

DYNAMICAL SYSTEMS ON DRAGON DOMAINS

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ABSTRACT

Dynamical systems on fractal domains are studied. These domains are called twindragon, tetradragon and cross dragon respectively.

1. INTRODUCTION

We can see the following fact in Knuth¹⁾: For any complex number there exists the zero-one sequence $a_k, a_{k-1}, \dots, a_0, a_{-1}, \dots$ such that

$$z = \sum_{-\infty \leq j \leq k} a_j (i-1)^j,$$

that is, every complex number has a "binary" representation with base $i-1$. This fact suggests an existence of a number theoretic dynamical system $(\hat{X}_{i-1}, \hat{T}_{i-1}, \hat{\mu})$ which induces the binary expansion. Actually if there exists a domain $\hat{X}_{i-1}$ and its partition $\{\hat{X}_{i-1,0}, \hat{X}_{i-1,1}\}$ such that

$$(i) \hat{X}_{i-1} = \hat{X}_{i-1,0} \cup \hat{X}_{i-1,1} \text{ and } \text{int}(\hat{X}_{i-1,0}) \cap \text{int}(\hat{X}_{i-1,1}) = \emptyset$$

$$(ii) \hat{X}_{i-1} = (i-1)\hat{X}_{i-1,0} = (i-1)\hat{X}_{i-1,1} - 1,$$

then the transformation $\hat{T}_{i-1}$ on $\hat{X}_{i-1}$ such that

$$\hat{T}_{i-1}z = (i-1)z - \hat{a}((i-1)z)$$

where $\hat{a}(z)=j$ if $z \in j + \hat{X}_{i-1,j}$, $j=0,1$, induces the binary expansion.

On the other hand we can see also the followings in Davis and Knuth²⁾: for any complex integer $m+in$, there exists a revolving sequence of finite length $\delta_1, \delta_2, \dots, \delta_k$ such that

$$m + in = \sum_{j=1}^k \delta_j (1+i)^{k-j}$$

where the revolving sequence $(\delta_1, \delta_2, \dots)$ is defined by the following conditions:

$$(i) \quad \delta_j \in \{0, 1, -i, -1, i\}$$

$$(ii) \quad \text{if } (\delta_1, \dots, \delta_j) \neq (0, \dots, 0)$$

$$\text{then } \delta_{j+1} = 0 \text{ or } (-i)\delta_{k_0} \text{ for all } j \in \mathbb{N}$$

$$\text{where } k_0 = \max\{k; \delta_k \neq 0, 1 \leq k \leq j\}$$

$$(iii) \quad \text{if } (\delta_1, \dots, \delta_j) = (0, \dots, 0)$$

$$\text{then } \delta_{j+1} \in \{0, \pm 1, \pm i\}.$$

This fact also suggests an existence of a number theoretic dynamical system (X, T, ν) which induces the revolving expansion

$$z = \sum_{k=1}^{\infty} \delta_k (1+i)^{-k}$$

We consider the existence problem of above dynamical systems $(\hat{X}_{i-1}, \hat{T}_{i-1}, \hat{\mu})$ and (X, T, ν) and show that the boundaries of these domains $\hat{X}_{i-1}$ and X , called the twindragon

and the tetradragon respectively, are indeed fractal sets ³⁾.

Moreover we propose a new construction of the dragon different from the paper folding process and consider a dynamical system (Y, S, λ) on a domain tiled by four dragon which is not the tetradragon, called a cross dragon. Suprisingly we can show that this cross dragon system ⁴⁾ is actually the dual system for a very simple group endomorphism T_L on $\mathbb{T}^2$ such that

$$T_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} [x-y] \\ [x+y] \end{pmatrix}.$$

2. BINARY EXPANSION ON TWINDRAGON

Firstly consider the binary expansion with base $(1+i)$;

$$z = \sum_{k=1}^{\infty} \varepsilon_k (1+i)^{-k},$$

where $\varepsilon_k \in \{0, i\}$ for all $k \in \mathbb{N}$. If there exist a dynamical system (X_{1+i}, T_{1+i}, μ) which induces this representation, then the domain must be the limit points of Q_n such that

$$Q_n = \{ \sum_{k=1}^n \varepsilon_k (1+i)^{-k}; \varepsilon_k \in \{0, i\} \}$$

and also $X_{1+i, \varepsilon}$, $\varepsilon = 0, i$, must be the limit point of $Q_{n, \varepsilon} = \{ \sum_{k=1}^n \varepsilon_k (1+i)^{-k}; \varepsilon_1 = \varepsilon \}$ in the Hausdorff metric space $(\mathcal{J}, d)$. For after discussions we put

$$P_{n+1} = (1+i)Q_n \quad \text{for } n \geq 1,$$

that is,

$$P_{n+1} = \{ \sum_{k=0}^{n-1} \varepsilon_k (1+i)^{-k}; \varepsilon_k \in \{0, i\} \}.$$

We consider the shape and properties of X_{1+i} such that $d(X_{1+i}, P_n) \rightarrow 0$ as $n \rightarrow \infty$. Let U be a closed square with vertices $0, 1, 1-i$ and $-i$, and for each point $x(\varepsilon_0, \dots, \varepsilon_{n-1}) \in P_{n+1}$ we prepare the neighborhood of a point $x(\varepsilon_0, \dots, \varepsilon_{n-1})$ such that

$$U_{x(\varepsilon_0, \dots, \varepsilon_{n-1})} = x(\varepsilon_0, \dots, \varepsilon_{n-1}) + (1+i)^{-(n-1)}U,$$

and let F_{n+1} and B_{n+1} be

$$F_{n+1} = \bigcup_{x \in P_{n+1}} U_{x(\varepsilon_0, \dots, \varepsilon_{n-1})}$$

and

$$B_{n+1} = \partial F_{n+1}$$

respectively. We call B_{n+1} a $(n+1)$ -step Bernoulli boundary (Fig.1(a)). We give the names for each side of B_{n+1} as a following way: For each $n \geq 1$ we name each side of the square $(1+i)^{-(n-1)}U$ A, B, A^{-1} and B^{-1} respectively, then we obtain names of each side of the neighborhood of point $x(\varepsilon_0, \dots, \varepsilon_{n-1})$ according to above namings. Therefore we can read a sequence of names for B_{n+1} as to be $A_{n+1,1}, A_{n+1,2}, \dots, A_{n+1,m(n)}$ where $A_{n+1,1}$ is a first name of a side $[0, (1+i)^{-(n-1)}(-i)]$ and $A_{n+1,k} \in \{A, A^{-1}, B, B^{-1}\}$ is a name of k -th side of B_{n+1} .

Lemma(2.1)

The names of each side of B_{n+1} are obtained from these of B_n by the substitution $\Theta: A \rightarrow AB, B \rightarrow A^{-1}B$, that is, the names of each side of $B_{n+1} = \Theta^n(ABA^{-1}B^{-1})$.

By the way, recall the notation by Dekking^{5),6)} for our purpose. Let G be a finite set of symbols, G^* the free

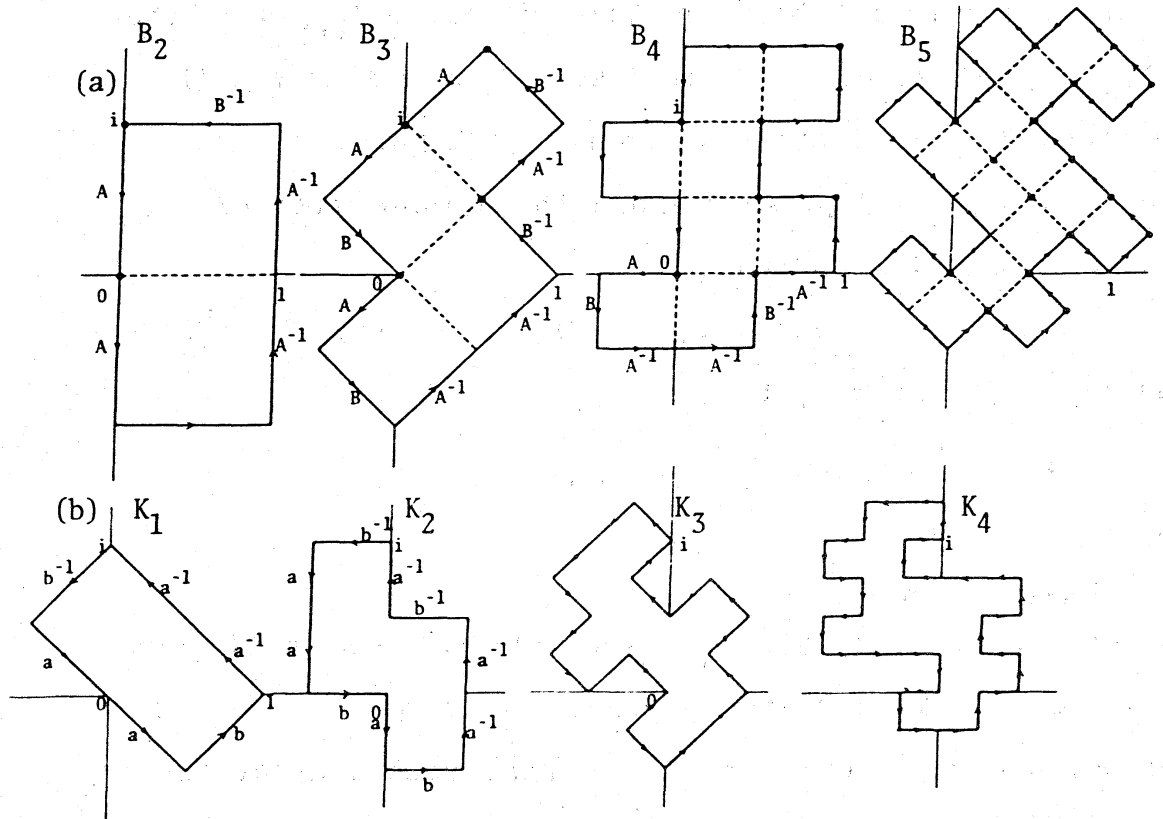


Fig.1: Bernoulli boundary B_n and Dragon boundary K_{n-1} .

semigroup generated by G and $\theta: G^* \rightarrow G^*$ a semigroup endomorphism. Let $f: G^* \rightarrow \mathbb{C}$ be a homeomorphism which satisfies

$$f(VW) = f(V) + f(W), \quad f(V^{-1}) = -f(V)$$

for all words $V, W \in G^*$. Define a map $K: S^* \rightarrow \mathbb{C}$, which satisfies

$$K[VW] = K[V] \cup (K[W] + f(V))$$

for all reduced words $V, W \in G^*$, by

$$K[s] = \{tf(s); 0 \leq t \leq 1\} \text{ for } s \in G.$$

This makes $K[s_1 \dots s_m]$ the polygonal line with vertices at $0, f(s_1), f(s_1) + f(s_2), \dots, f(s_1) + \dots + f(s_m)$.

Especially we consider here a following case.

$$G = \{a, b\}, f(a) = 1, f(b) = i,$$

and

$$\theta: \quad \theta(a) = ab, \quad \theta(b) = a^{-1}b.$$

Then the following relation holds

$$f\theta = (1+i)f.$$

We put

$$K_n = (1+i)^{-n} K[\theta^n(aba^{-1}b^{-1})],$$

and call K_n a n -step dragon boundary (Fig.1(b)).

Theorem (Dekking^{5),6)}

(1) There exists a closed curve K_θ such that

$$(1+i)^{-n} K[\theta^n(aba^{-1}b^{-1})] \rightarrow K_\theta \quad \text{as } n \rightarrow \infty$$

in the Hausdorff metric,

(2) $\dim_H K_\theta = 2 \log \beta_0 / \log 2$, where β_0 is a unique real root of $\beta^3 - \beta^2 - 2 = 0$.

K_θ is called a dragon boundary or a twindragon skin because of lemma(3.2).

We obtain a following relation between B_n and K_n .

Lemma(2.2)

$$B_{n+1} = 2(1+i)^{-1} (K_{n-1}).$$

Corollary(2.3)

Let X_{1+i} and $X_{1+i,\varepsilon}$, $\varepsilon=0,1$, be convergent sets of Q_n and $Q_{n,\varepsilon}$ ($\varepsilon=0,1$) in the Hausdorff metric (Fig.2), then

- (1) ∂X_{1+i} is similar to the dragon boundary.
- (2) $X_{1+i} = X_{1+i,0} \cup X_{1+i,i}$.
- (3) $X_{1+i} = (1+i)X_{1+i,0} = (1+i)X_{1+i,i} - i$.
- (4) $\dim_H(X_{1+i,0} \cap X_{1+i,i}) = 2\log\beta_0/\log 2$.

Thus we can define a transformation T_{1+i} on X_{1+i} by

$$T_{1+i}z = (1+i)z - [(1+i)z]_{1+i}$$

where a digit $[z]_{1+i}$ be

$$[w]_{1+i} = \begin{cases} 0 & \text{if } w \in X_{1+i,0} \\ i & \text{if } w \in i + X_{1+i,i} \end{cases}$$

Then we obtain

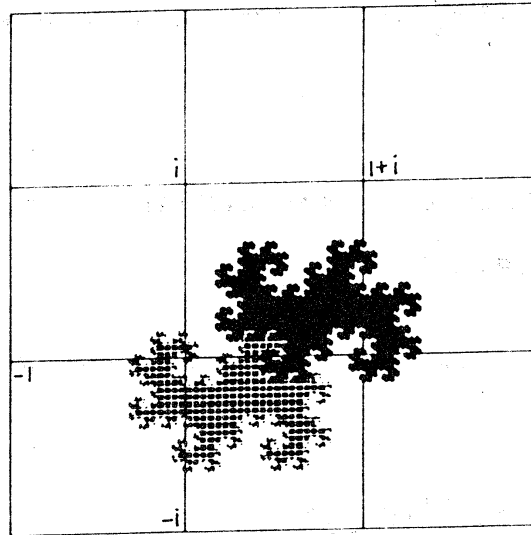


Fig.2: Domain X_{1+i} .

Theorem(2.1)

(1) The transformation (X_{1+i}, T_{1+i}) induces the complex binary expansion for a.e. $z \in X_{1+i}$ such that

$$z = \sum_{k=1}^{\infty} a_k(z) (1+i)^{-k}$$

where $a_k(z) = [(1+i)T_{1+i}^{k-1}z]_{1+i}$.

(2) The Lebesgue measure μ is invariant with respect to (X_{1+i}, T_{1+i}) and the dynamical system (X_{1+i}, T_{1+i}, μ) is isomorphic to the two states Bernoulli system.

Remark:

(i) Put

$$X_{1-i} = \overline{X_{1+i}}, \quad [w]_{1-i} = \overline{[w]_{1+i}},$$

where $\overline{\quad}$ means to take a complex conjugate, and

$$T_{1-i}z = (1-i)z - [(1-i)z]_{1-i} \quad \text{for } z \in X_{1-i}.$$

Then dynamical system (X_{1-i}, T_{1-i}, μ) induces the complex binary expansion with base $(1-i)$.

(ii) Putting

$$X_{i-1} = \frac{1-2i}{5} + X_{1-i},$$

$$X_{i-1, \varepsilon} = \frac{1-2i}{5} + X_{1-i, \varepsilon}, \quad \varepsilon = 0, -i.$$

and

$$T_{i-1}z = (i-1)z - [(i-1)z]_{i-1},$$

where $[w]_{i-1} = \varepsilon$ if $w \in \varepsilon + X_{i-1}$, then (X_{i-1}, T_{i-1}, μ) is well defined and induces the complex binary expansion with base $(i-1)$.

(iii) Taking a complex conjugate of (X_{i-1}, T_{i-1}, μ) , then the dynamical system $(X_{-1-i}, T_{-1-i}, \mu)$ is obtained and induces the

complex binary expansion with base $(-1-i)$.

We remark that the sets X_{i-1} and X_{-1-i} include the origin as an internal point respectively.

(iv) The set of the twin dragons $\{X_{1+i}+m+in; m+in \in \mathbb{Z}(i)\}$ tiles the whole plane, that is,

$$\bigcup_{m+in} X_{1+i}+m+in = \mathbb{C}$$

and

$$\mu\left(\bigcup_{m+in} \partial(X_{1+i}+m+in)\right) = 0.$$

3. REVOLVING EXPANSION ON TETRADRAGON

Let $M=(M_{j,k})$, $j,k \in \{0,1,2,3\}$, be a 0-1 matrix such that

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

and (X_M, σ_M) a Markov subshift (topological Markov chain) for the structure matrix M . Define a coding function Ψ_0 and a isomorphism Ψ on X_M by

$$\Psi_0(\varepsilon_1, \varepsilon_2) = \delta_1 = \begin{cases} 0 & \text{for } \varepsilon_1 - \varepsilon_2 = 0 \\ 1 & \text{for } \varepsilon_1 = 0 \\ -i & \text{for } \varepsilon_1 = 1 \\ -1 & \text{for } \varepsilon_1 = 2 \\ i & \text{for } \varepsilon_1 = 3 \end{cases} \quad \text{and } \varepsilon_1 - \varepsilon_2 \neq 0,$$

and for each $\omega \in X_M$

$$\Psi(\omega) = \{\Psi_0(\sigma_M^{n-1}\omega)\}_{n=1}^{\infty}.$$

Then we obtain,

Proposition(3.1)

Let W be a set of the revolving sequences. Then the map Ψ is one-one onto from $X_M \setminus \{(\varepsilon_1, \varepsilon_2, \dots) : \varepsilon_j = a \text{ for all } j \text{ and } a \in \{0, 1, 2, 3\}\}$ to $W \setminus \{(0, 0, \dots)\}$, and satisfies a commutative relation

$$\sigma \cdot \Psi = \Psi \cdot \sigma_M.$$

Now denote a set of all finite revolving sequences with length n by $W^{(n)}$ and the decomposition of $W^{(n)}$ by

$$W_{\varepsilon}^{(n)} = \{ \{ \Psi_0(\varepsilon_j, \varepsilon_{j+1}) \}_{j=1}^n : \varepsilon_1 = \varepsilon \text{ and } (\varepsilon_1, \dots, \varepsilon_{n+1}) \text{ is } M\text{-admissible} \},$$

and

$$W_{(\varepsilon, \delta)}^{(n)} = \{ (\delta_1, \dots, \delta_n) \in W_{\varepsilon}^{(n)} : \delta_1 = \delta \}.$$

Then we obtain.

Proposition(3.2)

$$(1) \quad W^{(n)} = \bigcup_{\varepsilon \in \{0, 1, 2, 3\}} W_{\varepsilon}^{(n)},$$

$$(2) \quad W_{\varepsilon}^{(n)} = W_{(\varepsilon, 0)}^{(n)} \cup W_{(\varepsilon, (-1)\varepsilon)}^{(n)},$$

$$(3) \quad \sigma W_{(\varepsilon, 0)}^{(n)} = W_{\varepsilon}^{(n-1)} \text{ and}$$

$$\sigma W_{(\varepsilon, (-1)\varepsilon)}^{(n)} = W_{\varepsilon+1 \pmod{4}}^{(n-1)}$$

$$(4) \quad (-1)W_{\varepsilon}^{(n)} = W_{\varepsilon+1 \pmod{4}}^{(n)},$$

$$(-1)W_{(\varepsilon, 0)}^{(n)} = W_{(\varepsilon+1 \pmod{4}, 0)}^{(n)},$$

and

$$(-i)w_{(\varepsilon, (-i)\varepsilon)}^{(n)} = w_{(\varepsilon+1 \pmod{4}, (-i)^{\varepsilon+1})}^{(n)}.$$

Let ϱ be a map from $w^{(n)}$ to a line segment such that

$$\varrho(\delta_1, \dots, \delta_n) = \text{segment which connects } p(\delta_1, \dots, \delta_n, 0) \text{ and } p(\delta_1, \dots, \delta_n, \delta_{n+1} \neq 0), \text{ where}$$

and $p(\delta_1, \dots, \delta_n, \delta_{n+1} \neq 0)$, where

$$p(\delta_1, \dots, \delta_n) = \sum_{k=1}^n \delta_k (1+i)^{-k}.$$

By the way, define a n -step twindragon curve D_n and a n -step dragon (paper folding) curve H_n (Fig.3(a)(b)) by

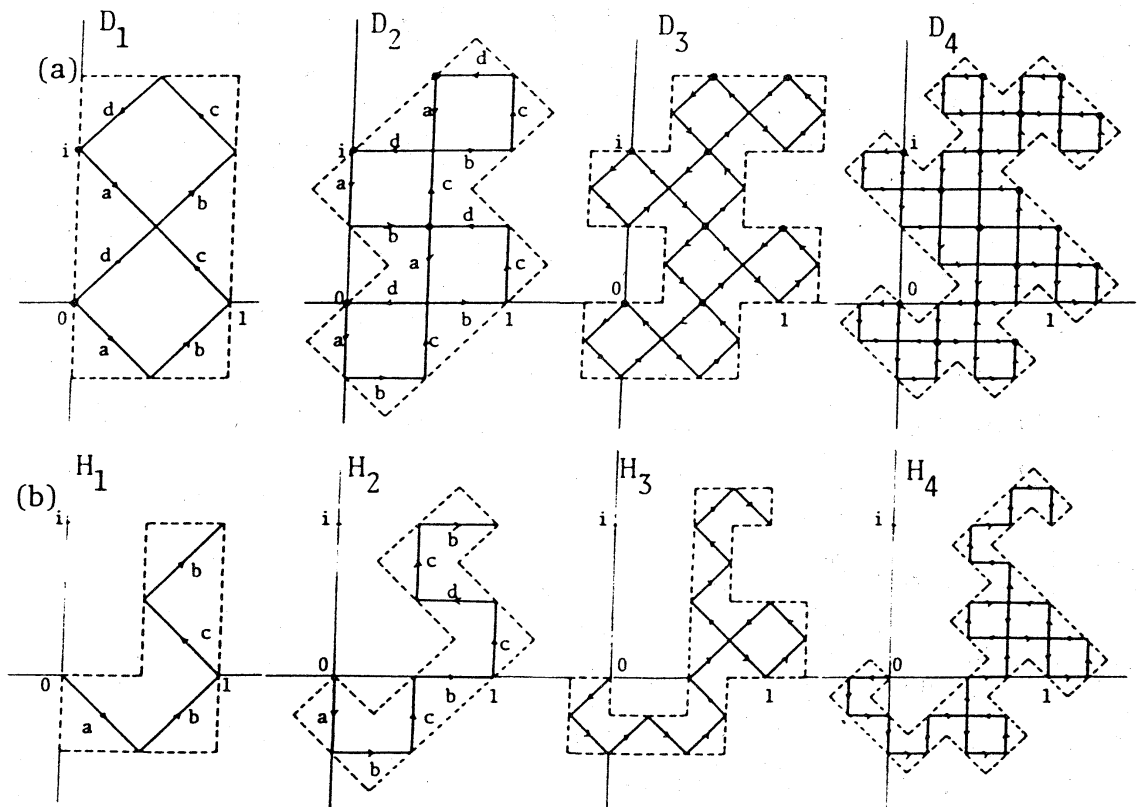


Fig.3: Twindragon D_n and Dragon H_n and their boundaries.

$$D_n = (1+i)^{-n} K[\theta_T^n(abcd)]$$

and

$$H_n = (1+i)^{-n} K[\theta_T^n(ab)],$$

where $G=\{abcd\}$ and a homeomorphism f is such that

$$f(a)=1=-f(c) \quad \text{and} \quad f(b)=i=-f(d),$$

and an endomorphism θ_T is defined by

$$\theta_T: a \rightarrow ab, b \rightarrow cb, c \rightarrow cd, d \rightarrow ad.$$

We notice the twindragon curve is tiled by two dragon curves.

that is,

$$D_n = H_n \cup (-H_n + 1 + i).$$

Lemma(3.1)

Let $\lambda_{\varepsilon}^{(n)}$ and $\lambda_{(\varepsilon, \delta)}^{(n)}$ be defined by

$$\lambda_{\varepsilon}^{(n)} = \bigcup_{(\delta_1, \dots, \delta_n) \in W_{\varepsilon}^{(n)}} \lambda(\delta_1, \dots, \delta_n)$$

and

$$\lambda_{(\varepsilon, \delta)}^{(n)} = \bigcup_{(\delta_1, \dots, \delta_n) \in W_{(\varepsilon, \delta)}^{(n)}} \lambda(\delta_1, \dots, \delta_n),$$

then $\lambda_{(\varepsilon, \delta)}^{(n)}$ and $\lambda_{\varepsilon}^{(n)}$ are similar to the $(n-2)$ -step and $(n-1)$ -step dragon curve respectively (Fig.4(a)(b)).

Let U be a closed square in section 2, U' a closed square such that $U' = U + i/2$ and B'_{n+1} defined by

$$B'_{n+1} = \partial \left(\bigcup_{x \in P_{n+1}} x(\varepsilon_0, \dots, \varepsilon_{n-1}) + (1+i)^{-(n-1)} U' \right),$$

then

Lemma(3.2)

(1) The n -step twindragon curve D_n is covered by a closed curve B'_{n+1} as an envelope (Fig.3(a)), that is.

$$d_0(D_n, B'_{n+1}) = \sup_{x \in B'_{n+1}} \inf_{y \in D_n} |x-y| = \left(\frac{1}{\sqrt{2}}\right)^{n+1}.$$

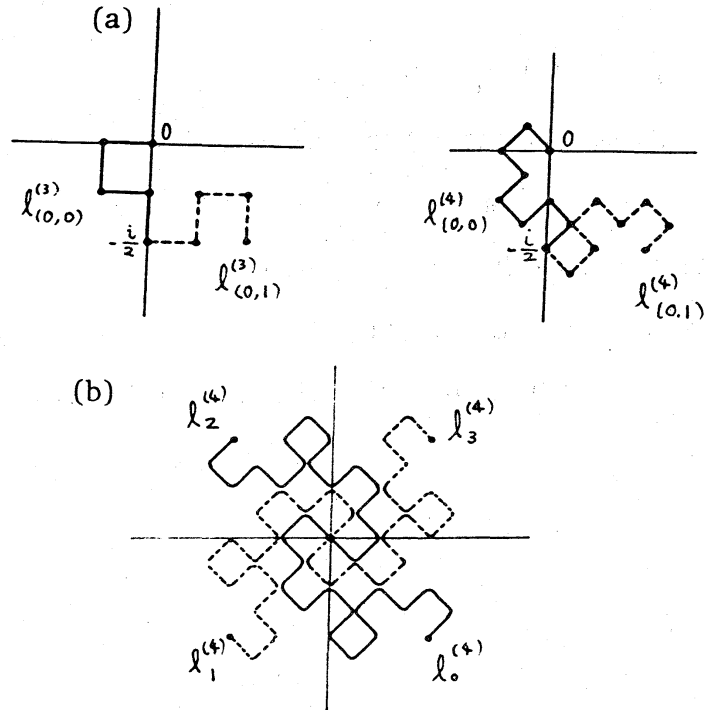


Fig.4: (a) Line segments $l_{(0,0)}^{(n)}$ and $l_{(0,1)}^{(n)}$ for $n=3,4$.
 (b) Line segments $\{l_{\epsilon}^{(n)}\}$ for $n=4$.

(2) The limit set D_T of $\{D_n\}_{n=1}$ has a dragon boundary as its boundary.

Moreover using above lemma we can prove that

Lemma(3.3)

Let H_T be the limit set of the paper folding curve H_n . Then the boundary of H_T consists of three parts of the dragon boundary. Therefore $\dim_H \partial H_T = \dim_H \partial D_T = 2 \log \beta_0 / \log 2$.

Put

$$X_{(\varepsilon, \delta)}^{(n)} = \{ \sum_{k=1}^n \delta_k (1+i)^{-k}; (\delta_1, \dots, \delta_n) \in W_{(\varepsilon, \delta)}^{(n)} \}$$

$$X_{\varepsilon}^{(n)} = \{ \sum_{k=1}^n \delta_k (1+i)^{-k}; (\delta_1, \dots, \delta_n) \in W_{\varepsilon}^{(n)} \},$$

and let $X_{(\varepsilon, \delta)}$ and X_{ε} be limit sets of $X_{(\varepsilon, \delta)}^{(n)}$ and $X_{\varepsilon}^{(n)}$ respectively (Fig.5). Thus we can prove that

Lemma(3.4)

- (1) $(1+i)X_{(\varepsilon, 0)} = X_{\varepsilon}$
- (2) $(1+i)X_{(\varepsilon, (-i)^{\varepsilon})} = X_{\varepsilon+1 \pmod{4}} + (-i)^{\varepsilon}$,
- (3) $\text{int}(X_{(\varepsilon, \delta)}) \cap \text{int}(X_{(\varepsilon', \delta')}) = \emptyset$ for $(\varepsilon, \delta) \neq (\varepsilon', \delta')$,

and $\partial X_{(\varepsilon, \delta)} \cap \partial X_{(\varepsilon', \delta')}$ consists of parts of the dragon boundary.

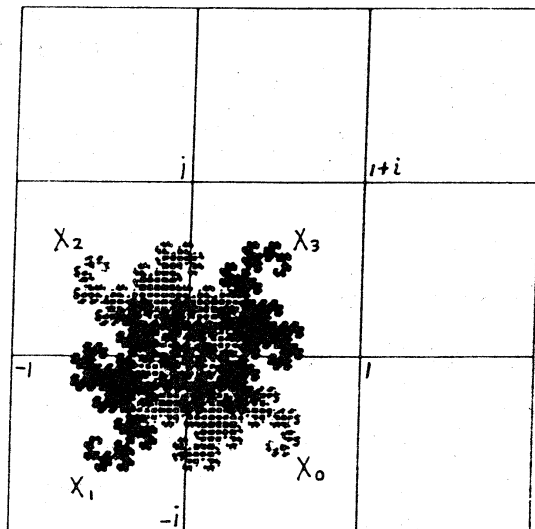


Fig.5: Tetrdragon X.

Then putting

$$\hat{U}_0 = X \text{ and } \hat{U}_\delta = \delta + X_{\varepsilon+1(\bmod 4)} \text{ for } \delta = (-i)^\varepsilon,$$

and let a map T on X be

$$Tz = (1+i)z - [z]_D,$$

where $[z]_D = \delta$ if $w \in \hat{U}_\delta$ for $\delta \in \{0, 1, -i, -1, i\}$,

then a transformation (X, T) is well defined and induces the revolving expansion.

Theorem(3.1)

- (1) The Lebesgue measure ν is invariant with respect to (X, T) .
- (2) the dynamical system (X, T, ν) is isomorphic to (X_M, σ_M, μ_M) , where μ_M is a stationary Markov measure such that

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix} \quad \Pi = (1/4, 1/4, 1/4, 1/4),$$

Remark 1:

The dual algorithm of (X, T, ν) is constructed by taking a complex conjugate, $X^* = \overline{X}$, and putting

$$\hat{U}_0^* = X^*, \quad \hat{U}_\delta^* = \delta + X_{\varepsilon-1(\bmod 4)}^* \text{ for } \delta \in \{0, 1, -i, -1, i\},$$

and

$$T^*z = (1-i)z - [(1-i)z]_{D^*},$$

where $[w]_{D^*} = \delta$ if $w \in \hat{U}_\delta^*$. Then a dynamical system

(X^*, T^*, ν) is the dual system for the system (X, T, ν) and

induces the "converse" revolving expansion,

$$z = \sum_{k=1}^{\infty} \delta_k^* (1-i)^{-k}.$$

Remark 2:

If we choose formally the dual domain $X^\#$ as

$$X^\# = \bigcup_{\varepsilon} X_{\varepsilon}^\#,$$

where

$$X_{\varepsilon}^\# = \left\{ \sum_{k=1}^{\infty} \delta_k^* (1+i)^{-k}; (\delta_1^*, \delta_2^*, \dots) \in W_{\varepsilon}^* \right\}.$$

Then we obtain an interesting picture (Fig.6). This selfsimilar fractal curve is already studied by P. Lévy in 1938⁸⁾.

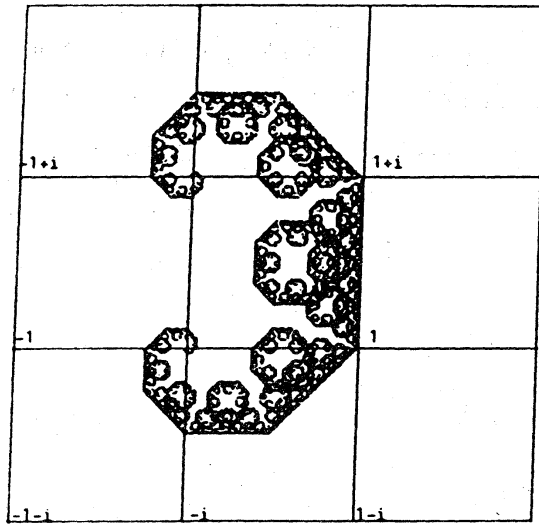


Fig.6: $X_0^\#$.

4. DUAL SYSTEM ON CROSS DRAGON

Let $E=(E_{j,k})$, $1 \leq j,k \leq 4$, be a matrix such that

$$E = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

We consider E as the structure matrix for states $\Gamma = \{0, i, -1+i, -1\}$ by a correspondence $\tau: \{1, 2, 3, 4\} \rightarrow \Gamma$ such that $\tau[1]=0$, $\tau[2]=i$, $\tau[3]=-1+i$ and $\tau[4]=-1$, that is, let V be a set of infinite sequences generated by the structure matrix E ,

$$V = \{(\gamma_1, \gamma_2, \dots); E_{\gamma_j, \gamma_{j+1}} = 1, \gamma_j \in \Gamma \text{ for all } j \in \mathbb{N}\}$$

and σ a shift on V . Then the system (V, σ) is a Markov subshift. Let $V^{(n)}$ be a set of E -admissible sequences with length n and $V_\gamma^{(n)}$ be

$$V_\gamma^{(n)} = \{(\gamma_1, \dots, \gamma_n) \in V^{(n)}; \gamma_1 = \gamma\}.$$

Notice that nonzero entries of the structure matrix can be written as $E_{\tau[k], \tau[(k+1) \bmod 4]} = E_{\tau[k], \tau[(k+2) \bmod 4]}$ and denote these two admissible states after $\gamma = \tau[k]$ by $\gamma[1] = \tau[(k+1) \bmod 4]$ and $\gamma[2] = \tau[(k+2) \bmod 4]$ respectively.

Property(4.1)

$$(1) \quad V^{(n)} = \bigcup_{\gamma \in \{0, i, -1+i, -1\}} V_\gamma^{(n)},$$

$$(2) \quad \sigma V_\gamma^{(n)} = V_{\gamma[1]}^{(n-1)} \cup V_{\gamma[2]}^{(n-1)},$$

$$(3) \quad i V_\gamma^{(n)} + i = V_{\gamma[1]}^{(n)}$$

and

$$-V_\gamma^{(n)} + (-1+i) = V_{\gamma[2]}^{(n)}.$$

We realize a sequence $(\gamma_1, \dots, \gamma_n)$ to a point $P(\gamma_1, \dots, \gamma_n)$ by

$$P(\gamma_1, \dots, \gamma_n) = \sum_{k=1}^n \gamma_k (1+i)^{-k}.$$

According to the set of sequence $V^{(n)}$ and $V_\gamma^{(n)}$ let $Y^{(n)}$ and $Y_\gamma^{(n)}$ be sets of points $\{P(\gamma_1, \dots, \gamma_n)\}$.

It is verified that

$$d(Y^{(n)}, Y^{(n+1)}) \leq \left(\frac{1}{\sqrt{2}}\right)^n,$$

in the Hausdorff metric. So $Y^{(n)}$ and $Y_\gamma^{(n)}$ converge to Y and Y_γ respectively as $n \rightarrow \infty$ (Fig.7).

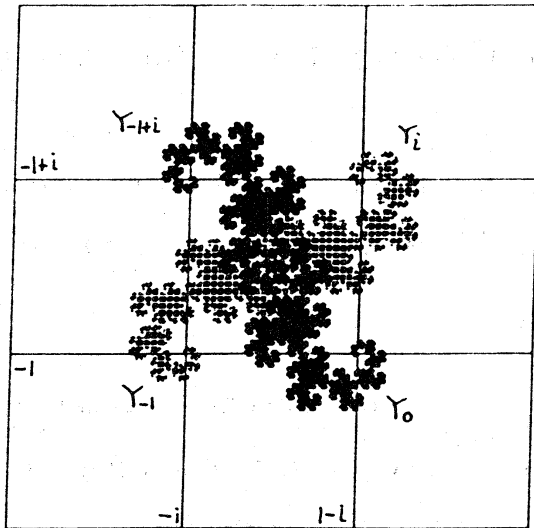


Fig.7: Cross dragon Y .

Lemma(4.1)

Let $Y = \{\sum_{k=1}^{\infty} r_k (1+i)^{-k} : (r_1, r_2, \dots) \in V\}$ and $Y_\gamma = \{\sum_{k=1}^{\infty} r_k (1+i)^{-k} : (r_1, r_2, \dots) \in V_\gamma\}$. Then the sets Y and Y_γ , $\gamma \in \Gamma$, satisfy following properties.

- (1) $Y = \bigcup_{\gamma \in \{0, i, -1+i, -1\}} Y_\gamma$.
- (2) $(1+i)Y_\gamma - \gamma = Y_{\gamma[1]} \cup Y_{\gamma[2]}$.
- (3) $iY_\gamma + 1 = Y_{\gamma[1]}$

and

$$-Y_\gamma + 1+i = Y_{\gamma[2]}.$$

- (4) $Y_\gamma = F_{0,\gamma}(Y_\gamma) \cup F_{1,\gamma}(Y_\gamma)$.

where $F_{0,\gamma}$ and $F_{1,\gamma}$ are contraction maps such that

$$F_{0,\gamma}(z) = (1+i)^{-1}(iz + \gamma + 1)$$

and

$$F_{1,\gamma}(z) = (1+i)^{-1}(-z + \gamma + 1+i) \quad \text{for each } \gamma \in \Gamma.$$

Recall another approach for selfsimilar fractal sets proposed by Hutchinson⁷⁾ using a set of contraction maps.

Theorem (Hutchinson⁷⁾)

Let $\mathcal{L}$ be a finite set of contraction maps $\{S_1, \dots, S_M\}$ on a metric space. Then there exists a unique closed bounded set K such that $K = \bigcup_{j=1}^M S_j(K)$. Moreover, let $\mathcal{L}(A) = \bigcup_{j=1}^M S_j(A)$ and $\mathcal{L}^p(A) = \mathcal{L}(\mathcal{L}^{p-1}(A))$ for arbitrary set A , then $\mathcal{L}^p(A) \rightarrow K$ in the Hausdorff metric as $p \rightarrow \infty$ for closed bounded set A .

Thus we can say that the limit sets $\{Y_\gamma\}$ are invariant sets for the contraction maps $\{F_{0,\gamma}, F_{1,\gamma}\}$. Notice that the set $\{X_\varepsilon^*\}$ in section 3 are the invariant set for the contraction maps $\{G_{0,\varepsilon}^*, G_{1,\varepsilon}^*\}$ for each $\varepsilon \in \{0, 1, 2, 3\}$, where

$$G_{0,\varepsilon}^*(z) = (1-i)^{-1}z \quad \text{and} \quad G_{1,\varepsilon}^*(z) = (1-i)^{-1}(iz + i^\varepsilon),$$

that is,

$$X_\varepsilon^* = G_{0,\varepsilon}^*(X_\varepsilon^*) \cup G_{1,\varepsilon}^*(X_\varepsilon^*)$$

and for $\mathcal{L} = \{G_{0,\varepsilon}^*, G_{1,\varepsilon}^*\}$

$$G_{0,\varepsilon}^*(\mathcal{L}^{n(0)}) = X_{(\varepsilon, 0)}^{*(n+1)}$$

and

$$G_{1,\varepsilon}^*(\mathcal{L}^{n(0)}) = X_{(\varepsilon, i^\varepsilon)}^{*(n+1)}.$$

Then we obtain

Theorem(4.1)

Let $\{Y_\gamma\}$ satisfy $Y_\gamma = F_{0,\gamma}(Y_\gamma) \cup F_{1,\gamma}(Y_\gamma)$ for each $\gamma \in \Gamma$, and $Y = \bigcup_{\gamma \in \{0, i, -1+i, -1\}} Y_\gamma$. Then

(1) Each set Y_γ is a dragon with end points 0 for Y_{-1} , 1 for Y_0 , $1+i$ for Y_i , i for Y_{-1+i} and $(1+i)/2$ in common.

(2) The set Y is tiled by four dragons $\{Y_\gamma\}$, that is,

$$Y = \bigcup_{\gamma \in \{0, i, -1+i, -1\}} Y_\gamma$$

and

$$\lambda(Y_\gamma \cap Y_{\gamma'}) = 0 \quad \text{for } \gamma \neq \gamma'.$$

We call the set Y a cross dragon.

Let a map S on Y be

$$Sz = (1+i)z - [(1+i)z]_{\mathbb{C}},$$

where $[w]_{\mathbb{C}} = \gamma$ if $w \in \gamma + (Y_{\gamma[1]} \cup Y_{\gamma[2]})$. Then (Y, S) is well defined and induces an expansion

$$z = \sum_{k=1}^{\infty} \gamma_k (1+i)^{-k} \quad \text{for a.e. } z \in Y.$$

Now let $Y^* = \{x+iy : 0 \leq x, y < 1\}$ and a map S^* be

$$S^*z = (1+i)z - [(1+i)z],$$

where $[w] = [\operatorname{Re}(w)] + i[\operatorname{Im}(w)]$ for $z \in \mathbb{C}$. This system is equivalent to a group endomorphism T_L on the torus $\mathbb{T}^2$ such that

$$T_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} [x-y] \\ [x+y] \end{pmatrix}.$$

Theorem(4.2)

- (1) The Lebesgue measure λ is invariant with respect to (Y, S) .
- (2) The cross dragon system (Y, S, λ) is actually the dual system for (Y^*, S^*, λ) .

Remark:

The cross dragon system (Y, S, λ) is isomorphic to a following map on the torus.

$$T^{\dagger} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} [x+y-1] \\ [-x+y+1] \end{pmatrix}.$$

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