

LINEAR COMBINATIONS OF STARLIKE AND CONVEX FUNCTIONS OF ORDER α WITH COMPLEX COEFFICIENTS

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1. INTRODUCTION.

Let A denote the class of functions $f(z)$, which are analytic in the unit disk U and normalized by $f(0)=0$, $f'(0)=1$. For $0 \leq \alpha < 1$, we define the subclasses of A ,

$$(1.1) \quad S^*(\alpha) = \{f(z) \in A; \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in U\},$$

$$(1.2) \quad K(\alpha) = \{f(z) \in A; \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, z \in U\}.$$

These are called the classes of starlike functions of order α and of convex functions of order α , respectively. It is easily seen that these are decreasing functions with respect to α and $S^*(\alpha) \supset K(\alpha)$. Let S be the class of normalized analytic and univalent functions in U . Then we know $K \subset S^* \subset S$, $K=K(0)$, $S^*=S^*(0)$.

Let us define another class $P(\alpha)$ for $\alpha < 1$.

$$(1.3) \quad P(\alpha) = \{p(z); \text{analytic in } U, p(0)=1, \operatorname{Re}\{p(z)\} > \alpha, z \in U\}.$$

The class has a well-known property by the subordination principle.

LEMMA 1. If $p(z) \in P(\alpha)$ for $z \in U$, then

$$(1.4) \quad \left| p(z) - \frac{1+(1-2\alpha)|z|^2}{1-|z|^2} \right| \leq \frac{2(1-\alpha)|z|^2}{1-|z|^2}.$$

It holds the equality of (1.4) by $p(z) = \frac{1+(1-2\alpha)z}{1-z}$.

If $f(z) \in S^*(\alpha)$, then $\frac{zf'(z)}{f(z)} = p(z) \in P(\alpha)$ and $p(z)$ is subordinate to $\frac{1+(1-2\alpha)z}{1-z}$, namely, we can write $\frac{zf'(z)}{f(z)} = \frac{1+(1-2\alpha)w(z)}{1-w(z)}$, where $w(z)$ is analytic in U with $w(0)=0$, $|w(z)| < 1$, $z \in U$.

LEMMA 2. [2] If $f(z) \in S^*(\alpha)$, then

$$(1.5) \quad \left| \arg \frac{f(z)}{z} \right| \leq 2(1-\alpha) \sin^{-1}|z|, \quad z \in U.$$

It holds the equality of (1.5) by $f(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$.

We have the following lemma from (1.5) and the fact that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$.

LEMMA 3. If $f(z) \in K(\alpha)$, then

$$(1.6) \quad \left| \arg f'(z) \right| \leq 2(1-\alpha) \sin^{-1}|z|, \quad z \in U.$$

It holds the equality of (1.6) by

$$(1.7) \quad f(z) = \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & (\alpha \neq \frac{1}{2}) \\ -\log(1-z) & (\alpha = \frac{1}{2}). \end{cases}$$

The following result has been shown by T.H. MacGregor[1] and completely proved by D.R. Wilken and J. Feng[4].

LEMMA 4. If $f(z) \in K(\alpha)$, then $f(z) \in S^*(\beta(\alpha))$, where

$$(1.8) \quad \beta(\alpha) = \begin{cases} \frac{2\alpha - 1}{2(1-2^{1-2\alpha})} & (\alpha \neq \frac{1}{2}) \\ \frac{1}{2 \log 2} & (\alpha = \frac{1}{2}) \end{cases}$$

2. THEOREMS AND PROOFS.

Lemma 5 is due to R.K. Stump[3] for which we will give a simple computation.

LEMMA 5. If $|u-a| \leq d$, $|v-a| \leq d$, where both u and v are complex variables and a and d are real constants such that $a > d > 0$, moreover, for $\rho > 0$ and $\theta \in [0, \pi)$ letting

$$(2.1) \quad W = u \cdot \frac{1}{1 + \rho e^{i\theta}} + v \cdot \frac{1}{1 + \rho^{-1} e^{-i\theta}}.$$

Then

$$(2.2) \quad \operatorname{Re} W \geq a - d \cdot \sec\left(\frac{\theta}{2}\right).$$

Proof. The condition $|u-a| \leq d$ implies $|uu_0 - au_0| \leq d|u_0|$ for some complex number u_0 . This shows $\operatorname{Re}(uu_0) \geq \operatorname{Re}(au_0) - d|u_0|$. At the same time, for a complex number v_0 , $\operatorname{Re}(vv_0) \geq \operatorname{Re}(av_0) - d|v_0|$.

Now, putting

$$(2.3) \quad u_0 = \frac{1}{1+\rho e^{i\theta}}, \quad v_0 = \frac{1}{1+\rho^{-1}e^{-i\theta}},$$

it holds that

$$(2.4) \quad \begin{aligned} \operatorname{Re} W &= \operatorname{Re}(uu_0) + \operatorname{Re}(vv_0) \\ &\geq a\{\operatorname{Re}(u_0) + \operatorname{Re}(v_0)\} - d(|u_0| + |v_0|). \end{aligned}$$

It follows from (2.3) that

$$(2.5) \quad \begin{aligned} \operatorname{Re}(u_0) &= \frac{1 + \rho \cos \theta}{1 + \rho^2 + 2\rho \cos \theta}, \quad \operatorname{Re}(v_0) = \frac{\rho^2 + \rho \cos \theta}{1 + \rho^2 + 2\rho \cos \theta}, \\ \operatorname{Re}(u_0) + \operatorname{Re}(v_0) &= 1. \end{aligned}$$

$$\text{Also, } |u_0| = \frac{1}{(1 + \rho^2 + 2\rho \cos \theta)^{1/2}}, \quad |v_0| = \frac{\rho}{(1 + \rho^2 + 2\rho \cos \theta)^{1/2}},$$

$$(2.6) \quad |u_0| + |v_0| = \frac{1 + \rho}{(1 + \rho^2 + 2\rho \cos \theta)^{1/2}} \leq \sec\left(\frac{\theta}{2}\right).$$

Therefore, we have $\operatorname{Re} W \geq a - d \cdot \sec\left(\frac{\theta}{2}\right)$. This completes the proof.

THEOREM 1. For a complex number λ , let $f_1(z) \in S^*(\alpha)$,

$f_2(z) \in S^*(\alpha)$ and

$$(2.7) \quad F(z) = \lambda f_1(z) + (1-\lambda)f_2(z), \quad z \in U,$$

where $0 \leq \delta = \arg \frac{\lambda}{1-\lambda} < \pi$. Then $F(z)$ is the starlike function of order μ for $|z| < \min\{\sin \frac{\pi - \delta}{4(1-\alpha)}, r_s\}$, where r_s is the smallest positive root of the equation,

$$(2.8) \quad \frac{1 + (1-2\alpha)r^2}{1 - r^2} - \frac{2(1-\alpha)r}{1 - r^2} \cdot \sec\left\{\frac{\delta}{2} + 2(1-\alpha)\sin^{-1}r\right\} = \mu.$$

Proof. From (2.7) we have

$$(2.9) \quad \frac{zF'(z)}{F(z)} = \frac{zf_2'(z)}{f_2(z)} \cdot [1 + \{\frac{\lambda}{1-\lambda} \cdot \frac{f_1(z)}{f_2(z)}\}]^{-1} + \frac{zf_1'(z)}{f_1(z)} \cdot [1 + \{\frac{\lambda}{1-\lambda} \cdot \frac{f_1(z)}{f_2(z)}\}^{-1}]^{-1}.$$

Putting in Lemma 5 $u = \frac{zf_2'(z)}{f_2(z)}$, $v = \frac{zf_1'(z)}{f_1(z)}$,

$$a = \frac{1+(1-2\alpha)|z|^2}{1-|z|^2}, \quad d = \frac{2(1-\alpha)|z|}{1-|z|^2} \quad \text{and} \quad \rho e^{i\theta} = \frac{\lambda}{1-\lambda} \cdot \frac{f_1(z)}{f_2(z)},$$

it follows from the result of Lemma 5

$$(2.10) \quad \operatorname{Re}\left\{\frac{zF'(z)}{F(z)}\right\} \geq \frac{1+(1-2\alpha)|z|^2}{1-|z|^2} - \frac{2(1-\alpha)|z|}{1-|z|^2} \cdot \sec\left(\frac{\gamma}{2}\right),$$

where

$$\gamma = \arg \frac{\lambda}{1-\lambda} \cdot \frac{f_1(z)}{f_2(z)} = \delta + \arg \frac{f_1(z)}{z} - \arg \frac{f_2(z)}{z}.$$

Hence, providing $|\gamma| \leq \delta + 4(1-\alpha)\sin^{-1}|z|$ from Lemma 2, $z \in U$ and

$$|z| \leq r < \sin \frac{\pi - \delta}{4(1-\alpha)}, \quad \text{then } 0 \leq |\gamma| < \pi \text{ and } \sec\left(\frac{\gamma}{2}\right) > 0. \quad \text{Now}$$

if we let r_s be the smallest positive root of (2.8), then

$\operatorname{Re}\left\{\frac{zF'(z)}{F(z)}\right\} > \mu$ for $|z| < \min\{\sin \frac{\pi - \delta}{4(1-\alpha)}, r_s\}$. This completes the proof.

THEOREM 2. Under the same notations as in Theorem 1, let $f_1(z) \in K(\alpha)$, $f_2(z) \in K(\alpha)$, and $F(z) = \lambda f_1(z) + (1-\lambda)f_2(z)$, $z \in U$. Then $F(z)$ is the convex function of order μ for $|z| < \min\{\sin \frac{\pi - \delta}{4(1-\alpha)}, r_c\}$, where r_c is the smallest positive root of (2.8).

Proof. We have from the definition of $F(z)$,

$$(2.11) \quad 1 + \frac{zF''(z)}{F'(z)} = \left\{1 + \frac{zf_1''(z)}{f_1'(z)}\right\} \left[1 + \left\{\frac{\lambda}{1-\lambda} \cdot \frac{f_1'(z)}{f_2'(z)}\right\}^{-1}\right]^{-1} \\ + \left\{1 + \frac{zf_2''(z)}{f_2'(z)}\right\} \left[1 + \left\{\frac{\lambda}{1-\lambda} \cdot \frac{f_1'(z)}{f_2'(z)}\right\}^{-1}\right]^{-1}.$$

Also, it holds from Lemma 5 that

$$(2.12) \quad \operatorname{Re}\left\{1 + \frac{zF''(z)}{F'(z)}\right\} \geq \frac{1+(1-2\alpha)|z|^2}{1-|z|^2} - \frac{2(1-\alpha)|z|}{1-|z|^2} \cdot \sec\left(-\frac{\gamma}{2}\right),$$

where $\gamma = \arg\left\{\frac{\lambda}{1-\lambda} \cdot \frac{f_1'(z)}{f_2'(z)}\right\} = \delta + \arg f_1'(z) - \arg f_2'(z)$.

Hence, providing $|\gamma| \leq \delta + 4(1-\alpha)\sin^{-1}|z|$ after using Lemma 3, then

$0 \leq |\gamma| < \pi$ and $\sec(-\frac{\gamma}{2}) > 0$ for $|z| < \sin \frac{\pi - \delta}{4(1-\alpha)}$. Therefore,

$\operatorname{Re}\left\{1 + \frac{zF''(z)}{F'(z)}\right\} > \mu$ for $|z| < \min\left\{\sin \frac{\pi - \delta}{4(1-\alpha)}, r_c\right\}$, where r_c is the smallest positive root of (2.8).

Theorem 3 follows by the above methods, therefore we omit the proof.

THEOREM 3. Let $f_1(z) \in K(\alpha)$, $f_2(z) \in K(\alpha)$ and

$F(z) = \lambda f_1(z) + (1-\lambda)f_2(z)$, $z \in U$, where $0 \leq \delta = \arg \frac{\lambda}{1-\lambda} < \pi$.

Then $F(z)$ is the starlike function of order μ

for $|z| < \min\left\{\sin \frac{\pi - \delta}{4(1-\beta(\alpha))}, r_s\right\}$, where $\beta(\alpha)$ is denoted in Lemma 4,

and r_s is the smallest positive root of

$$(2.13) \quad \frac{1+(1-2\alpha)r^2}{1-r^2} - \frac{2(1-\beta(\alpha))r}{1-r^2} \cdot \sec\left\{\frac{\gamma}{2} + 2(1-\beta(\alpha)) \cdot \sin^{-1}r\right\} \\ = \mu.$$

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