

A bilinear estimate for commutators of fractional integral operators

By

Yoshihiro SAWANO*, Satoko SUGANO** and Hitoshi TANAKA***

Abstract

The aim of this paper is to investigate the Morrey norm boundedness of commutators generated by $BMO(\mathbb{R}^n)$ -functions and the Riesz kernel. A bilinear estimate is the focus of this paper, which cannot be obtained from a mere combination of the boundedness of commutators and the Hölder inequality. As a key tool, a decomposition using dyadic cubes is employed.

§ 1. Introduction

The aim of this paper is to investigate a bilinear estimate generated by commutators. Let $a \in BMO(\mathbb{R}^n)$ and $0 < \alpha < n$. Let $m \in \mathbb{N}$. The m -fold commutator $[I_\alpha, a]^{(m)}$ is given by

$$[I_\alpha, a]^{(m)} f(x) \equiv \int_{\mathbb{R}^n} \frac{(a(x) - a(y))^m}{|x - y|^{n-\alpha}} f(y) dy.$$

Here and below we assume that the functions are real-valued and measurable. We recall the definition of $BMO(\mathbb{R}^n)$ in Section 2. As is verified in (3.4), we shall consider

$$x \mapsto \int_{\mathbb{R}^n} \frac{|a(x) - a(y)|^m}{|x - y|^{n-\alpha}} f(y) dy$$

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*Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1 Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan.

e-mail: ysawano@tmu.ac.jp

**Kobe City College of Technology, 8-3 Gakuen-higashimachi, Nishi-ku, Kobe 651-2194, Japan.

e-mail: sugano@kobe-kosen.ac.jp

***Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo, 153-8914, Japan.

e-mail: htanaka@ms.u-tokyo.ac.jp

and hence we may assume that the integral defining $[I_\alpha, a]^{(m)}f(x)$ converges for a.e. $x \in \mathbb{R}^n$.

In the present paper we investigate the boundedness of the operator given by

$$(f, g) \mapsto g \cdot [I_\alpha, a]^{(m)}f$$

on Morrey spaces. Here, we shall adopt the following definition of the Morrey space $\mathcal{M}_p^{p_0}(\mathbb{R}^n)$, $1 < p \leq p_0 < \infty$: First we define $\mathcal{D}$ as the set of all dyadic cubes (see (2.1) below). For a measurable function f we define

$$\|f\|_{\mathcal{M}_p^{p_0}} \equiv \sup_{Q \in \mathcal{D}} |Q|^{\frac{1}{p_0} - \frac{1}{p}} \left(\int_Q |f(y)|^p dy \right)^{\frac{1}{p}}.$$

The function space $\mathcal{M}_p^{p_0}(\mathbb{R}^n)$ is the set of all measurable functions f for which the norm $\|f\|_{\mathcal{M}_p^{p_0}}$ is finite.

Now we present our main result of the present paper.

Theorem 1.1. *Suppose we are given parameters $\alpha, p_0, p, q_0, q, r_0, r$ satisfying*

$$0 < \alpha < n, 1 < p \leq p_0 < \infty, 1 < q \leq q_0 < \infty, 1 < r \leq r_0 < \infty$$

and

$$q > r, \frac{1}{p_0} > \frac{\alpha}{n} \geq \frac{1}{q_0}.$$

Assume in addition that

$$\frac{r}{r_0} = \frac{p}{p_0}, \frac{1}{p_0} + \frac{1}{q_0} - \frac{\alpha}{n} = \frac{1}{r_0}.$$

Then we have

$$(1.1) \quad \|g \cdot [I_\alpha, a]^{(m)}f\|_{\mathcal{M}_r^{r_0}} \leq C \|a\|_{\text{BMO}}^m \|f\|_{\mathcal{M}_p^{p_0}} \|g\|_{\mathcal{M}_q^{q_0}}$$

for all $a \in \text{BMO}(\mathbb{R}^n)$, $f \in \mathcal{M}_p^{p_0}(\mathbb{R}^n)$ and $g \in \mathcal{M}_q^{q_0}(\mathbb{R}^n)$.

The method of the proof of Theorem 1.1 also covers a classical theorem of the commutator: It corresponds to the case $q_0 = \infty$ and $g \equiv 1$.

Corollary 1.2. *Suppose we are given parameters α, p_0, p, r_0, r satisfying*

$$0 < \alpha < n, 1 < p \leq p_0 < \infty, 1 < r \leq r_0 < \infty.$$

Assume in addition that

$$\frac{r}{r_0} = \frac{p}{p_0}, \frac{1}{p_0} - \frac{\alpha}{n} = \frac{1}{r_0}.$$

Then we have

$$\|[I_\alpha, a]^{(m)}f\|_{\mathcal{M}_r^{r_0}} \leq C \|a\|_{\text{BMO}}^m \|f\|_{\mathcal{M}_p^{p_0}}$$

for all $a \in \text{BMO}(\mathbb{R}^n)$ and $f \in \mathcal{M}_p^{p_0}(\mathbb{R}^n)$.

The proof is left for interested readers.

This inequality (1.1) dates back to the one obtained in [9], which deals with the operator I_α given by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Proposition 1.3. *Suppose we are given parameters $\alpha, p_0, p, q_0, q, r_0, r$ satisfying*

$$0 < \alpha < n, 1 < p \leq p_0 < \infty, 1 < q \leq q_0 < \infty, 1 < r \leq r_0 < \infty$$

and

$$q > r, \frac{1}{p_0} > \frac{\alpha}{n} \geq \frac{1}{q_0}.$$

Assume in addition that

$$\frac{r}{r_0} = \frac{p}{p_0}, \frac{1}{p_0} + \frac{1}{q_0} - \frac{\alpha}{n} = \frac{1}{r_0}.$$

Then we have

$$(1.2) \quad \|g \cdot I_\alpha f\|_{\mathcal{M}_r^{r_0}} \leq C \|f\|_{\mathcal{M}_p^{p_0}} \|g\|_{\mathcal{M}_q^{q_0}}$$

for all $f \in \mathcal{M}_p^{p_0}(\mathbb{R}^n)$ and $g \in \mathcal{M}_q^{q_0}(\mathbb{R}^n)$.

As was discussed in [10, p.7], Proposition 1.3 is not an immediate consequence of the well-known boundedness of I_α due to Adams [1] and the Hölder inequality. The same can be said for Theorem 1.1; it cannot be deduced directly from Corollary 1.2 and the Hölder inequality. The inequality (1.2) is called the Olsen inequality and investigated initially in [7].

Several people have tried to extend the original results (see [3, 15] for more details). In [9] we proved that the condition on $q > r$ is sharp. Although we mean (1.2) by the Olsen inequality (see [3, 9, 10, 17], for example), we overlooked the original paper [2]. In [2], on $\mathbb{R}^3$, Conlon and Redondo considered the following equation:

$$\begin{cases} (-\Delta - \mathbf{b}(x) \cdot \nabla)u(x) = f(x) & (|x| < R), \\ u(x) = 0 & (|x| = R), \end{cases}$$

for $R > 0$. If $\mathbf{b}$ is smooth, then as is described in [12], we have an expression of the solution;

$$u(x) = E_x \left[\int_0^\tau f(X_{\mathbf{b}}(t)) dt \right],$$

where $X_{\mathbf{b}}(t)$ is a Brownian motion starting from $x \in \{|y| < R\}$ with drift $\mathbf{b}$, E_x denotes the expectation with respect to $X_{\mathbf{b}}(t)$ and τ is the first hitting time on the boundary $|x| = R$. Conlon and Redondo proved Proposition 1.3 with $n = 3$ essentially.

One of the reasons why (1.2) holds is that in the Adams theorem I_α is not surjective. Indeed, we have;

Proposition 1.4. *Suppose $0 < \alpha < n$, $1 < p \leq p_0 < \infty$, $1 < r \leq r_0 < \infty$. Assume that*

$$\frac{r}{r_0} = \frac{p}{p_0}, \quad \frac{1}{r_0} = \frac{1}{p_0} - \frac{\alpha}{n}.$$

Then, I_α is bounded from $\mathcal{M}_p^{p_0}(\mathbb{R}^n)$ to $\mathcal{M}_r^{r_0}(\mathbb{R}^n)$.

Proposition 1.5. *In Proposition 1.4, I_α is not surjective from $\mathcal{M}_p^{p_0}(\mathbb{R}^n)$ to $\mathcal{M}_r^{r_0}(\mathbb{R}^n)$.*

Proposition 1.5 was proven as [11, Corollary 3.6]. However, in Section 4, we give an alternative proof. Recently, in [13, 14] an inequality dealing with I_α and intersection of Morrey spaces was considered. In this note, by using this new type of inequality, we reprove Proposition 1.5.

Seemingly, Theorem 1.1 is a consequence of Corollary 1.2 and the following lemma:

Lemma 1.6. *Let $1 < q_1 \leq p_1 < \infty$ and $1 < q_2 \leq p_2 < \infty$. Define*

$$p \equiv \frac{p_1 p_2}{p_1 + p_2}, \quad q \equiv \frac{q_1 q_2}{q_1 + q_2}.$$

Then

$$\|f \cdot g\|_{\mathcal{M}_q^p} \leq \|f\|_{\mathcal{M}_{q_1}^{p_1}} \|g\|_{\mathcal{M}_{q_2}^{p_2}}$$

for all $f \in \mathcal{M}_{q_1}^{p_1}(\mathbb{R}^n)$ and $g \in \mathcal{M}_{q_2}^{p_2}(\mathbb{R}^n)$.

However, this is not the case; a mere combination of Proposition 1.4 and Lemma 1.6 does not give Theorem 1.1. Indeed, Morrey spaces are nested:

$$\mathcal{M}_{p_1}^{p_0}(\mathbb{R}^n) \subset \mathcal{M}_{p_2}^{p_0}(\mathbb{R}^n)$$

for all $1 < p_2 \leq p_1 < \infty$. The following example shows that the inclusion is strict:

Example 1.7. For $r < 1/2$, and $\vec{e} \in \{0, 1\}^n$, we define

$$S_{r, \vec{e}}(x) = rx + (1 - r)\vec{e} \quad (x \in \mathbb{R}^n).$$

Define inductively $\{E_j\}_{j=0}^\infty$ by

$$E_0 = [0, 1]^n, \quad E_j = \bigcup_{\vec{e} \in \{0, 1\}^n} S_{r, \vec{e}}(E_{j-1}) \quad (j = 1, 2, \dots).$$

Then we have

$$\|\chi_{E_j}\|_{\mathcal{M}_q^p} \simeq \max(\|\chi_{[0, r]^n}\|_{L^p}, \|\chi_{E_j}\|_{L^q})$$

for all j , where the implicit constants in $\simeq$ do not depend upon j and r .

A detailed calculation in [10, p.6] shows that the case when $\frac{p}{p_0}r_0 < q < \frac{p}{p_0}q_0$ is beyond the reach of the combination of Corollary 1.2 and Lemma 1.6.

We can pass our result to the operator given by

$$[I_\alpha, (a_1, a_2, \dots, a_m)]f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \prod_{j=1}^m (a_j(x) - a_j(y)) dy.$$

Theorem 1.8. *Suppose we are given parameters $\alpha, p_0, p, q_0, q, r_0, r$ satisfying*

$$0 < \alpha < n, 1 < p \leq p_0 < \infty, 1 < q \leq q_0 < \infty, 1 < r \leq r_0 < \infty$$

and

$$q > r, \frac{1}{p_0} > \frac{\alpha}{n} \geq \frac{1}{q_0}.$$

Assume in addition that

$$\frac{r}{r_0} = \frac{p}{p_0}, \frac{1}{p_0} + \frac{1}{q_0} - \frac{\alpha}{n} = \frac{1}{r_0}.$$

Then we have

$$\|g \cdot [I_\alpha, (a_1, a_2, \dots, a_m)]f\|_{\mathcal{M}_r^{r_0}} \leq C \left(\prod_{j=1}^m \|a_j\|_{\text{BMO}} \right) \|f\|_{\mathcal{M}_p^{p_0}} \|g\|_{\mathcal{M}_q^{q_0}}$$

for all $a_1, a_2, \dots, a_m \in \text{BMO}(\mathbb{R}^n)$, $f \in \mathcal{M}_p^{p_0}(\mathbb{R}^n)$ and $g \in \mathcal{M}_q^{q_0}(\mathbb{R}^n)$.

Theorem 1.8 follows from Theorem 1.1, a homogeneity argument and the following lemma, the proof of which will be given in the appendix:

Lemma 1.9. *For all $m \in \mathbb{N}$, the polynomial $x_1 x_2 \cdots x_m$ is in the linear span of the set*

$$\mathcal{V}_m \equiv \{(a_1 x_1 + a_2 x_2 + \cdots + a_m x_m)^m : a_1, a_2, \dots, a_m \in \mathbb{R}\}.$$

§ 2. Notations and preliminaries

Here we fix some notations.

1. We define the set of all dyadic cubes as follows:

$$(2.1) \quad \mathcal{D} \equiv \left\{ \prod_{j=1}^n \left[\frac{m_j}{2^\nu}, \frac{m_j+1}{2^\nu} \right) : \nu \in \mathbb{Z}, m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n \right\}.$$

If a dyadic cube Q has volume $2^{-\nu n}$, then we say that Q is of the ν -th generation. We also write $\mathcal{D}_\nu$ the set of all dyadic cubes of the ν -th generation. If $Q \in \mathcal{D}_\nu$, then define $\ell(Q) = 2^{-\nu}$. Observe that

$$(2.2) \quad \sum_{Q \in \mathcal{D}_\nu} \chi_Q \equiv 1.$$

2. The open ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$ will be denoted by $B(x, r)$.
3. Given a function f and a dyadic cube $Q \in \mathcal{D}$, we set $m_Q(f) \equiv \frac{1}{|Q|} \int_Q f(x) dx$.
4. By a cube we mean a compact cube whose edges are parallel to the coordinate axes. The set $\mathcal{Q}$ denotes the totality of all cubes. For a point $x \in \mathbb{R}^n$, we write $\mathcal{Q}(x)$ for the set of all cubes in $\mathcal{Q}$ containing x .
5. The function space $\text{BMO}(\mathbb{R}^n)$ is the set of all measurable functions f for which the quantity

$$\|f\|_{\text{BMO}} \equiv \sup_{Q \in \mathcal{Q}} m_Q(|f - m_Q(f)|)$$

is finite.

6. The maximal operator is defined by

$$Mf(x) \equiv \sup_{Q \in \mathcal{Q}(x)} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

There are several variants: write

$$\begin{aligned} M^{(u)}f(x) &\equiv \sup_{Q \in \mathcal{Q}(x)} \left(\frac{1}{|Q|} \int_Q |f(y)|^u dy \right)^{\frac{1}{u}} \quad (u \in (1, \infty)), \\ M_\alpha f(x) &\equiv \sup_{Q \in \mathcal{Q}(x)} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy \quad (\alpha \in [0, n)). \end{aligned}$$

We shall recall some fundamental facts of the maximal operator M_α above and $\text{BMO}(\mathbb{R}^n)$ -functions.

Lemma 2.1. *Suppose the parameters α, p_0, p, r_0, r satisfy*

$$0 \leq \alpha < n, \quad 1 < p \leq p_0 < \infty, \quad 1 < r \leq r_0 < \infty.$$

Assume in addition that

$$\frac{r}{r_0} = \frac{p}{p_0}, \quad \frac{1}{p_0} - \frac{\alpha}{n} = \frac{1}{r_0}, \quad \frac{1}{p_0} > \frac{\alpha}{n}.$$

Then we have

$$\|M_\alpha f\|_{\mathcal{M}_r^{r_0}} \leq C \|f\|_{\mathcal{M}_p^{p_0}}.$$

Here it will be understood that M_0 denotes the Hardy-Littlewood maximal operator M .

Lemma 2.2 (The John-Nirenberg inequality). *Let $1 \leq p < \infty$ and let Q be a cube. Then there exists a constant $c > 0$ such that*

$$\left(\frac{1}{|Q|} \int_Q |a(x) - m_Q(a)|^p dx \right)^{\frac{1}{p}} \leq c \|a\|_{\text{BMO}}$$

for all $a \in \text{BMO}(\mathbb{R}^n)$.

We also need a decomposition result about cubes. Let Q_0 be a cube and let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We set

$$\mathcal{D}(Q_0) \equiv \{Q \in \mathcal{D} : Q \subset Q_0\}.$$

We write $3Q_0$ for its triple, that is, the unique cube concentric to Q_0 and having the volume $3^n|Q_0|$. Letting $\gamma_0 \equiv m_{3Q_0}(f)$ and $A = 2 \cdot 18^n$, we set, for $k = 1, 2, \dots$,

$$D_k \equiv \bigcup \{Q : Q \in \mathcal{D}(Q_0), m_{3Q}(f) > \gamma_0 A^k\}.$$

Considering the maximal cubes with respect to inclusion, we can write

$$D_k = \bigcup_j Q_{k,j},$$

where the cubes $\{Q_{k,j}\}_j \subset \mathcal{D}(Q_0)$ are nonoverlapping. That is, $\{Q_{k,j}\}_j$ is a family of cubes satisfying

$$\sum_j \chi_{Q_{k,j}} \leq \chi_{Q_0}$$

for almost everywhere. By the maximality of $Q_{k,j}$ we see that

$$(2.3) \quad \gamma_0 A^k < m_{3Q_{k,j}}(f) \leq 2^n \gamma_0 A^k.$$

Let

$$E_0 \equiv Q_0 \setminus D_1, \quad E_{k,j} \equiv Q_{k,j} \setminus D_{k+1}.$$

We need the following properties:

Lemma 2.3. ([5]) *The set $\{E_0\} \cup \{E_{k,j}\}$ forms a disjoint family of sets, which decomposes Q_0 , and satisfies*

$$(2.4) \quad |Q_0| \leq 2|E_0|, \quad |Q_{k,j}| \leq 2|E_{k,j}|.$$

For the sake of completeness we recall the proof here.

Proof. By (2.3) we see that

$$Q_{k,j} \cap D_{k+1} \subset \{x \in Q_{k,j} : M[\chi_{3Q_{k,j}} f](x) > \gamma_0 A^{k+1}\}.$$

Using the weak-(1, 1) boundedness of M , we have

$$(2.5) \quad |Q_{k,j} \cap D_{k+1}| \leq \frac{3^n}{\gamma_0 A^{k+1}} \int_{3Q_{k,j}} |f(y)| dy \leq \frac{6^n}{A} |3Q_{k,j}| = \frac{18^n}{A} |Q_{k,j}| = \frac{1}{2} |Q_{k,j}|,$$

where we have used again (2.3). Similarly, we see that

$$(2.6) \quad |D_1| \leq \frac{1}{2} |Q_0|.$$

Clearly, (2.5) and (2.6) imply (2.4). □

§ 3. Proof of Theorem 1.1

We depend on the method of Li and Perez [6, 8]. Here and below we can assume that f and g are positive.

§ 3.1. Set up

We set

$$\begin{aligned} C_1[f, g](x) &\equiv g(x) \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) \int_{3Q} |m_Q(a) - a(y)|^m f(y) dy \\ C_2[f, g](x) &\equiv g(x) \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) |a(x) - m_Q(a)|^m \int_{3Q} f(y) dy. \end{aligned}$$

We decompose $C_2[f, g]$ according to Q_0 : We write

$$\begin{aligned} C_{21}[f, g](x) &\equiv 3^n g(x) \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu, Q \supset Q_0} \ell(Q)^\alpha \chi_Q(x) |a(x) - m_Q(a)|^m m_{3Q}(f) \\ C_{22}[f, g](x) &\equiv 3^n g(x) \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu, Q \not\supset Q_0} \ell(Q)^\alpha \chi_Q(x) |a(x) - m_Q(a)|^m m_{3Q}(f). \end{aligned}$$

Note that $C_2[f, g] = C_{21}[f, g] + C_{22}[f, g]$.

Let us recall the notation of Lemma 2.3. We set

$$\mathcal{D}_0(Q_0) \equiv \{Q \in \mathcal{D}(Q_0) : m_{3Q}(f) \leq \gamma_0 A\}$$

and

$$\mathcal{D}_{k,j}(Q_0) \equiv \{Q \in \mathcal{D}(Q_0) : Q \subset Q_{k,j}, \gamma_0 A^k < m_{3Q}(f) \leq \gamma_0 A^{k+1}\}.$$

Then we obtain

$$(3.1) \quad \mathcal{D}(Q_0) = \mathcal{D}_0(Q_0) \cup \bigcup_{k,j} \mathcal{D}_{k,j}(Q_0).$$

Next, we shall choose $\theta \in (1, p)$ and $s \in (r, q)$ so that

$$(3.2) \quad s\theta < q$$

and that

$$(3.3) \quad s'\theta < r.$$

Write

$$\theta' \equiv \frac{\theta}{\theta - 1}.$$

§ 3.2. Decomposition of the operator $[I_\alpha, a]^{(m)} f(x)$

We first obtain a pointwise estimate of $[I_\alpha, a]^{(m)} f(x)$; by using

$$(3.4) \quad |[I_\alpha, a]^{(m)} f(x)| \leq \int_{\mathbb{R}^n} \frac{|a(x) - a(y)|^m}{|x - y|^{n-\alpha}} f(y) dy$$

we obtain

$$\begin{aligned} |[I_\alpha, a]^{(m)} f(x)| &\leq \sum_{\nu \in \mathbb{Z}} \int_{2^{-\nu-1} < |x-y| < 2^{-\nu}} \frac{|a(x) - a(y)|^m}{|x - y|^{n-\alpha}} f(y) dy \\ &\leq C \sum_{\nu \in \mathbb{Z}} 2^{\nu(n-\alpha)} \int_{2^{-\nu-1} < |x-y| < 2^{-\nu}} |a(x) - a(y)|^m f(y) dy. \end{aligned}$$

Now that $\mathcal{D}_\nu$ partitions $\mathbb{R}^n$ according to (2.2), we have

$$\begin{aligned} |[I_\alpha, a]^{(m)} f(x)| &\leq C \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) \int_{2^{-\nu-1} < |x-y| < 2^{-\nu}} |a(x) - a(y)|^m f(y) dy \\ &\leq C \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) \int_{B(x, 2^{-\nu})} |a(x) - a(y)|^m f(y) dy. \end{aligned}$$

We recall that we denote by $3Q$ the triple of a dyadic cube Q ; $3Q$ is made up of 3^n dyadic cubes of equal size and the center of $3Q$ is that of Q . A geometric observation shows that $B(x, 2^{-\nu}) \subset 3Q$ if $x \in Q \in \mathcal{D}_\nu$. Consequently we obtain

$$|[I_\alpha, a]^{(m)} f(x)| \leq C \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) \int_{3Q} |a(x) - a(y)|^m f(y) dy.$$

Recall that $m_Q(a)$ denotes the average of a over a cube Q . Using $m_Q(a)$, we shall decompose

$$\begin{aligned} |g(x)[I_\alpha, a]^{(m)} f(x)| &\leq C |g(x)| \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) \int_{3Q} |m_Q(a) - a(y)|^m f(y) dy \\ &\quad + C |g(x)| \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} 2^{\nu(n-\alpha)} \chi_Q(x) |a(x) - m_Q(a)|^m \int_{3Q} f(y) dy \\ &= CC_1[f, g](x) + CC_2[f, g](x) \\ &= CC_1[f, g](x) + CC_{21}[f, g](x) + CC_{22}[f, g](x). \end{aligned}$$

Hence we have

$$(3.5) \quad |g(x)[I_\alpha, a]^{(m)}f(x)| \leq C(C_1[f, g](x) + C_{21}[f, g](x) + C_{22}[f, g](x)).$$

Thus, we are led to analyzing three operators $C_1[f, g]$, $C_{21}[f, g]$ and $C_{22}[f, g]$.

§ 3.3. Estimate for $C_1[f, g]$

The analysis of $C_1[f, g]$ depends on (1.2): First, we choose θ slightly larger than 1. By the John-Nirenberg inequality (see Lemma 2.2), we have

$$(3.6) \quad \begin{aligned} & \frac{1}{|Q|} \int_{3Q} |m_Q(a) - a(y)|^m f(y) dy \\ & \leq \frac{1}{|Q|} \left(\int_{3Q} |m_Q(a) - a(y)|^{m\theta'} dy \right)^{\frac{1}{\theta'}} \left(\int_{3Q} f(y)^\theta dy \right)^{\frac{1}{\theta}} \\ & \leq C \|a\|_{\text{BMO}}^m \inf_{y \in Q} M^{(\theta)} f(y). \end{aligned}$$

Consequently, by inserting (3.6) to $C_1[f, g]$, we are led to a pointwise estimate:

$$C_1[f, g](x) \leq C \|a\|_{\text{BMO}}^m g(x) I_\alpha [M[M^{(\theta)} f]](x).$$

If we use (1.2) and $\theta < p$, then we have

$$(3.7) \quad \|C_1[f, g]\|_{\mathcal{M}_r^{p_0}} \leq C \|a\|_{\text{BMO}}^m \|f\|_{\mathcal{M}_p^{p_0}} \|g\|_{\mathcal{M}_q^{q_0}}.$$

This is an estimate we are looking for.

§ 3.4. Estimate for $C_{21}[f, g]$

We aim to estimate

$$\Pi_1 = |Q_0|^{\frac{1}{r_0} - \frac{1}{r}} \left(\int_{Q_0} |C_{21}[f, g](x)|^r dx \right)^{\frac{1}{r}}.$$

The estimate for $C_{21}[f, g](x)$ is simple. Let us denote by Q_k the unique cube containing Q_0 and satisfying $|Q_k| = 2^{kn}|Q_0|$. By the Hölder inequality, if we set $\nu \equiv -\log_2 |Q_k|^{\frac{1}{n}}$,

we have

$$\begin{aligned}
\Pi_1 &= |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} \left\{ 2^{\nu(n-\alpha)} g(x) \chi_{Q_k}(x) |a(x) - m_{Q_k}(a)|^m \int_{3Q_k} f(y) dy \right\}^r dx \right)^{\frac{1}{r}} \\
&= \int_{3Q_k} f(y) dy \times |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} \left\{ 2^{\nu(n-\alpha)} g(x) |a(x) - m_{Q_k}(a)|^m \right\}^r dx \right)^{\frac{1}{r}} \\
&\leq C \int_{3Q_k} f(y) dy \times |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} \left\{ 2^{\nu(n-\alpha)} g(x) |a(x) - m_{Q_0}(a)|^m \right\}^r dx \right)^{\frac{1}{r}} \\
&\quad + C |m_{Q_0}(a) - m_{Q_k}(a)|^m \int_{3Q_k} f(y) dy \times |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} \left\{ 2^{\nu(n-\alpha)} g(x) \right\}^r dx \right)^{\frac{1}{r}} \\
&\leq C 2^{-k\left(\frac{n}{p_0}-\alpha\right)} (1+k^m) \|f\|_{\mathcal{M}_p^{p_0}} \|g\|_{\mathcal{M}_q^{q_0}} \|a\|_{\text{BMO}}^m.
\end{aligned}$$

Here for the last inequality we have invoked the fact that, for every $k \in \mathbb{Z}$, we have $|m_Q(a) - m_R(a)| \leq C \|a\|_{\text{BMO}}$, if $Q \in \mathcal{D}_k$ is engulfed by $R \in \mathcal{D}_{k-1}$. Assuming that $p_0 < \frac{n}{\alpha}$, we see that this estimate is summable over $k \in \mathbb{N} \cup \{0\}$. Hence, we have

$$(3.8) \quad \Pi_1 = |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} |C_{21}[f, g](x)|^r dx \right)^{\frac{1}{r}} \leq C \|f\|_{\mathcal{M}_p^{p_0}} \|g\|_{\mathcal{M}_q^{q_0}} \|a\|_{\text{BMO}}^m.$$

Thus, the control of $C_{21}[f, g](x)$ is valid.

§ 3.5. Estimate for $C_{22}[f, g]$

The heart of the matter, as is the case with the operator $g \cdot [I_\alpha, a]^{(m)} f$, is to estimate $C_{22}[f, g]$.

Finally, we aim to estimate

$$\Pi_2 = |Q_0|^{\frac{1}{r_0}-\frac{1}{r}} \left(\int_{Q_0} |C_{22}[f, g](x)|^r dx \right)^{\frac{1}{r}}.$$

To investigate $C_{22}[f, g](x)$, we linearize the estimate: Choose a positive element $w \in L^{r'}(Q_0)$ with norm 1 so that

$$(3.9) \quad \left(\int_{Q_0} |C_{22}[f, g](x)|^r dx \right)^{\frac{1}{r}} \leq 2 \int_{Q_0} C_{22}[f, g](x) w(x) dx.$$

Using (3.1), we decompose the right-hand side of (3.9) as follows:

$$\begin{aligned}
&\int_{Q_0} C_{22}[f, g](x) w(x) dx \\
&= \sum_{Q \in \mathcal{D}_0(Q_0)} \ell(Q)^\alpha \left(\int_Q |a(x) - m_Q(a)|^m g(x) w(x) dx \right) m_{3Q}(f) \\
(3.10) \quad &+ \sum_{k,j} \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \ell(Q)^\alpha \left(\int_Q |a(x) - m_Q(a)|^m g(x) w(x) dx \right) m_{3Q}(f).
\end{aligned}$$

From the Hölder inequality and Lemma 2.2, we have

$$\begin{aligned}
& \int_Q |a(x) - m_Q(a)|^m g(x) w(x) dx \\
& \leq \left(\int_Q |a(x) - m_Q(a)|^{m\theta'} dx \right)^{\frac{1}{\theta'}} \left(\int_Q (g(x)w(x))^\theta dx \right)^{\frac{1}{\theta}} \\
& = |Q|^{\frac{1}{\theta'}} \left(\frac{1}{|Q|} \int_Q |a(x) - m_Q(a)|^{m\theta'} dx \right)^{\frac{1}{\theta'}} \left(\int_Q (g(x)w(x))^\theta dx \right)^{\frac{1}{\theta}} \\
& \leq C \|a\|_{\text{BMO}}^m |Q| \left(\frac{1}{|Q|} \int_Q (g(x)w(x))^\theta dx \right)^{\frac{1}{\theta}} \leq C \|a\|_{\text{BMO}}^m \int_Q M^{(\theta)}[gw](x) dx.
\end{aligned}$$

This implies

$$\begin{aligned}
& \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \ell(Q)^\alpha \left(\int_Q |a(x) - m_Q(a)|^m g(x) w(x) dx \right) m_{3Q}(f) \\
(3.11) \quad & \leq C \|a\|_{\text{BMO}}^m \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \ell(Q)^\alpha \left(\int_Q M^{(\theta)}[gw](x) dx \right) m_{3Q}(f).
\end{aligned}$$

It follows from the definition of $\mathcal{D}_{k,j}(Q_0)$, $\alpha/n < 1$, support condition and (2.3) that

$$\begin{aligned}
& \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \ell(Q)^\alpha \left(\int_Q M^{(\theta)}[gw](x) dx \right) m_{3Q}(f) \\
& = \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} |Q|^{\frac{\alpha}{n}} \left(\int_Q M^{(\theta)}[gw](x) dx \right) m_{3Q}(f) \\
& \leq C |Q_{k,j}|^{\frac{\alpha}{n}} \left(\int_{Q_{k,j}} M^{(\theta)}[gw](x) dx \right) m_{3Q_{k,j}}(f) \\
& = C |Q_{k,j}|^{\frac{\alpha}{n}} m_{Q_{k,j}}(M^{(\theta)}[gw]) m_{3Q_{k,j}}(f) |Q_{k,j}| \\
(3.12) \quad & \leq C |Q_{k,j}|^{\frac{\alpha}{n}} m_{Q_{k,j}}(M^{(\theta)}[gw]) m_{3Q_{k,j}}(f) |E_{k,j}|.
\end{aligned}$$

The Hölder inequality gives

$$M^{(\theta)}[gw] \leq M^{(s'\theta)}w \cdot M^{(s\theta)}g$$

and hence

$$(3.13) \quad m_{Q_{k,j}}(M^{(\theta)}[gw]) \leq \left(m_{Q_{k,j}}((M^{(s'\theta)}w)^{q'}) \right)^{\frac{1}{q'}} \left(m_{Q_{k,j}}((M^{(s\theta)}g)^q) \right)^{\frac{1}{q}}.$$

Lemma 2.1 with $\alpha = 0$, the Morrey boundedness of M enables us that, noticing (3.2),

$$(3.14) \quad |Q_{k,j}|^{\frac{1}{q_0}} \left(m_{Q_{k,j}}((M^{(s\theta)}g)^q) \right)^{\frac{1}{q}} \leq C \|g\|_{\mathcal{M}_q^{q_0}}.$$

(3.11)–(3.14) yield

$$\begin{aligned}
 & \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \ell(Q)^\alpha \left(\int_Q |a(x) - m_Q(a)|^m g(x) w(x) dx \right) m_{3Q}(f) \\
 (3.15) \quad & \leq C \|a\|_{\text{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} |Q_{k,j}|^{\frac{\alpha}{n} - \frac{1}{q_0}} \left(m_{Q_{k,j}}((M^{(s'\theta)} w)^{q'}) \right)^{\frac{1}{q'}} m_{3Q_{k,j}}(f) |E_{k,j}|.
 \end{aligned}$$

Similarly, we see that

$$\begin{aligned}
 & \sum_{Q \in \mathcal{D}_0(Q_0)} \ell(Q)^\alpha \left(\int_Q |a(x) - m_Q(a)|^m g(x) w(x) dx \right) m_{3Q}(f) \\
 (3.16) \quad & \leq C \|a\|_{\text{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} |Q_0|^{\frac{\alpha}{n} - \frac{1}{q_0}} \left(m_{Q_0}((M^{(s'\theta)} w)^{q'}) \right)^{\frac{1}{q'}} m_{3Q_0}(f) |E_0|.
 \end{aligned}$$

Gathering all factors (3.10), (3.15) and (3.16) and using the fact that $\{E_0\} \cup \{E_{k,j}\}$ forms a disjoint family of sets, which decomposes Q_0 , we see that the right-hand side of (3.9) is bounded by constant times

$$\begin{aligned}
 & \|a\|_{\text{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} \int_{Q_0} M^{(q')} [M^{(s'\theta)} w](x) M_{\alpha-(n/q_0)} f(x) dx \\
 & \leq C \|a\|_{\text{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} \left(\int_{Q_0} M^{(q')} [M^{(s'\theta)} w](x)^{r'} dx \right)^{\frac{1}{r'}} \left(\int_{Q_0} M_{\alpha-(n/q_0)} f(x)^r dx \right)^{\frac{1}{r}}.
 \end{aligned}$$

Recall that θ is slightly larger than 1. Since $\frac{r'}{q'} > 1$ and $\frac{r'}{s'\theta} > 1$, we have

$$\left(\int_{Q_0} M^{(q')} [M^{(s'\theta)} w](x)^{r'} dx \right)^{\frac{1}{r'}} \leq C.$$

Thus,

$$\begin{aligned}
 & |Q_0|^{\frac{1}{r_0} - \frac{1}{r}} \left(\int_{Q_0} |C_{22}[f, g](x)|^r dx \right)^{\frac{1}{r}} \\
 & \leq C \|a\|_{\text{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} |Q_0|^{\frac{1}{r_0} - \frac{1}{r}} \left(\int_{Q_0} M_{\alpha-(n/q_0)} f(x)^r dx \right)^{\frac{1}{r}}
 \end{aligned}$$

Finally, Lemma 2.1 gives

$$(3.17) \quad \Pi_2 = |Q_0|^{\frac{1}{r_0} - \frac{1}{r}} \left(\int_{Q_0} |C_{22}[f, g](x)|^r dx \right)^{\frac{1}{r}} \leq C \|a\|_{\text{BMO}}^m \|g\|_{\mathcal{M}_q^{q_0}} \|f\|_{\mathcal{M}_p^{p_0}}.$$

From (3.5), (3.7), (3.8) and (3.17), we conclude the proof of Theorem 1.1.

§ 4. Proof of Proposition 1.5

We recall the following estimate by Hedberg:

Lemma 4.1. *Suppose the parameters α, p, s satisfy*

$$0 < \alpha < n, \quad 1 < p < s < \infty, \quad \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}.$$

Let f be a positive measurable function. Then

$$I_\alpha f(x) \leq CM f(x)^{\frac{p}{s}} \|f\|_{\mathcal{M}_1^p}^{1-\frac{p}{s}}.$$

Proof. The proof is well-known but for the sake of completeness, we supply it. By the Fubini theorem, we obtain

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \left(\int_{|x-y|}^{\infty} \frac{n-\alpha}{\ell^{n-\alpha+1}} d\ell \right) f(y) dy = (n-\alpha) \int_0^\infty \left(\frac{1}{\ell^{n-\alpha+1}} \int_{B(x,\ell)} f(y) dy \right) d\ell.$$

If we insert the estimates

$$\frac{1}{\ell^n} \int_{B(x,\ell)} f(y) dy \leq CM f(x), \quad \frac{1}{\ell^{n-n/p}} \int_{B(x,\ell)} f(y) dy \leq C \|f\|_{\mathcal{M}_1^p},$$

then we obtain

$$I_\alpha f(x) \leq C \int_0^\infty \min(\ell^{\alpha-1} M f(x), \ell^{\alpha-n/p-1} \|f\|_{\mathcal{M}_1^p}) d\ell = CM f(x)^{\frac{p}{s}} \|f\|_{\mathcal{M}_1^p}^{1-\frac{p}{s}}.$$

□

Corollary 4.2. *Let $1 < t \leq s < \infty$ and $1 < q \leq p < \infty$ satisfy*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t}{s} = \frac{q}{p}.$$

Then

$$\|I_\alpha f\|_{\mathcal{M}_t^s} \leq C \|f\|_{\mathcal{M}_q^p}^{\frac{p}{s}} \|f\|_{\mathcal{M}_1^p}^{1-\frac{p}{s}}.$$

We prove Proposition 1.5.

Proof. Since I_α is known to be injective, if I_α were surjective, then by virtue of the open mapping theorem, $I_\alpha : \mathcal{M}_q^p(\mathbb{R}^n) \rightarrow \mathcal{M}_t^s(\mathbb{R}^n)$ would be isomorphic. So, we would have a constant C such that

$$C^{-1} \|f\|_{\mathcal{M}_q^p} \leq \|I_\alpha f\|_{\mathcal{M}_t^s} \leq C \|f\|_{\mathcal{M}_q^p}.$$

If we combine this with Corollary 4.2, then we obtain

$$C^{-1} \|f\|_{\mathcal{M}_q^p} \leq C \|f\|_{\mathcal{M}_q^p}^{\frac{p}{s}} \|f\|_{\mathcal{M}_1^p}^{1-\frac{p}{s}}.$$

This implies that $\mathcal{M}_1^p(\mathbb{R}^n) \subset \mathcal{M}_q^p(\mathbb{R}^n)$. Since $\mathcal{M}_1^p(\mathbb{R}^n) \supset \mathcal{M}_q^p(\mathbb{R}^n)$ is known to hold, it follows that $\mathcal{M}_1^p(\mathbb{R}^n) = \mathcal{M}_q^p(\mathbb{R}^n)$ with norm equivalence. This is a contradiction to Example 1.7. $\square$

§ 5. Appendix: The proof of Lemma 1.9

In what follows we prove Lemma 1.9. When $m = 2, 3$, this is true as the following identities show:

$$\begin{aligned} x_1 x_2 &= \frac{1}{4}(x_1 + x_2)^2 - \frac{1}{4}(x_1 - x_2)^2. \\ x_1 x_2 x_3 &= \frac{1}{24}(x_1 + x_2 + x_3)^3 \\ &\quad - \frac{1}{24}(x_1 + x_2 - x_3)^3 - \frac{1}{24}(x_1 - x_2 + x_3)^3 - \frac{1}{24}(-x_1 + x_2 + x_3)^3. \end{aligned}$$

Suppose that Lemma 1.9 is correct for $m = m_0$. Then by the induction assumption, it suffices to prove that $\xi^{m_0} x_{m_0+1}$ is in the linear span of $\mathcal{V}_{m_0+1}$, where $\xi \equiv a_1 x_1 + a_2 x_2 + \cdots + a_{m_0} x_{m_0}$.

Consider an $m_0 + 2$ matrix

$$A \equiv \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & m_0 + 2 \\ 1^2 & 2^2 & 3^2 & \cdots & (m_0 + 2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1^{m_0+1} & 2^{m_0+1} & 3^{m_0+1} & \cdots & (m_0 + 2)^{m_0+1} \end{pmatrix}.$$

Then, by virtue of the Vandermonde determinant, A becomes invertible. We now set

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{m_0+2} \end{pmatrix} \equiv A^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We notice that

$$\sum_{j=1}^{m_0+2} b_j j^k = \begin{cases} 1 & (k = 1), \\ 0 & (k = 0, 2, 3, \dots, m_0 + 1). \end{cases}$$

This implies

$$\sum_{j=1}^{m_0+2} b_j (\xi + j x_{m_0+2})^{m_0+1} = (m_0 + 1) \xi^{m_0} x_{m_0+2}.$$

So, Lemma 1.9 is correct for $m = m_0 + 1$.

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