

NOTE ON QUESTIONS OF ANDERSON'S *

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Let G be a torsion-free abelian (additive) group and S a subsemigroup $\not\supseteq \{0\}$ of G . Then S is called a grading monoid. A non-empty subset I of S is called an ideal of S if $S + I \subset I$. If $I = S + x = (x)$ for some $x \in S$, then I is called a principal ideal of S . If $I + J_1 = I + J_2$ (for ideals J_i) implies $J_1 = J_2$, then I is called a cancellation ideal of S .

Let D be an integral domain and I an ideal of D . If $I = Dx = (x)$ for some $x \in D$, then I is called a principal ideal of D . If $IJ_1 = IJ_2$ (for ideals J_i) implies $J_1 = J_2$, then I is called a cancellation ideal of D .

A domain with a unique maximal ideal is called a local domain.

[AA]D.D.Anderson-D.F.Anderson, Math. Japon 29(1984);
posed the following,

(D)([AA]). Is a cancellation ideal of a local domain principal?

We may pose

(S)(A semigroup version of (D)). Is a cancellation ideal of a grading monoid S principal?

I proved the following,

Theorem 1. Every cancellation ideal of S is principal.

$q(S) = \{x - y \mid x, y \in S\}$ is called the quotient group of S . If $n\alpha \in S$ ($n \in \mathbb{N}$, $\alpha \in q(S)$) implies $\alpha \in S$, then S is called integrally closed.

For subsets I_1, I_2 and I of $q(S)$, we put $I_1 :_I I_2 = \{x \in I \mid x + I_2 \subset I_1\}$. And we put $S :_{q(S)} I = I^{-1}$ and $(I^{-1})^{-1} = I^v$. We have, for an ideal

*This is an abstract and the details will appear elsewhere.

$I, S = S^{-1} \subset I^{-1}$ and $S \supset I^v \supset I$.

[A] D.D.Anderson, Comm. in Alg. 16(1988);
posed the following,

(D')([A]). Characterize the integrally closed domains D for which $(A \cap B)^v = A^v \cap B^v$ for all non-zero ideals A, B of D .

We may pose

(S')(A semigroup version of (D')). Characterize the integrally closed S for which $(A \cap B)^v = A^v \cap B^v$ for all ideals A, B of S .

For each maximal ideal M of S , since $S \supset M^v \supset M$, we have either $M^v = M$ or $M^v = S$.

I proved the following,

Theorem 2. (1) Assume that S has a maximal ideal M such that $M^v = M$. If $(A \cap B)^v = A^v \cap B^v$ for all ideals A, B of S , then $I^v = I$ for all ideals I of S .

(2) Assume that for each maximal ideal M of D , $M^v = M$. If $(A \cap B)^v = A^v \cap B^v$ for all non-zero ideals A, B of D , then $I^v = I$ for all non-zero ideals I of D .

Lemma 1. Let $a \in S$. If there exists an ideal which does not contain a , then there exists a maximum ideal which does not contain a .

Proof. Let $\{J_\lambda \mid \lambda\}$ be the set of all ideals which does not contain a . Then $\bigcup_\lambda J_\lambda$ is a desired ideal.

Proof of Theorem 1. Let I be a cancellation ideal. Suppose that I is not principal.

If $I = S$, $I = (0)$ is principal; a contradiction. Hence $I \subsetneq S = (0)$. Let M be the maximum ideal which does not contain 0. Then $I = I + S \supset I + M$. If $I + S = I + M$, since I is cancellation, we have $S = M$;

a contradiction. Hence $I \not\supseteq I + M$. Choose $I \ni x \notin I + M$. Then $(x) = S + x \subset I$. Since I is not principal, $(x) \subsetneq I$. Choose $I \ni y \notin (x)$. Put $x + y = a$.

If $a \in (2x)$, $x + y = 2x + s (s \in S)$; $y = x + s \in (x)$; a contradiction. Hence $a \notin (2x)$. Let J be the maximum ideal which does not contain a . Then $2x \in J$.

(1) Let $I \ni b \in (x)$: Then $b + a = x + s + x + y = (s + y) + 2x \in I + J$.

(2) Let $I \ni b \notin (x)$: If $a \in (b + y)$, then $x + y = b + y + s$ and $x = b + s$. Since $x \notin I + M$, $s \notin M$. By the choice of M , $(s) \ni 0$. Then $0 = s + s'$ for some $s' \in S$. Then $b = x - s = x + s' \in (x)$; a contradiction. Hence $a \notin (b + y)$. Therefore $b + y \in J$. Then $b + a = x + (b + y) \in I + J$.

By (1) and (2), we have $I + a \subset I + J$. Hence $a \in J$; a contradiction.

An ideal I of S is called a quasi-cancellation ideal if $I + J_1 = I + J_2$ for finitely generated ideals J_1 and J_2 of S implies $J_1 = J_2$.

Let v be a mapping of a torsion-free abelian (additive) group G to a totally ordered abelian (additive) group. If $v(x + y) = v(x) + v(y)$ for all $x, y \in G$, then v is called a valuation on G . $\{x \in G \mid v(x) \geq 0\}$ is called the valuation semigroup belonging to v .

[SM] Sugatani-Matsuda, Proc. 19th Sympos. on Semigroups, Languages and their related Fields, 1995; showed that there is a pseudo-valuation semigroup S that is not a valuation semigroup and has a quasi-cancellation ideal which is not a cancellation ideal.

It follows

Remark 1. A quasi-cancellation ideal of S need not be principal.

Lemma 2([SM, Prop. 5]). All propositions of [AA] hold for S .

For example, we have

Lemma 3. Every ideal of S is cancellation if and only if either S is a group or S is a $\mathbf{Z}$ -valued valuation semigroup.

Lemma 4. Let A be a flat ideal of S . Then the following conditions are equivalent:

- (1) A is cancellation.
- (2) A is principal.
- (3) A is faithfully flat.

Now we have

Corollary 1. Assume that S has the ascending chain condition on principal ideals (ab. a.c.c.p.). If S has a maximal ideal M which is cancellation, then S is a $\mathbf{Z}$ -valued valuation semigroup.

Proof. By Theorem 1, $M = (x)$ is principal. Then $\{(0), (x), (2x), (3x), \dots\}$ is the set of all ideals of S . By Lemma 3, S is a $\mathbf{Z}$ -valued valuation semigroup.

Remark 2. Assume that S is a valuation semigroup and has a maximal ideal which is principal. Then S need not be a $\mathbf{Z}$ -valued valuation semigroup.

For example, put $G = \mathbf{Z} \oplus \mathbf{Z}$ with $(1, 0) < (0, 1)$. Let v be the identity mapping of G to G . Let S be the valuation semigroup belonging to v . Then the principal ideal $M = S + (1, 0)$ is a maximal ideal of S .

Remark 3. (1) Let A be an ideal of S . Then A is faithfully flat if and only if A is principal.

(2) A flat ideal of S need not be principal.

For example of (2), put $S = \mathbf{Q}_0$. Then the maximal ideal M of S is flat. But M is not principal.

There is a conjecture ([M]) which says: Almost all propositions in multiplicative ideal theory for D hold for S . And it is usually expected that ideal theory of S is simpler than that of D .

Next, a D -submodule A of $K = q(D)$ is called a fractional ideal of D , if $dA \subset D$ for some non-zero $d \in D$. Let $F(D)$ be the set of all non-zero fractional ideals of D . Let $f(D)$ be the set of finitely generated members of $F(D)$. A mapping $A \mapsto A^*$ of $F(D)$ to $F(D)$ is called a star-operation on D if the following conditions hold for all $a \in K - \{0\}$ and $A, B \in F(D)$:

- (1) $(a)^* = (a)$, $(aA)^* = aA^*$;
- (2) $A \subset A^*$; if $A \subset B$, then $A^* \subset B^*$;
- (3) $(A^*)^* = A^*$.

$A \in F(D)$ is called a $*$ -ideal if $A^* = A$. An ideal of D is also called an integral ideal of D . An ideal properly contained in D is called a proper integral ideal of D .

Let I be an ideal of D and let $A \in F(D)$. Then the subset $\{x \in K \mid sx \in A \text{ for some } s \in D - I\}$ of K is denoted by A_I , where $K = q(D)$.

Lemma 5. Let $*$ be a star-operation on D . Let $\{I_\lambda \mid \lambda\}$ be the set of proper integral $*$ -ideals of D . Then we have $\bigcap_\lambda A_{I_\lambda} \subset A^*$ for every $A \in F(D)$.

Proof. Let $0 \neq x \in \bigcap_\lambda A_{I_\lambda}$. Then we have $(A :_D x) \not\subset I_\lambda$ for every λ . Hence $(A^* :_D x) \not\subset I_\lambda$. We have $(A^* :_D x) = x^{-1}A^* \cap D$. It follows that $(A^* :_D x)^* = (x^{-1}A^* \cap D)^* \subset (x^{-1}A^*)^* \cap D^* = x^{-1}A^* \cap D = (A^* :_D x)$. Namely $(A^* :_D x)$ is a $*$ -ideal of D . It follows that $(A^* :_D x) = D$. $1 \in (A^* :_D x)$ implies $x \in A^*$. We have proved $\bigcap_\lambda A_{I_\lambda} \subset A^*$.

Theorem 3. Let $*$ be a star-operation on D . Then the following conditions are equivalent:

- (1) $(A \cap B)^* = A^* \cap B^*$ for all $A, B \in F(D)$.
- (2) $(A \cap D)^* = A^* \cap D$ for all $A \in F(D)$.
- (3) $(A :_D B)^* = (A^* :_D B^*)$ for all $A \in F(D)$ and $B \in f(D)$,
- (4) Let $S = \{I_\lambda \mid \lambda\}$ be the set of proper integral $*$ -ideals of D . Then $A^* = \bigcap_\lambda A_{I_\lambda}$ for all $A \in F(D)$.
- (5) There is a collection $S = \{I_\lambda \mid \lambda\}$ of proper integral $*$ -ideals of D with the property that every proper integral $*$ -ideal of D is contained in some I_λ , such that $A^* = \{x \in K \mid (A :_D x) \not\subset I_\lambda \text{ for every } \lambda\}$ for every

$A \in F(D)$.

Proof. (2) $\implies$ (4): By Lemma 5, we have $\bigcap_{\lambda} A_{I_{\lambda}} \subset A^*$. Conversely, let $0 \neq x \in A^*$. Then

$$(x) = (x) \cap A^* = (x)(D \cap x^{-1}A^*) = (x)(D \cap x^{-1}A)^*.$$

Hence $D = (D \cap x^{-1}A)^*$. Therefore $D \cap x^{-1}A \not\subset I_{\lambda}$ for each λ . It follows $x \in A_{I_{\lambda}}$ for each λ .

(1) $\implies$ (3): Set $B = \sum_i b_i D$. Then we have

$$(A^* :_D B^*) = \bigcap_i (b_i^{-1} A^* \cap D) = (\bigcap_i b_i^{-1} A \cap D)^* = (A :_D B)^*.$$

(5) $\implies$ (1): Let $x \in A^* \cap B^*$. There exist $s_{\lambda}, t_{\lambda} \in D - I_{\lambda}$ such that $s_{\lambda}x \in A, t_{\lambda}x \in B$ for each λ . Then $t_{\lambda}s_{\lambda}x \in A \cap B$. Then $s_{\lambda}x \in (A \cap B)^*$ for each λ . It follows $x \in ((A \cap B)^*)^*$. Hence $x \in (A \cap B)^*$.

Remark 4([M]). [A] holds for S .

Proposition. Theorem 3 holds for S .

The proof of Proposition is a semigroup version of that of Theorem 3.

Corollary 2. (1) Assume that S has a maximal ideal M such that $M^* = M$. If $(A \cap B)^* = A^* \cap B^*$ for all $A, B \in F(D)$. Then $I^* = I$ for all $I \in F(D)$.

(2) Assume that, for each maximal ideal M of D , $M^* = M$. If $(A \cap B)^* = A^* \cap B^*$ for all $A, B \in F(D)$. Then $I^* = I$ for all $I \in F(D)$.

Proof. (2) Let $\{M_{\lambda} \mid \lambda\}$ be the set of maximal ideals of D . By Theorem 3, $I^* \subset \bigcap_{\lambda} I_{M_{\lambda}} = \bigcap_{\lambda} ID_{M_{\lambda}} = I$. Hence $I^* = I$.

The proof of (1) is a semigroup version of that of (2).

Theorem 2 is a special case of Corollary 2.

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