

Critically finite maps on projective spaces (II)

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1 Introduction

Critically finite maps provide interesting examples of complex dynamics on projective spaces which can be fairly well analyzed. For one dimensional case such maps are first investigated by Thurston. A holomorphic map f from the Riemann sphere $\mathbb{P}^1$ onto itself, i.e., a rational function of one variable, is said to be critically finite if every critical point of f is (pre-)periodic. It is called strictly critically finite if every critical point is preperiodic but not periodic. Thurston's theorem asserts that the Julia set for a strictly critically finite map coincides with the whole $\mathbb{P}^1$.

Generalizations of critically finite maps on projective spaces of general dimension were first studied by Fornæss-Sibony [FS1, FS2, FS3] (see also [U1]). In [U3] we showed, for the case of dimension 2, that strictly critically finite maps have empty Fatou set. Further, Jonsson [J] showed that the support of the invariant measure is all of $\mathbb{P}^2$.

In this article we propose a definition of strictly critically finite map and develop the method introduced in [U3, U4, U5] further. This may be regarded as a generalization of Thurston's theorem for strictly critically finite maps on projective spaces of any dimension.

2 Definitions and main results

Let $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be a holomorphic map of degree $d \geq 2$ and let C denote its critical set. The map f is a d^n -fold branched covering over $\mathbb{P}^n$ whose branch locus lies over the set $f(C)$. For every $i \geq 1$, the critical set of the iterate f^i is $\bigcup_{k=0}^{i-1} f^{-k}(C)$ and f^i is a d^{ni} -fold branched covering whose branch locus lies over $\bigcup_{k=1}^i f^k(C)$.

We define the postcritical set of f by $D = \bigcup_{i \geq 1} f^i(C)$. We will say that f is critically finite if D is an algebraic subset of $\mathbb{P}^n$. For such a map f , the iterates $f^i : \mathbb{P}^n \rightarrow \mathbb{P}^n$ ($i \geq 1$) are branched covering whose branch locus lies only over the postcritical set D .

We will say that f is strictly critically finite if the branching order of the map f^i is everywhere bounded by some number independent of i . We note that, this definition of strictly critically finite map reduces to the original in the case of dimension one, and coincides with the definition of n -critically maps given by Jonsson [J] (see also [Rn]).

Our main result is the following theorem.

Theorem 2.1 *Let $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be a strictly critically finite map and let K be a compact connected subset of $\mathbb{P}^n$ containing at least two points. Then there is no subsequence of $\{f^i\}$ that is uniformly convergent on K .*

As consequences of this theorem, we have the following theorems.

Theorem 2.2 *For a strictly critically finite map $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$, all periodic points of are repelling. Further the set of all (repelling) periodic points is dense in $\mathbb{P}^n$.*

Theorem 2.3 *If $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ is a strictly critically finite map, then, for any point $a \in \mathbb{P}^n$, the set $\bigcup_{j=1}^{\infty} f^{-j}(a)$ is dense in $\mathbb{P}^n$. There exists no closed subset of $\mathbb{P}^n$ that is backward invariant under f except for the empty set and the whole $\mathbb{P}^n$.*

In the proof of the theorem, we will use the concepts of Fatou maps and branched coverings.

3 Outline of the proof

3.1 Fatou maps

Let $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be a holomorphic map of degree $d \geq 2$ and let $\varphi : X \rightarrow \mathbb{P}^n$ be a holomorphic map from a connected complex analytic space X into $\mathbb{P}^n$. We say that φ is a Fatou map for f if the sequence $\{f^i \circ \varphi\}_i$ is a normal family. This may be considered as a generalization of the Fatou set, and admits a characterization using the Green function, similar to that of Fatou sets.

For a holomorphic map $\varphi : X \rightarrow \mathbb{P}^n$, we say that a holomorphic map $\psi : X \rightarrow \mathbb{P}^n$ is a holomorphic lift of φ by an iterate f^i of f if $f^i \circ \psi = \varphi$ holds. We note that, when X is an open subset of $\mathbb{P}^n$ and φ is the inclusion map, such a lift ψ is a branch on X of the inverse of f^i .

If $\varphi : X \rightarrow \mathbb{P}^n$ is a holomorphic map, then the set of all possible lifts $\psi : X \rightarrow \mathbb{P}^n$ by some iterate f^i forms a normal family (Theorem 2.1 in

[U3]). Further, we can prove easily that the limit of any locally uniformly convergent sequence of lifts is a Fatou map.

If X is compact, then there exists no nonconstant Fatou map from X . The following theorem generalizes this fact.

Theorem 3.1 *Let φ be a holomorphic map from an irreducible complex space X of positive dimension into $\mathbb{P}^n$. Suppose that, for every point $p_0 \in \mathbb{P}^n$, there exists a neighborhood U of p_0 such that either $\varphi^{-1}(U)$ is empty or every connected component of $\varphi^{-1}(U)$ is relatively compact in X . Then φ is not a Fatou map for any (not necessarily critically finite) holomorphic map $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ of degree > 2 .*

It turns out that a strictly critically finite map admits no nonconstant Fatou map.

To prove the main result, we suppose the existence of a non-trivial connected compact set K and construct a Fatou map that contradicts the above theorem.

3.2 Branched coverings

Let B be a connected and locally connected Hausdorff space. A continuous map η from a Hausdorff space Y onto B is called an unbranched covering if for any point $b \in B$ there is a connected neighborhood V of b such that each connected component of $\eta^{-1}(V)$ is mapped homeomorphically onto V .

Now let A be a connected complex space. A holomorphic map ξ from a connected complex space X onto A is called a branched covering over A , if the following condition is satisfied: For any point $a \in A$, there exists a neighborhood U of a such that the restriction of ξ to each connected component of $\xi^{-1}(U)$ is a finite proper map. Let D be an analytic subset of A . A branched covering $\xi : X \rightarrow A$ will be called a D -branched covering if the restriction of ξ to $X \setminus \xi^{-1}(D)$ is an unbranched covering over $A \setminus D$.

If f is a strictly critically finite map, then the iterates $f^i : \mathbb{P}^n \rightarrow \mathbb{P}^n$ ($i = 1, 2, \dots$) constitute a family of coverings that are branched only over the postcritical set D . The following theorem asserts that we can construct a branched covering that dominates this family of coverings.

Theorem 3.2 *Let $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be a strictly critically finite map with postcritical set D . Then there exists a D -branched covering $\xi : X \rightarrow \mathbb{P}^n$ with the following property: For any $i \geq 1$ and any pair of points $p \in \mathbb{P}^n$ and $x \in X$ with $f^i(p) = \xi(x)$, there exists a branched covering map $\varphi : X \rightarrow \mathbb{P}^n$ such that $f^i \circ \varphi = \xi$ and that $\varphi(x) = p$.*

This is a consequence of the following lemma that deals with a general situation of families of branched coverings.

Lemma 3.3 *Let $\xi_\lambda : X_\lambda \rightarrow A$ ($\lambda \in \Lambda$) be a family of D -branched coverings. Suppose that there is a constant m such that, for any $\lambda \in \Lambda$ and for any point $x \in X_\lambda$, the branching order $\text{ord}(\xi_\lambda, x)$ is bounded by m .*

Then there exists a normal D -branched covering $\hat{\xi} : \hat{X} \rightarrow A$ with the following property: For any $\lambda \in \Lambda$ and any pair of points $x \in X_\lambda$ and $\hat{x} \in \hat{X}$ with $\xi_\lambda(x) = \hat{\xi}(\hat{x})$, there is a $\xi_\lambda^{-1}(D)$ -branched covering $\psi : \hat{X} \rightarrow X_\lambda$ such that $\xi_\lambda \circ \psi = \hat{\xi}$ and that $\psi(\hat{x}) = x$. Further there exists a minimal such D -branched covering determined uniquely up to isomorphism.

3.3 Proof of the main theorem

Let $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be a strictly critically finite map. Suppose that there exists a connected compact subset K of $\mathbb{P}^n$ containing at least two points and a subsequence of the iterates f^i uniformly convergent on K , and let $h : K \rightarrow \mathbb{P}^n$ be the limit of the sequence. We take the branched covering $\xi : X \rightarrow \mathbb{P}^n$ that dominates the iterates f^i (Theorem 3.2).

First we show that the map h is nonconstant. We let $\hat{K}$ be a connected component of $\xi^{-1}(h(K))$. We choose a sequence $\{\psi_\nu\}_\nu$ of lifts $\psi_\nu : X \rightarrow \mathbb{P}^n$ of ξ by some $f^{i(\nu)}$ that converges to a holomorphic map $\psi_* : X \rightarrow \mathbb{P}^n$. The sequence $\{\psi_\nu\}_\nu$ can be so chosen that ψ_* and ξ coincide on $\hat{K}$. Let Z be the connected component of the analytic set $\{x \in X \mid \psi_*(x) = \xi(x)\}$ containing $\hat{K}$. Then the map $\psi_*|_Z = \xi|_Z : Z \rightarrow \mathbb{P}^n$ satisfies the condition of Theorem 3.1 and this contradicts that this map is a Fatou map.

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