.p.) isometric embedding $j_{x,v} : (D(1), 0) \rightarrow (T_x M, 0)$ with $j_{x,v}(1, 1) = v$ and define $i_{x,v} \in \mathcal{E}(X, M)_0$ by $i_{x,v} = \exp_x(\varepsilon(x)j_{x,v}|_X)$. Theorem 3 is verified by constructing a strong deformation retraction of $\mathcal{E}(X, M)_0$ onto $S(TM)$. To deform any topological embedding of (X, x_0) into $(D(1), 0)$ to a rotation around 0 canonically, we need $SO(2)$ -equivariant canonical extension of embeddings of X into $D(1)$. This is obtained by using the conformal mapping theorem in the complex function theory [6].

3. SKETCH OF PROOF

Let M and X be as in Section 2.1. By $\mathcal{H}_X(M)$ we denote the group of homeomorphisms h of M onto itself with $h|_X = id$, equipped with the compact-open topology, and by $\mathcal{H}_X(M)_0$ we denote the identity component of $\mathcal{H}_X(M)$. When K is a subpolyhedron of X , let $\mathcal{E}_K(X, M)$ denote the subspace of $\mathcal{E}(X, M)$ consisting of embeddings $f : X \hookrightarrow M$ with $f|_K = id$. and let $\mathcal{E}_K(X, M)_0$ denote the connected component of the inclusion map $i_X : X \subset M$ in $\mathcal{E}_K(X, M)$.

By pulling M into $\text{Int } M$ with using a collar of M , it is shown that the inclusion $\mathcal{E}(X, \text{Int } M) \subset \mathcal{E}(X, M)$ is a homotopy equivalence. Thus there is no loss of generality under the following assumption.

Assumption 2. Below we assume that $\partial M = \emptyset$.

3.1. Homotopy types of connected components of homeomorphism groups of surfaces.

When M is a surface of finite type, the homotopy type of $\mathcal{H}_X(M)_0$ is well known. When M is a surface of infinite type, the homotopy type of $\mathcal{H}_X(M)_0$ is classified as follows [9]:

Proposition 1. *Suppose M is a noncompact connected 2-manifold and X is a compact subpolyhedron of M .*

- (i) $\mathcal{H}_X(M)_0 \simeq \mathbb{S}^1$ if $(M, X) \cong (\mathbb{R}^2, \emptyset)$, $(\mathbb{R}^2, 1pt)$, $(\mathbb{S}^1 \times \mathbb{R}^1, \emptyset)$, $(\mathbb{S}^1 \times [0, 1], \emptyset)$ or $(\mathbb{P}^2 \setminus 1pt, \emptyset)$.
- (ii) $\mathcal{H}_X(M)_0 \simeq *$ otherwise.

3.2. Bundle Theorem.

The homeomorphism group $\mathcal{H}_K(M)_0$ and the embedding space $\mathcal{E}_K(X, M)_0$ are joined by the restriction map $\pi : \mathcal{H}_K(M)_0 \rightarrow \mathcal{E}_K(X, M)_0$, $\pi(f) = f|_X$. In [8] we have investigated some extension property of embeddings of a compact polyhedron into a 2-manifold, based upon the conformal mapping theorem. The result is summarized as follows [9]:

Proposition 2. (i) *The restriction map $\pi : \mathcal{H}_K(M)_0 \rightarrow \mathcal{E}_K(X, M)_0$ is a principal bundle with fiber $\mathcal{G} \equiv \mathcal{H}_K(M)_0 \cap \mathcal{H}_X(M)$, where the group $\mathcal{G}$ acts on $\mathcal{H}_K(M)_0$ by right composition.*

(ii) Suppose $K \subset Y$ are compact subpolyhedra of X . Then the restriction map $p : \mathcal{E}_K(X, M)_0 \rightarrow \mathcal{E}_K(Y, M)_0$, $p(f) = f|_Y$ is a fiber bundle with fiber $\mathcal{F} = \mathcal{E}_K(X, M)_0 \cap \mathcal{E}_Y(X, M)$.

The next proposition provides a sufficient condition for the fiber $\mathcal{G}$ to be connected [9]. ($\#A$ denotes the cardinality of a set A .)

Proposition 3. Suppose N is a compact 2-submanifold of M and Y is a subset of N . If (M, N, Y) satisfies the following conditions, then $\mathcal{H}_Y(M)_0 \cap \mathcal{H}_N(M) = \mathcal{H}_N(M)_0$.

- (i) (a) If H is a disk component of N , then $\#(H \cap Y) \geq 2$.
- (b) If H is an annulus or Möbius band component of N , then $H \cap Y \neq \emptyset$.
- (ii) (a) If L is a disk component of $cl(M \setminus N)$, then $\#(L \cap Y) \geq 2$.
- (b) If L is a Möbius band component of $cl(M \setminus N)$, then $L \cap Y \neq \emptyset$.

3.3. Embedding spaces of regular neighborhoods.

Suppose N is a regular neighborhood of X in M . By Proposition 2 (ii) we have the fiber bundle

$$\mathcal{F} \equiv \mathcal{E}(N, M)_0 \cap \mathcal{E}_X(N, M) \hookrightarrow \mathcal{E}(N, M)_0 \xrightarrow{p} \mathcal{E}(X, M)_0, \quad p(f) = f|_X.$$

Consider the following conditions:

- (i) X is an arc and M is nonorientable. (ii) X is an orientation reversing (o.r.) circle.

Proposition 4.

- (1) If (M, X) is neither in the case (i) nor (ii), then $\mathcal{F} = \mathcal{E}_X(N, M)_0 \simeq *$ and the map p is a homotopy equivalence.
- (2) In the case (i) or (ii) $\mathcal{E}(N, M)_0$ has a natural $\mathbb{Z}_2$ -action and the map p factors as

$$p : \mathcal{E}(N, M)_0 \xrightarrow{\pi} \mathcal{E}(N, M)_0 / \mathbb{Z}_2 \xrightarrow{q} \mathcal{E}(X, M)_0.$$

The map π is a double cover and the map q is a homotopy equivalence.

3.4. Proof of Theorem 1.

Once we show that the restriction map $p : \mathcal{H}(M)_0 \rightarrow \mathcal{E}(X, M)_0$ is a homotopy equivalence, then the conclusion follows from the homotopy type of $\mathcal{H}(M)_0$.

Let N be a regular neighborhood of X and let N_1 be the union of N and the disk or Möbius band components of $cl(M \setminus N)$. The map p factors to the restriction maps

$$\mathcal{H}(M)_0 \xrightarrow{p_1} \mathcal{E}(N_1, M)_0 \xrightarrow{p_2} \mathcal{E}(N, M)_0 \xrightarrow{p_3} \mathcal{E}(X, M)_0.$$

By Proposition 4 (1) the map p_3 is a homotopy equivalence. The map p_2 is also a homotopy equivalence since $\mathcal{H}_\partial(E) \simeq *$ if E is a disk or a Möbius band. The map p_1 is a principal bundle with fiber $\mathcal{G} = \mathcal{H}(M)_0 \cap \mathcal{H}_{N_1}(M)$. By Proposition 3 we have $\mathcal{G} = \mathcal{H}_{N_1}(M)_0 \simeq *$ and so p_1 is a homotopy equivalence. Therefore the map p is a homotopy equivalence. $\square$

3.5. Simplification of embedded polyhedra — Elimination of circles.

In this section we apply Proposition 4 to modify the polyhedron X without changing the homotopy type of $\mathcal{E}(X, M)_0$.

Proposition 5.

- (1) If E is a disk or a Möbius band in M and $\partial E \subset X$, then the restriction map $p : \mathcal{E}(X \cup E, M)_0 \rightarrow \mathcal{E}(X, M)_0$ is a homotopy equivalence.
- (2) Suppose $X = Y \cup C$ is a one point union of an inessential circle C and a compact connected subpolyhedron Y which satisfies the condition of Proposition 4 (1). Then the restriction map $p : \mathcal{E}(X, M)_0 \rightarrow \mathcal{E}(Y, M)_0$ is a homotopy equivalence.
- (3) Suppose $X = Y \cup C_1 \cup C_2$ is a one point union of two essential circles C_1 and C_2 and a compact connected subpolyhedron Y ($\neq 1\text{pt}$), where if one of C_1 and C_2 is an o.r. circle, we relabel them so that C_2 is an o.r. circle. If $G = \text{Im}(i_X)_*$ is a cyclic group, then the restriction map $p : \mathcal{E}(X, M)_0 \rightarrow \mathcal{E}(Y \cup C_2, M)_0$ is a homotopy equivalence.

3.6. Proof of Theorem 2.

We treat the generic case (1). So we assume that $M \not\cong \mathbb{P}^2, \mathbb{T}^2, \mathbb{K}^2$ and show that $\mathcal{E}(C, M)_0 \simeq \mathbb{S}^1$. The remaining cases are treated separately.

[1] Case where X is a circle (cf. [7]):

Suppose C is an essential circle in M . Fix a base point $x \in C$ and let $\alpha \in \pi_1(M, x)$ be the element represented by C with an appropriate orientation. By $\langle \alpha \rangle$ we denote the subgroup of $\pi_1(M, x)$ generated by α . Consider the following fiber bundles:

$$\begin{aligned} \mathcal{F} &\equiv \mathcal{E}(C, M)_0 \cap \mathcal{E}_x(C, M) \subset \mathcal{E}(C, M)_0 \xrightarrow{p} M & : p(f) = f(x), \\ \mathcal{G} &\equiv \mathcal{H}_x(M)_0 \cap \mathcal{H}_C(M) \subset \mathcal{H}_x(M)_0 \xrightarrow{q} \mathcal{E}_x(C, M)_0 & : q(h) = h|_C. \end{aligned}$$

Inspecting these bundles, we see that

- (i) $\mathcal{E}_x(C, M)_0 \simeq *$.
- (ii) $\pi_k(\mathcal{E}(C, M)_0) = 0$ ($k \geq 2$) and $p_* : \pi_1(\mathcal{E}(C, M)_0, i_C) \xrightarrow{\cong} \text{Im } p_* \subset \pi_1(M, x)$.
- (iii) (a) $\alpha \in \text{Im } p_* \subset \pi_1(M, x)$, (b) $\alpha\beta = \beta\alpha$ ($\beta \in \text{Im } p_*$).
- (iv) $\text{Im } p_* = \langle \alpha \rangle \cong \mathbb{Z}$.

Since $\mathcal{E}(C, M)_0$ is an ANR and $K(\mathbb{Z}, 1)$, it follows that $\mathcal{E}(C, M)_0 \simeq \mathbb{S}^1$.

[2] Case where X is not a circle:

This case reduces to the circle case through the following argument: Let N be a regular neighborhood of X . By Assumption 1 N has a boundary and includes a subpolyhedron Y such that N is a regular neighborhood of Y in M and $Y = A \cup (\cup_{i=1}^m C_i) \cup (\cup_{j=1}^n C'_j)$ is a one point union of essential circles C_i ($i = 1, \dots, m$) ($m \geq 1$), inessential circles C'_j ($j = 1, \dots, n$) ($n \geq 0$) and an arc A . Let $Y_1 = A \cup (\cup_{i=1}^m C_i)$. By Propositions 4 (1) and 5 (2) the following restriction maps are homotopy equivalences:

$$\mathcal{E}(X, M)_0 \longleftarrow \mathcal{E}(N, M)_0 \longrightarrow \mathcal{E}(Y, M)_0 \longrightarrow \mathcal{E}(Y_1, M)_0$$

Since $i_{Y_1*}\pi_1(Y_1) = i_{X*}\pi_1(X)$ is a cyclic subgroup of $\pi_1(M)$, by the repeated application of Proposition 5 (3) we can find some C_k such that the restriction map

$$\mathcal{E}(Y_1, M)_0 \longrightarrow \mathcal{E}(A \cup C_k, M)_0$$

is a homotopy equivalence.

Let N_1 be a regular neighborhood of $A \cup C_k$. Then N_1 is a regular neighborhood of C_k and it is an annulus or a Möbius band. We set $C = C_k$ when N_1 is an annulus and $C = \partial N_1$ when N_1 is a Möbius band. The restriction maps

$$\mathcal{E}(A \cup C_k, M)_0 \longleftarrow \mathcal{E}(N_1, M)_0 \longrightarrow \mathcal{E}(C, M)_0$$

are homotopy equivalences. We have the required conclusion by applying Case [1] to the circle C . $\square$

3.7. Proof of Theorem 3.

For the sake of simplicity, below we assume that M is oriented and X is not an arc. We choose a smooth structure and a Riemannian metric on M . Let d denote the distance function induced from this Riemannian metric. The tangent bundle $q : TM \rightarrow M$ is a 2-dim oriented vector bundle with an inner product. By the assumption X has a disk neighborhood D , which inherits a natural orientation from M . Fix a base point x_0 of X .

Notation 1. For the embedding space, the symbol “+” denotes “orientation preserving”. For example, when E is an oriented disk, $Y \subset E$ and N is an oriented surface, we define as follows:

$$\mathcal{E}^+(E, N) = \{f \in \mathcal{E}(E, N) \mid f \text{ preserves the orientations}\}$$

$$\mathcal{E}^+(Y, N) = \{f \in \mathcal{E}(Y, N) \mid f \text{ admits an extension } \bar{f} \in \mathcal{E}^+(E, N)\}$$

For $X \subset D \subset M$, we have $\mathcal{E}(D, M)_0 = \mathcal{E}^+(D, M)$ and $\mathcal{E}(X, M)_0 = \mathcal{E}^+(X, M)$.

3.7.1. Spaces of ε -embeddings.

For $x \in M$ and $r > 0$, let $U_x(r) = \{y \in M \mid d(x, y) < r\}$ and $O_x(r) = \{v \in T_x M \mid |v| < r\}$. If $\varepsilon : M \rightarrow (0, \infty)$ is a sufficiently small continuous function, then at each point $x \in M$ the exponential map $\exp_x$ defines an o.p. diffeomorphism $\exp_x : O_x(\varepsilon(x)) \xrightarrow{\cong} U_x(\varepsilon(x))$. Since $\exp_x$ is smooth in $x \in M$, if we set

$$O_{TM}(\varepsilon) = \bigcup_{x \in M} O_x(\varepsilon(x)) \subset TM \quad U_M(\varepsilon) = \bigcup_{x \in M} \{x\} \times U_x(\varepsilon(x)) \subset M \times M$$

then we obtain a f.p. diffeomorphism over M :

$$\exp : O_{TM}(\varepsilon) \rightarrow U_M(\varepsilon), \quad \exp(v) = (x, \exp_x(v)) \quad (v \in O_x(\varepsilon(x))).$$

Next consider the following subspaces of $\mathcal{E}(X, TM)$ and $\mathcal{E}^+(X, M)$ defined by

$$\mathcal{E}_q^+(X, x_0; O_{TM}(\varepsilon), 0) = \bigcup_{x \in M} \mathcal{E}^+(X, x_0; O_x(\varepsilon(x)), 0) \subset \mathcal{E}(X, TM),$$

$$\text{where } \mathcal{E}^+(X, x_0; O_x(\varepsilon(x)), 0) = \{f \in \mathcal{E}^+(X; O_x(\varepsilon(x))) \mid f(x_0) = 0\}$$

$$\mathcal{E}_\varepsilon^+(X, M) = \{f \in \mathcal{E}^+(X, M) \mid f(X) \subset U_{f(x_0)}(\varepsilon(f(x_0)))\} \subset \mathcal{E}^+(X, M).$$

The space $\mathcal{E}_q^+(X, x_0; O_{TM}(\varepsilon), 0)$ has a natural projection onto M . The projection $p : \mathcal{E}^+(X, M) \rightarrow M$, $p(f) = f(x_0)$, induces the projection $p : \mathcal{E}_\varepsilon^+(X, M) \rightarrow M$.

Lemma 1. *The f.p. diffeomorphism $\exp$ induces a f.p. homeomorphism over M*

$$\text{Exp} : \mathcal{E}_q^+(X, x_0; O_{TM}(\varepsilon), 0) \cong \mathcal{E}_\varepsilon^+(X, M), \quad \text{Exp}(f) = \exp_x \circ f \quad (f \in \mathcal{E}^*(X, x_0; O_x(\varepsilon(x)), 0)) \square$$

Remark 1. With multiplying $\varepsilon(x)$ on $T_x M$, we see that $O_{TM}(\varepsilon)$ and $O_{TM}(1)$ are f.p. homeomorphic over M . Thus $\mathcal{E}_q^+(X, x_0; O_{TM}(\varepsilon), 0)$ and $\mathcal{E}_q^+(X, x_0; O_{TM}(1), 0)$ are also f.p. homeomorphic over M .

Lemma 2. *The inclusion $\mathcal{E}_\varepsilon^+(X, M) \subset \mathcal{E}^+(X, M)$ is a f.p. homotopy equivalence over M .*

This lemma is verified by extending $f \in \mathcal{E}^+(X, M)$ to $\bar{f} \in \mathcal{E}^+(D, M)$ canonically and shrinking $\bar{f}(D)$ towards $f(x_0)$.

3.7.2. Reduction to the complex plane.

By the argument in the previous section it remains to construct a f.p. homotopy equivalence $\mathcal{E}_q^+(X, x_0; O_{TM}(1), 0) \simeq S(TM)$. Since $\mathcal{E}_q^+(X, x_0; O_{TM}(1), 0)$ is locally trivial, it suffices to construct a canonical homotopy equivalence $\mathcal{E}^+(X, x_0; O_V(1), 0) \simeq S(V)$ for any 2-dim oriented vector space V with an inner product.

First we work on the complex plane $\mathbb{C}$. Let $D(r)$, $O(r)$ and $C(r)$ denote the closed disk, the open disk and the circle in $\mathbb{C}$ with the center 0 and the radius r . We fix an o.p. homeomorphism $(D, x_0) \cong (D(1), 0)$ and regard as $0 \in X \subset D(1)$. Let O_2 and SO_2 denote the orthogonal group and the rotation group on $\mathbb{R}^2$ respectively. SO_2 acts on $\mathcal{E}^+(X, x_0; O(1), 0)$ by the left composition. For each $z \in C(1)$, we have the rotation θ_z of $\mathbb{C}$ defined by $\theta_z(w) = z \cdot w$, by which we can identify $C(1)$ with SO_2 . The circle $C(1)$ is naturally embedded into $\mathcal{E}^+(X, x_0; O(1), 0)$ by $z \mapsto \theta_z|_X$. The next proposition is verified in the next section.

Proposition 6. *There exists a canonical SO_2 -equivariant strong deformation retraction F_t of $\mathcal{E}^+(X, x_0; O(1), 0)$ onto $C(1)$.*

Suppose V is any oriented 2-dim vector space with an inner product and let $O_V(1)$ and $C_V(1) (= S(V))$ denote the open disk and the circle in V with the center 0 and the radius 1. For each $v \in C_V(1)$ there exists a unique o.p. linear isometry $\alpha_v : \mathbb{C} \cong V$ such that $\alpha_v(1) = v$. $C_V(1)$ can be embedded naturally into $\mathcal{E}^+(X, x_0; O_V(1), 0)$ by $C_V(1) \ni v \mapsto \alpha_v|_X$.

Choose any o.p. linear isometry $\alpha : \mathbb{C} \cong V$. Then we can define a strong deformation retraction φ_t^V of $\mathcal{E}^+(X, x_0; O_V(1), 0)$ onto $C_V(1)$ by the following formula:

$$\varphi_t^V(f) = \alpha F_t(\alpha^{-1}f).$$

This definition is independent of the choice of α due to the SO_2 -equivariance of F_t . When X is an arc, it suffices to consider the case where $X = [-1/2, 1/2] \subset O(2)$. In this case, O_2 acts on $\mathcal{E}(X, 0; O(1), 0)$ and we can construct an O_2 -equivariant strong deformation retraction

F_t of $\mathcal{E}(X, 0; O(1), 0)$ onto $C(1)$ (cf. §3.8.3 Lemma 5). Therefore, even if V is not oriented, any linear isometry $\alpha : \mathbb{C} \cong V$ can be used to define a strong deformation retraction φ_t^V of $\mathcal{E}(X, x_0; O_V(1), 0)$ onto $C_V(1)$. Thus, when X is an arc, we need no assumption on orientation.

We have completed the proof of Theorem 3 except Proposition 6.

3.8. Canonical extension and deformation of embeddings into a disk.

We identify the complex plane $\mathbb{C}$ with the plane $\mathbb{R}^2$. Let $A(r, 1)$ ($0 < r < 1$) denote the annulus region in $\mathbb{R}^2$ between the circles $C(r)$ and $C(1)$ and $\lambda_r : A(1/2, 1) \rightarrow A(r, 1)$ the natural radial homeomorphism. We fix a tuple of three points $a_0 = (-i, 1, i)$ on $C(1)$.

Below we use the conformal mapping theorem to give a canonical parametrization of $O(1) - X$ (§3.8.1), and construct a canonical extension $\Phi(f)$ of $f \in \mathcal{E}^+(X, O(1))$ (§3.8.2). The extension map Φ is SO_2 -equivariant, and using this property, we construct a SO_2 -equivariant canonical deformation $F_t(f)$ of f to a rotation (§3.8.3).

3.8.1. Canonical parametrization.

We show that $O(1) - X$ has a canonical parametrization under normalization data. In general, when G is a compact graph, $V(G)$ denotes the set of points of G which has no neighborhood homeomorphic to $\mathbb{R}$. Each point of $V(G)$ is called a vertex of G and the closure of each component of $G - V(G)$ in G is called an edge of G .

Suppose X is a compact connected polyhedron ($\neq 1$ pt) topologically embedded in $O(1)$. Then $O(1) - X$ is a disjoint union of an open annulus U and finitely many open disks U_i ($i = 1, \dots, m$). Since the frontier $\text{Fr}_{O(1)}U$ is a compact connected graph, there exists a unique cyclic chain of oriented edges $e_1, \dots, e_n$ of $\text{Fr } U$ such that if we move on these edges in this order, we obtain a unique loop ℓ_U which runs on $\text{Fr } U$ once in the “counterclockwise” orientation with seeing U in the right-hand side. Similarly, each $\text{Fr}_{O(1)}U_i$ is a compact connected graph and we can find a unique cyclic chain of oriented edges $e_1^i, \dots, e_n^i$ of $\text{Fr } U_i$ such that if we move on these edges in this order, we obtain a loop ℓ_{U_i} which runs on $\text{Fr } U_i$ once in the “counterclockwise” orientation with seeing U in the left-hand side. To normalize the data for U_i , we choose an ordered set $a_i = (x_i, y_i, z_i)$ of three distinct points lying on the loop ℓ_{U_i} in the positive order. The data $a = (a_i)$ is called normalization data for X .

By the conformal mapping theorem (existence and boundary behaviour) [6] we have canonical parametrizations of U and U_i 's.

Lemma 3. (1) *For the annulus component U , there exists a unique $r \in (0, 1)$ and a unique o.p. map $g : A(r, 1) \rightarrow \text{cl } U \subset D(1)$ such that g maps $\text{Int } A(r, 1)$ conformally onto U and $g(1) = 1$. Furthermore, g satisfies the following conditions:*

- (i) g maps $C(1)$ homeomorphically onto $C(1)$,
- (ii) (a) $g(C(r)) = \text{Fr } U$ and
- (b) *There exists a unique collection of points $\{u_1, \dots, u_n\}$ lying on $C(r)$ in counterclockwise order such that g maps each positively oriented circular arc $[u_j u_{j+1}]$ on*

$C(r)$ onto the oriented edge e_j in o.p. way and maps $\text{Int}[u_j u_{j+1}]$ homeomorphically onto $e_j - V(\text{Fr } U)$. (Here $u_{n+1} = u_1$, and when $n = 1$, we mean that $[u_1 u_1] = C(r)$.)

(2) For each (U_i, a_i) there exists a unique o.p. map $g_i : D(1) \rightarrow cl U_i \subset D(1)$ such that g_i maps $O(1)$ conformally onto U_i and $g(a_0) = a_i$. Furthermore, g_i satisfies the following conditions:

- (a) $g(C(1)) = \text{Fr } U_i$ and
- (b) There exists a unique collection of points $\{u_1^i, \dots, u_{n_i}^i\}$ lying on $C(1)$ in counterclockwise order such that g_i maps each positively oriented circular arc $[u_j^i u_{j+1}^i]$ on $C(1)$ onto the oriented edge e_j^i in o.p. way and maps $\text{Int}[u_j^i u_{j+1}^i]$ homeomorphically onto $e_j^i - V(\text{Fr } U_i)$. (Here $u_{n_i+1}^i = u_1^i$, and when $n_i = 1$, we mean that $[u_1^i u_1^i] = C(1)$.)

We set $g_0 = g\lambda_r : A(1/2, 1) \rightarrow cl U$. The collection of maps $(g_0, (g_i))$ obtained in Lemma 3 is called the canonical parametrization of $O(1) - X$ with respect to the normalization data $a = (a_i)$.

3.8.2. Canonical extensions.

We fix normalization data $a = (a_i)$ for X . We show that each $f \in \mathcal{E}^+(X, O(1))$ has a canonical extension $\Phi(f) \in \mathcal{H}^+(D(1))$. The map f has an extension $\bar{f} \in \mathcal{H}^+(D(1))$. Corresponding with the connected components $U, (U_i)$ of $O(1) - X$ and normalization data $a = (a_i)$ of X , we obtain the connected components $V = \bar{f}(U), (V_i) = (\bar{f}(U_i))$ of $O(1) - f(X)$ and normalization data $f(a) = (f(a_i))$ of $f(X)$. (These are independent of the choice of $\bar{f}$.) Applying the argument in § 3.8.1 to $(f(X), f(a))$, we obtain the canonical parametrization $(h_0, (h_i))$ of $O(1) - f(X)$ with respect to the normalization data $f(a)$.

(1) For (U, V) there exists a unique $\theta(f) \in \mathcal{H}^+(C(1/2))$ such that $h_0\theta(f) = fg_0$. Let $\Theta(f) \in \mathcal{H}^+(A(1/2, 1))$ denote the radial extension of $\theta(f)$. Then there exists a unique homeomorphism $\varphi(f) : cl U \rightarrow cl V$ such that $h_0\Theta(f) = \varphi(f)g_0$. The map $\varphi(f)$ is an extension of $f : \text{Fr } U \cong \text{Fr } V$.

(2) For each (U_i, V_i) ($i = 1, \dots, m$), there exists a unique $\theta_i(f) \in \mathcal{H}^+(C(1))$ such that $h_i\theta_i(f) = fg_i$. Let $\Theta_i(f) \in \mathcal{H}^+(D(1))$ denote the conical extension of $\theta_i(f)$. Then there exists a unique homeomorphism $\varphi_i(f) : cl U_i \rightarrow cl V_i$ such that $h_i\Theta_i(f) = \varphi_i(f)g_i$. The map $\varphi_i(f)$ is an extension of $f : \text{Fr } U_i \cong \text{Fr } V_i$.

Finally we define $\Phi(f) \in \mathcal{H}^+(D(1))$ by

$$\Phi(f) = \begin{cases} f & \text{on } X \\ \varphi(f) & \text{on } cl U \\ \varphi_i(f) & \text{on } cl U_i \end{cases}$$

The map $\Phi = \Phi_{(X,a)} : \mathcal{E}^+(X, O(1)) \rightarrow \mathcal{H}^+(D(1))$ is continuous since conformal mappings depend upon their ranges continuously.

3.8.3. Symmetry of the extension map Φ .

Next we study the naturality and symmetry properties of the extension map $\Phi_{(X,a)} : \mathcal{E}^+(X, O(1)) \rightarrow \mathcal{H}^+(D(1))$. We use the following notations: The rotation group SO_2 acts on $\mathcal{E}^+(X, O(1))$ and $\mathcal{H}^+(D(1))$ by the left composition, and SO_2 is naturally embedded into these spaces by

$\gamma \mapsto \gamma|_X, \gamma|_{D(1)}$. Let $\eta : \mathbb{R}^2 \cong \mathbb{R}^2$ denote the reflection $\eta(x, y) = (x, -y)$. The restriction of $\gamma \in O_2$ onto $D(r)$, $O(r)$ etc. are denoted by the same symbol γ .

- Lemma 4.** (1) $\Phi_{(X,a)}(gf) = \Phi_{(f(X),a(f))}(g)\Phi_{(X,a)}(f) \quad (f \in \mathcal{E}^+(X, O(1)), g \in \mathcal{E}^+(f(X), O(1)))$.
 (2) $\Phi_{(X,a)}(\gamma|_X) = \gamma \quad (\gamma \in SO_2)$.
 (3) $\Phi_{(X,a)} : \mathcal{E}^+(X, O(1)) \rightarrow \mathcal{H}^+(D(1))$ is SO_2 -equivariant.

When X is an arc, by the similar argument, we can construct the following canonical extension map. We set $\mathcal{E}^*(X, O(1)) = \mathcal{E}(X, O(1)) \times \{\pm\}$. For each $(f, \delta) \in \mathcal{E}^*(X, O(1))$ we obtain a canonical extension $\Phi(f, \delta) \in \mathcal{H}^\delta(D(1))$. The canonical extension map $\Phi = \Phi_X : \mathcal{E}^*(X, O(1)) \rightarrow \mathcal{H}(D(1))$ is continuous and has the following properties:

Lemma 5. (Case where X is an arc)

- (1) $\Phi_X(gf, \varepsilon\delta) = \Phi_{f(X)}(g, \varepsilon)\Phi_X(f, \delta) \quad ((f, \delta) \in \mathcal{E}^*(X, O(1)), (g, \varepsilon) \in \mathcal{E}^*(f(X), O(1)))$.
 (2) $\Phi_X(\gamma|_X, \delta(\gamma)) = \gamma \quad (\gamma \in O_2)$.
 (3) (i) $\Phi_X : \mathcal{E}^*(X, O(1)) \rightarrow \mathcal{H}(D(1))$ is O_2 -equivariant.
 (ii) $\Phi_{\eta(X)}(\eta f \eta|_{\eta(X)}, \delta) = \eta \Phi_X(f, \delta) \eta \quad ((f, \delta) \in \mathcal{E}^*(X, O(1)))$.

Proof of Proposition 6. Using the Alexander trick, we can construct a SO_2 -equivariant strong deformation retraction H_t of $\mathcal{H}_0^+(D(1))$ onto SO_2 . Then we can define a SO_2 -equivariant strong deformation retraction F_t of $\mathcal{E}^+(X, 0; O(1), 0)$ onto $SO_2 \cong C(1)$ by $F_t(f) = H_t(\Phi_X(f))|_X$. $\square$

Now we have completed the proofs of Theorems 1-3.

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