

THE ROLE OF BOUNDARY HARNACK PRINCIPLE

IN THE STUDY OF PICARD PRINCIPLE

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A nonnegative locally Hölder continuous function P on $0 < |z| \leq 1$ will be referred to as a *density* on $\Omega: 0 < |z| < 1$. A density on Ω gives rise to an elliptic operator L_P on Ω defined by

$$(1) \quad L_P u = \Delta u - Pu, \quad \Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2.$$

We say that the *Picard principle* (abbreviated as PP) is valid for P , rather for L_P , at $z=0$ if the dimension of the half module of nonnegative solutions of $L_P u = 0$ on Ω with vanishing boundary values on $\partial\Omega - \{z=0\}$ is 1. With the operator L_P we associate an elliptic operator $\hat{L}_P$ on Ω , referred to as the *associate operator* to L_P , given by

$$(2) \quad \hat{L}_P v = \Delta v + 2V \log e_P \cdot \nabla v, \quad \nabla = (\partial / \partial x, \partial / \partial y),$$

where e_P , referred to as the *P-unit* on Ω , is the unique bounded solution of $L_P u = 0$ on Ω with boundary values 1 on $\partial\Omega - \{z=0\}$. We also say that the *Riemann theorem* (abbreviated as RT) is valid for $\hat{L}_P$ at $z=0$ if the limit $\lim_{z \rightarrow 0} v(z)$ exists for every bounded solution v of $\hat{L}_P v = 0$ on Ω . Then we have the *duality theorem* (cf. Heins [3], Hayashi [2], Nakai [8]): The Picard principle is valid for L_P at $z=0$ if and only if the Riemann theorem is valid for $\hat{L}_P$ at $z=0$. As a sufficient condition for the Riemann theorem for $\hat{L}_P$ at $z=0$ we have, what we call, the *boundary Harnack principle* (abbreviated as BHP) for L_P at $z=0$ (Kawamura [6]):

$$(3) \quad \left\{ \begin{array}{l} \text{For every Jordan region } U \text{ in } |z| < 1 \text{ containing } z=0 \text{ there exists a} \\ \text{Jordan region } V_U \text{ containing } z=0 \text{ such that } \bar{V}_U \subset U \text{ and } u(\zeta) \leq C u(\xi) \text{ for} \\ \text{every nonnegative bounded solution } u \text{ of } L_P u = 0 \text{ on } U - \{z=0\} \text{ and } \zeta, \xi \\ \text{in } \partial V_U, \text{ where } C \text{ is a positive constant independent of } U, u, \zeta, \text{ and } \xi. \end{array} \right.$$

In fact (3) implies the *boundary Harnack principle* for $\hat{L}_p$ at $z=0$ which is formulated in the same fashion as it is done for L_p originally considered by Kawamura [6] and then the Riemann theorem for $\hat{L}_p$ at $z=0$ is deduced from the boundary Harnack principle for $\hat{L}_p$ at $z=0$ ([6]). In short it has been known that the following string of implications holds:

$$(4) \quad \text{BHP for } L_p \Rightarrow \text{BHP for } \hat{L}_p \Rightarrow \text{RT for } \hat{L}_p \Leftrightarrow \text{PP for } L_p.$$

The purpose of this lecture is to show that the Picard principle for L_p conversely implies the boundary Harnack principle for L_p ([9]). Therefore we can conclude that properties appearing in (4) are in fact all equivalent to each other :

$$(5) \quad \text{BHP for } L_p \Leftrightarrow \text{BHP for } \hat{L}_p \Leftrightarrow \text{RT for } \hat{L}_p \Leftrightarrow \text{PP for } L_p.$$

We will also give an example of a density satisfying the boundary Harnack principle at $z=0$ ([9]) : If a density P on Ω satisfies $Q(z) \leq P(z) \leq Q(z) + C/|z|^2$ for a positive constant C and a *rotation free* density Q on Ω , i.e. a density satisfying $Q(z) = Q(|z|)$, for which the Picard principle is valid at $z=0$, then the boundary Harnack principle is valid for L_p at $z=0$ so that the Picard principle is valid for L_p at $z=0$.

1. The Harnack principle.

We will define a Harnack constant $C(K, \Omega_a, P)$ and deduce the *ordinary* Harnack principle. For a density P on Ω and a real number a in $(0,1]$ we denote by $G_p^{\Omega_a}$ the P -Green's function on $\Omega_a = \{0 < |z| < a\}$, i.e. the Green's function on Ω_a with respect to the equation $L_p u = 0$. We consider a Harnack constant $C(K, \Omega_a, P)$ of a compact subset K of Ω_a defined by

$$C(K, \Omega_a, P) = \max \left\{ \frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \xi)} ; |z| = a \text{ and } \zeta, \xi \text{ are in } K \right\},$$

where $\partial/\partial n_z$ means the inner normal derivative. Then the integral representation of a bounded solution of $L_P u = 0$ in terms of the inner normal derivative of the P-Green's function yields the following Harnack principle: for any nonnegative bounded solution u of $L_P u = 0$ on $\bar{\Omega}_a - \{z=0\}$ and ζ, ξ in K we have

$$u(\zeta) \leq C(K, \Omega_a, P) u(\xi).$$

2. The boundary Harnack principle.

We will show that the Picard principle for L_P implies the boundary Harnack principle for L_P , and hence they are equivalent. Let P be a density on Ω such that the Picard principle is valid for L_P at $z=0$. Then the function $G_P^{\Omega_a}(z, \zeta)/e_P(\zeta)$ in z converges uniformly on every compact subset of $\bar{\Omega}_a - \{z=0\}$ as $\zeta \rightarrow 0$, and hence the inner normal derivative $\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)/e_P(\zeta)$ converges to a positive continuous function on $\partial\Omega_a - \{z=0\}$ (cf Itô [5]). In order to show (3) we consider two cases separately: $\limsup_{\zeta \rightarrow 0} e_P(\zeta) = 0$ and > 0 .

First we consider the case $\limsup_{\zeta \rightarrow 0} e_P(\zeta) = 0$, i.e. $\lim_{\zeta \rightarrow 0} e_P(\zeta) = 0$. For every λ in $(0,1)$ let A_λ be a connected component of $\{\zeta \in \Omega; e_P(\zeta) < \lambda\}$ such that $z=0$ is an isolated boundary point of A_λ . Observe that $\bar{A}_\lambda \downarrow \{z=0\}$ as $\lambda \rightarrow 0$ and

$$\frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \xi)} = \frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)}{e_P(\zeta)} \frac{e_P(\xi)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \xi)}$$

for ζ, ξ in $\partial A_\lambda - \{z=0\}$. Then we have

$$\lim_{\lambda \rightarrow 0} C(\partial A_\lambda - \{z=0\}, \Omega_a, P) = 1$$

so that for every subregion U of $\{|z| < 1\}$ containing $z=0$ we can take a_U, λ_U in $(0,1)$ with $\Omega_{a_U} \subset U$ and $C(\partial A_{\lambda_U} - \{z=0\}, \Omega_{a_U}, P) < 2$. Therefore (3) is valid for $C = 2$ and $V_U = A_{\lambda_U} \cup \{z=0\}$.

Assume next that $\limsup_{\zeta \rightarrow 0} e_P(\zeta) \equiv \delta > 0$. There exists a closed set E thin at $z=0$ in Ω such that $e_P(\zeta) \rightarrow \delta$ as $\zeta \rightarrow 0$ with $\zeta \notin E$ (cf BreLOT [1]). Then we can take a decreasing sequence $\{\lambda_n\}_1^\infty$ in $(0,1)$ with $E \cap \bigcup_1^\infty (\partial \Omega_{\lambda_n} - \{z=0\}) = \emptyset$ and $\lim \lambda_n = 0$. Observe that $e_P(\zeta) \rightarrow \delta$ as $\zeta \rightarrow 0$ with $\zeta \in \bigcup_1^\infty (\partial \Omega_{\lambda_n} - \{z=0\})$ and

$$\frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \xi)} = \frac{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta)}{e_P(\zeta)} \frac{e_P(\xi)}{\frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \xi)} \frac{e_P(\zeta)}{e_P(\xi)}$$

for ζ, ξ in $\partial \Omega_{\lambda_n} - \{z=0\}$. Then we have

$$\lim_{n \rightarrow \infty} C(\partial \Omega_{\lambda_n} - \{z=0\}, \Omega_a, P) = 1.$$

Therefore (3) is valid for $C = 2$ and $V_U = \Omega_{\lambda_n} \cup \{z=0\}$ for some n depending on U .

3. Fundamental properties of units.

We now recall some of fundamental properties of the Q_n -unit. Let Q be a *rotation free* density on Ω , i.e. a density satisfying $Q(z) = Q(|z|)$. We consider a rotation free density $Q_n(z) = Q(z) + n^2/|z|^2$ on Ω for every nonnegative integer n and the Q_n -unit $f_n(z, a)$ on Ω_a , i.e. an unique bounded solution of $L_{Q_n} u = 0$ on Ω_a with boundary values 1 on $\partial \Omega_a - \{z=0\}$, where we follow the convention $Q_0 = Q$ and $f_0(z, 1) = e_Q(z)$. Then $f_n(z, a)$ is rotation free and $f_n(r, a)$ is an unique bounded solution of

$$(6) \quad \ell_n \psi(r) \equiv \ell_{Q_n} \psi(r) \equiv \frac{d^2}{dr^2} \psi(r) + \frac{1}{r} \frac{d}{dr} \psi(r) - Q_n(r) \psi(r) = 0$$

on $(0, a)$ with boundary values 1 at $r = a$. We have the following properties of $f_n(r, a)$ (cf Nakai [7]):

$$(7) \quad f_n(r, \rho) = \frac{f_n(r, a)}{f_n(\rho, a)} \quad (0 < r \leq a, r \leq \rho \leq a);$$

$$(8) \quad f_n(r, a) > f_{n+1}(r, a) \quad (0 < r < a);$$

$$(9) \quad \frac{f_{n+1}(r, a)}{f_n(r, a)} \geq \frac{f_{n+2}(r, a)}{f_{n+1}(r, a)} \quad (0 < r \leq a);$$

$$(10) \quad \left\{ \frac{f_{n+1}(r, a)}{f_n(r, a)} \right\}^3 \leq \frac{f_{n+2}(r, a)}{f_{n+1}(r, a)} \quad (0 < r \leq a);$$

the Picard principle is valid for L_Q at $z = 0$ if and only if

$$(11) \quad \lim_{r \rightarrow 0} \frac{f_1(r, a)}{f_0(r, a)} = 0$$

for some a , and hence by (7) any a in $(0, 1]$. For another rotation free density R on Ω with $Q \leq R$ we have also (cf Imai [4])

$$(12) \quad \frac{f_{n+1}(r, a)}{f_n(r, a)} \leq \frac{g_{n+1}(r, a)}{g_n(r, a)} \quad (0 < r \leq a),$$

where $R_n(z) = R(z) + n^2/|z|^2$ and $g_n(z, a)$ is the R_n -unit on Ω_a ($n=0, 1, \dots$).

4. Fourier coefficients of solutions.

We consider Fourier coefficients

$$\begin{cases} c_0(r, w) = \frac{1}{2\pi} \int_0^{2\pi} w(re^{i\theta}) d\theta, \\ a_n(r, w) = \frac{1}{\pi} \int_0^{2\pi} w(re^{i\theta}) \cos n\theta d\theta, \end{cases}$$

$$b_n(r, w) = \frac{1}{\pi} \int_0^{2\pi} w(re^{i\theta}) \sin n\theta d\theta$$

for a continuous function $w(z)$ on $\bar{\Omega}_a - \{z=0\}$. Here and hereafter let Q be a rotation free density on Ω and $f_n(z, a)$ the Q_n -unit on Ω_a . If w is further a bounded solution of $L_Q u = 0$ on Ω_a , then the Fourier coefficients of w are bounded solutions of (6):

$$\ell_0 c_0(r, w) = \ell_n a_n(r, w) = \ell_n b_n(r, w) = 0.$$

Therefore they are represented in terms of Q_n -units:

$$\begin{cases} c_0(r, w) = c_0(a, w) f_0(r, a), \\ a_n(r, w) = a_n(a, w) f_n(r, a), \\ b_n(r, w) = b_n(a, w) f_n(r, a) \end{cases}$$

$(0 < r \leq a; n = 1, 2, \dots)$.

5. Normal derivatives of Green's functions.

We expand the inner normal derivative of the Q -Green's function into its Fourier series. For any τ in $[0, 2\pi)$ we denote by w_τ a bounded solution of $L_Q u = 0$ on Ω_a with boundary values 1 on $\{ae^{i\theta}; 0 < \theta < \tau\}$ and 0 on $\{ae^{i\theta}; \tau < \theta < 2\pi\}$. Then w_τ is represented in an integral form:

$$w_\tau(se^{i\sigma}) = \frac{1}{2\pi} \int_0^\tau \left[-\frac{\partial}{\partial r} G_Q^a(re^{i\theta}, se^{i\sigma}) \right]_{r=a} a d\theta$$

for any $se^{i\sigma}$ in Ω_a . On the other hand w_τ is represented in a Fourier series:

$$\begin{aligned} w_\tau(se^{i\sigma}) &= c_0(a, w_\tau) f_0(s, a) \\ &+ \sum_{n=1}^{\infty} \{a_n(a, w_\tau) \cos n\sigma + b_n(a, w_\tau) \sin n\sigma\} f_n(s, a). \end{aligned}$$

Since by (8) and (9) we have

$$(13) \quad \frac{f_1(s,a)}{f_0(s,a)} < 1, \quad f_n(s,a) \leq f_0(s,a) \left\{ \frac{f_1(s,a)}{f_0(s,a)} \right\}^n,$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial \tau} w_\tau(se^{i\sigma}) &= \frac{\partial}{\partial \tau} c_0(a, w_\tau) f_0(s, a) \\ &+ \sum_{n=1}^{\infty} \frac{\partial}{\partial \tau} \{a_n(a, w_\tau) \cos n\sigma + b_n(a, w_\tau) \sin n\sigma\} f_n(s, a). \end{aligned}$$

Observe that

$$\frac{\partial}{\partial \tau} c_0(a, w_\tau) = \frac{\partial}{\partial \tau} \frac{1}{2\pi} \int_0^\tau d\theta = \frac{1}{2\pi},$$

$$\frac{\partial}{\partial \tau} a_n(a, w_\tau) = \frac{\partial}{\partial \tau} \frac{1}{\pi} \int_0^\tau \cos n\theta d\theta = \frac{1}{\pi} \cos n\tau,$$

and

$$\frac{\partial}{\partial \tau} b_n(a, w_\tau) = \frac{1}{\pi} \sin n\tau.$$

Then we expand the inner normal derivative of the Q-Green's function into the following Fourier series :

$$\left[-\frac{\partial}{\partial r} G_Q^\Omega a(re^{i\tau}, se^{i\sigma}) \right]_{r=a} = \frac{1}{a} \{f_0(s, a) + 2 \sum_{n=1}^{\infty} f_n(s, a) \cos n(\sigma - \tau)\}.$$

Estimating the right hand side of this equality by using (13) we have the following inequalities :

$$(14) \quad \left[-\frac{\partial}{\partial r} G_Q^\Omega a(re^{i\tau}, se^{i\sigma}) \right]_{r=a} \leq \frac{1}{a} f_0(s, a) \left\{ 1 + \frac{f_1(s, a)}{f_0(s, a)} \right\} \left\{ 1 - \frac{f_1(s, a)}{f_0(s, a)} \right\}^{-1}$$

and

$$(15) \quad \left[-\frac{\partial}{\partial r} G_Q^\Omega a(re^{i\tau}, se^{i\sigma}) \right]_{r=a} \geq \frac{1}{a} f_0(s, a) \left\{ 1 - 3 \frac{f_1(s, a)}{f_0(s, a)} \right\} \left\{ 1 - \frac{f_1(s, a)}{f_0(s, a)} \right\}^{-1}.$$

6. The Picard principle.

We give an example of a density on Ω satisfying the boundary Harnack

principle, and hence the Picard principle. Let P be a general and Q a rotation free density on Ω such that the Picard principle is valid for L_C at $z=0$ and

$$Q(z) \leq P(z) \leq Q(z) + \frac{C}{|z|^2}$$

for a positive constant C . We take a positive integer k with $9k^2 > C$ and consider a rotation free density $R(z) = Q(z) + 9k^2/|z|^2$ on Ω . First we evaluate the inner normal derivative of the P -Green's function in terms of Q_n -unit $f_n(z,a)$ and R_n -unit $g_n(z,a)$ on Ω_a . Since the P -Green's function satisfies

$$G_R^{\Omega_a} \leq G_P^{\Omega_a} \leq G_Q^{\Omega_a}$$

we have

$$\begin{aligned} \left[-\frac{\partial}{\partial r} G_R^{\Omega_a}(re^{i\tau}, se^{i\sigma}) \right]_{r=a} &\leq \left[-\frac{\partial}{\partial r} G_P^{\Omega_a}(re^{i\tau}, se^{i\sigma}) \right]_{r=a} \\ &\leq \left[-\frac{\partial}{\partial r} G_Q^{\Omega_a}(re^{i\tau}, se^{i\sigma}) \right]_{r=a} \end{aligned}$$

for every τ in $[0, 2\pi)$ and $se^{i\sigma}$ in Ω_a . Then by (12), (14), and (15) we obtain

$$(16) \quad \frac{\left[-\frac{\partial}{\partial r} G_P^{\Omega_a}(re^{i\tau}, se^{i\alpha}) \right]_{r=a}}{\left[-\frac{\partial}{\partial r} G_P^{\Omega_a}(re^{i\tau}, se^{i\beta}) \right]_{r=a}} \leq \frac{f_0(s,a)}{g_0(s,a)} \frac{1 + \frac{g_1(s,a)}{g_0(s,a)}}{1 - 3 \frac{g_1(s,a)}{g_0(s,a)}}$$

for any α, β in $[0, 2\pi)$ if $g_1(s,a)/g_0(s,a) < 1/3$.

Next we evaluate $f_0(s,a)/g_0(s,a)$ in terms of $g_1(s,a)/g_0(s,a)$. From (10) it follows that

$$(17) \quad \frac{g_{4k}(s,a)}{g_0(s,a)} \geq \left\{ \frac{g_1(s,a)}{g_0(s,a)} \right\}^{(81^k - 1)/2}$$

and

$$(18) \quad \frac{f_{3k}(s,a)}{f_0(s,a)} \geq \left\{ \frac{f_1(s,a)}{f_0(s,a)} \right\}^{(27^k - 1)/2}.$$

Observe that $g_0 = f_{3k}$ and $g_{4k} = f_{5k}$. Then (17), (18), and

$$(19) \quad \frac{f_{5k}(s,a)}{f_{3k}(s,a)} \leq \left\{ \frac{f_1(s,a)}{f_0(s,a)} \right\}^{2k}$$

yield an evaluation

$$\frac{f_0(s,a)}{g_0(s,a)} \leq \left\{ \frac{g_0(s,a)}{g_1(s,a)} \right\}^{\alpha_k},$$

where $\alpha_k = (81^k - 1)(27^k - 1)/8k$.

Now we show the boundary Harnack principle (3) for L_p at $z=0$. We have by (17) and (19)

$$\frac{g_1(s,a)}{g_0(s,a)} \leq \left\{ \frac{f_1(s,a)}{f_0(s,a)} \right\}^{4k/(81^k - 1)}.$$

Then by (11) we can take s_a in $(0,a)$ such that $g_1(s_a,a)/g_0(s_a,a) = 1/4$.

Therefore by (16) we obtain

$$C(\partial\Omega_{s_a} - \{z=0\}, \Omega_a, P) \leq 5 \cdot 4^{\alpha_k}.$$

Thus (3) is valid for $C = 5 \cdot 4^{\alpha_k}$ and $V_U = \Omega_{s_a} \cup \{z=0\}$, where a is a positive number with $\Omega_a \subset U$, so that the Picard principle is valid for L_p at $z=0$.

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