

On the algebraic geometry of Kac–Moody groups

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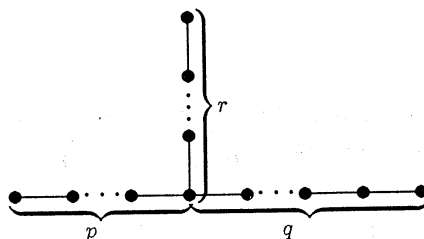
These notes are a slightly elaborated version of a talk given at the RIMS–Symposium on "Topological Field Theory and Related Topics", Kyoto, December 1996. Their aim is to give a survey of the main results obtained by Claus Møller in his dissertation at Hamburg University ([8], October 1996) pertaining to a natural semigroup completion of Kac–Moody groups.

1. "Abstract" Kac–Moody groups

Starting point for the construction of Kac–Moody Lie algebras and associated groups is a generalized Cartan matrix, i.e. an $l \times l$ -matrix $A = ((a_{ij})) \in M_l(\mathbb{Z})$ satisfying

$$\begin{aligned} a_{ii} &= 2 \\ a_{ij} &\leq 0 \quad i \neq j \\ a_{ij} &= 0 \Rightarrow a_{ji} = 0 \end{aligned}$$

We shall assume, in addition, that A is symmetrizable (cf. [2]). In fact, one might take A to be symmetric for simplicity. Also, the generalized Cartan matrices arising in singularity theory and providing the original motivation for our research in Kac–Moody groups (cf. [11], [13]) are symmetric, e.g. the matrix of type T_{pqr} encoded by the Coxeter–Dynkin diagram



Whereas the Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ is essentially generated by l copies of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$,

$$\langle e_i, h_i, f_i \rangle, \quad i = 1, \dots, l,$$

subject to relations derived from A , the corresponding Kac–Moody group $G = G(A)$ is essentially generated by l copies of the Lie group $SL_2(\mathbb{C})$. Here, the relations are either imposed abstractly (Tits, cf. [15], [16]) or by the "integration" of G from the integrable representations of $\mathfrak{g}$ (Moody–Teo, Marcuson, Garland, and, in the most thorough way, Kac–Peterson [10], [3], [4]).

The most important result about G as an abstract group is the existence of a "twin" BN -pair or "twin" Tits system (B^+, B^-, N, S) in G providing us, among others, with

- *positive and negative Borel subgroups* B^+ and B^- ,
- *a maximal torus* $T = B^+ \cap B^- = N \cap B^+ = N \cap B^-$,
- *a Weyl group* $W = N/T$ with generating set S ,
- *Bruhat decompositions*

$$G = \bigcup_{w \in W} B^+ w B^+ = \bigcup_{w \in W} B^- w B^-,$$

and a *Birkhoff-decomposition*

$$G = \bigcup_{w \in W} B^- w B^+.$$

Similarly, as in the case of the Lie algebra $\mathfrak{g}$ where one usually adjoins additional derivations to a "minimal" Kac–Moody algebra, the precise structure of G depends on slightly finer data than A . These data are given by an *integral realization* (H, Π, Π^*) of A which fixes the size of the maximal torus T and its position inside G .

Here, H is the lattice of algebraic one-parameter subgroups $\mathbb{C}^* \rightarrow T$ into T with dual $P = H^* = \text{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$, the lattice of algebraic characters $T \rightarrow \mathbb{C}^*$, and

$$\Pi = \{\alpha_1, \dots, \alpha_e\} \subset P, \quad \Pi^* = \{h_1, \dots, h_l\} \subset H$$

are free subsets of *simple roots* in P , resp. of *simple coroots* in H , related by

$$\alpha_i(h_j) = a_{ij}.$$

More explicitly, Π and Π^* are given in our context as follows:

Let $\kappa_i : SL_2(\mathbb{C}) \rightarrow G$, $i = 1, \dots, l$ denote the basic homomorphisms of $SL_2(\mathbb{C})$ into G , and let

$$\begin{aligned} h_i : \mathbb{C}^* &\rightarrow G \\ u_i : \mathbb{C} &\rightarrow G \end{aligned}$$

be given by

$$h_i(s) := \kappa_i\left(\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}\right), \quad s \in \mathbb{C}^*,$$

$$u_i(c) := \kappa_i\left(\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\right), \quad c \in \mathbb{C},$$

Then $h_i(\mathbb{C}^*) \subset T$, i.e. $h_i \in H$, and there is a character $\alpha_i \in P$ such that

$$t u_i(c) t^{-1} = u_i(\alpha_i(t)c)$$

for all $t \in T$, $c \in \mathbb{C}$.

By its natural action on T and P , the Weyl group $W = N/T$ is identified with the subgroup of $\text{Aut}_{\mathbb{Z}}(P)$ generated by the reflections $S = \{s_1, \dots, s_l\}$

$$s_i(\omega) = \omega - \omega(h_i)\alpha_i, \quad \omega \in P.$$

Also, s_i is given by the class of

$$\kappa_i\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \text{ in } N/T.$$

We can also make the groups B^+ and B^- more explicit:

Let U_i denote the subgroup $u_i(\mathbb{C})$ and, for any *real root* $\gamma = w(\alpha_i)$ ($w \in W$), put

$$U_\gamma := w U_i w^{-1}.$$

The set $\Sigma^{\text{real}} = W(\Pi)$ of all real roots divides naturally into positive and negative roots,

$$\Sigma^{\text{real}} = \Sigma^{\text{real},+} \cup \Sigma^{\text{real},-}$$

where $\Sigma^{\text{real},-} = -\Sigma^{\text{real},+}$, and if we put

$$U^{\pm} = \langle U_{\gamma} | \gamma \in \Sigma^{\text{real},\pm} \rangle$$

($\langle a, b, \dots \rangle$ denoting the group generated by $a, b, \dots$) we have

$$B^+ = T \bowtie U^+, \quad B^- = T \bowtie U^-.$$

Finally, the anti-involution

$$\begin{array}{ccc} SL_2(\mathbb{C}) & \longrightarrow & SL_2(\mathbb{C}) \\ g & \longmapsto & {}^t g \end{array}$$

can be lifted to all of G , i.e. there is an anti-involution $*$: $G \rightarrow G$ such that

- $*(t) = t$, for all $t \in T$
- $*(\kappa_i(g)) = \kappa_i({}^t g)$, for all $g \in SL_2(\mathbb{C})$.

In particular, one has $*(U^+) = U^-$, $*(U^-) = U^+$.

2. "Algebraic" Kac-Moody groups

If A is a Cartan matrix of "finite type" (i.e. all components are of type $A_n, B_n, \dots, F_n$, or G_2) then G , as described in the last section, is a reductive algebraic group over $\mathbb{C}$. The algebra $\mathbb{C}[G]$ of regular functions on G is then a Hopf algebra, and the group G can be completely recovered from the Hopf algebra $\mathbb{C}[G]$, in particular

$$G = \text{Specmax } \mathbb{C}[G] = \text{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[G], \mathbb{C}).$$

If A is a proper generalized Cartan matrix, then the associated algebra $\mathfrak{g}$ is of infinite dimension over $\mathbb{C}$. Thus, also G should be infinite-dimensional. A proposal for an algebra of "strongly regular" functions on G was made by Kac and Peterson in 1983 ([3]). As in the finite-dimensional case, this algebra is

generated by the matrix coefficients of a suitable representation. Let us therefore recall some basic facts about the irreducible highest weight representations of G .

To simplify the presentation, we shall assume that G is of "simply-connected type", i.e. that the coroot lattice $Q^\vee = \mathbb{Z} \cdot \Pi^\vee$ is a direct summand of H

$$H = Q^\vee \oplus D.$$

Then the set $P^+ = \{\omega \in P \mid \omega(h_i) \geq 0, i = 1, \dots, l\}$ of *dominant weights* can be written as a direct sum

$$P^+ = P^0 \oplus \bigoplus_{i=1}^l \mathbb{N} \cdot \Lambda_i$$

where

$$P^0 = \{\omega \in P \mid \omega(h_i) = 0, i = 1, \dots, l\} \cong D^*$$

and where $\Lambda_i, i = 1, \dots, l$, are *fundamental dominant weights*

$$\Lambda_i(h_j) = \delta_{ij}, i, j = 1, \dots, l,$$

uniquely determined modulo P^0 .

As in the finite-dimensional case there is a bijection of P^+ onto the set of isomorphism classes of irreducible highest weight representations L of G

$$\Lambda \in P^+ \longleftrightarrow L(\Lambda)$$

determined by $L(\Lambda)$ having a unique (up to scalars) highest weight vector $v_\Lambda \in L(\Lambda) \setminus \{0\}$ of weight Λ . (If $\Lambda \in P^0$, the module $L(\Lambda)$ will be one-dimensional.)

Any such module carries a nondegenerate contravariant form (essentially unique), i.e. a symmetric bilinear form

$$\langle \ , \ \rangle : L(\Lambda) \times L(\Lambda) \rightarrow \mathbb{C}$$

such that $\langle v, gw \rangle = \langle g^* v, w \rangle$ for all $v, w \in L(\Lambda), g \in G$, and $g^* = *(g)$ the anti-involution on G .

Let us call the function

$$c_{v,w} : G \rightarrow \mathbb{C}$$

given by $c_{v,w}(g) = \langle v, gw \rangle$ for some $v, w \in L(\Lambda)$ a *matrix coefficient* of G (in the representation $L(\Lambda)$). Kac and Peterson now define

$$\mathbb{C}[G] := \left(\begin{array}{l} \mathbb{C}\text{-algebra generated by} \\ \text{the matrix coefficients} \\ c_{v,w} \text{ for all } v, w \in L(\Lambda) \\ \text{and all } \Lambda \in P^+ \end{array} \right)$$

and they prove the following

"Peter–Weyl"–Theorem: The map

$$\bigoplus_{\Lambda \in P^+} L(\Lambda) \otimes L(\Lambda) \rightarrow \mathbb{C}[G]$$

induced by $v \otimes w \mapsto c_{v,w}$ is an isomorphism of $G \times G$ -modules.

Here, the action of $G \times G$ on $\mathbb{C}[G]$ is given by $((g, h)f)(x) = f(g^*xh)$. Alternatively, one might use the usual action of $G \times G$ on functions on G and let act G on the first factor $L(\Lambda)$ by the contragredient action

$$(g, v) \mapsto (g^*)^{-1}v.$$

It turned out that $\mathbb{C}[G]$ is not a Hopf algebra. There is neither a co-multiplication nor an antipode (basically due to the infinite-dimensionality of the $L(\Lambda)$ and the inequivalence between highest weight and lowest weight representations). Even worse, Kac and Peterson exhibited elements in $\text{Specmax } \mathbb{C}[G]$ not contained in G (which injects into $\text{Specmax } \mathbb{C}[G]$) (cf. [3] Remark 2.2). Thus they formulated the following problem (loc. cit., 4H b)):

Determine $\text{Specmax } \mathbb{C}[G]$ (possibly with respect to a topological structure on the algebra $\mathbb{C}[G]$)!

Inspired by the deformation theory of certain singularities (cf. [13]) we conjectured

$$\overline{G} := \text{Specmax } \mathbb{C}[G] = G.\overline{T}.G$$

where $\overline{T}$ is the closure of T in $\overline{G}$ realized as the torus embedding

$$T = \text{Specmax } \mathbb{C}[P] \subset \text{Specmax } \mathbb{C}[P \cap I] = \overline{T}$$

for $I \subset P \otimes_{\mathbb{Z}} \mathbb{R}$, the Tits cone attached to G . This embedding, or rather a domain $\mathcal{T} \subset \overline{\mathcal{T}}$ of discontinuity for the action of W , had been studied before by Looijenga and the quotient $\overline{\mathcal{T}}/W$ had turned out to be the base space of a semiuniversal deformation for certain isolated singularities (cf. [6], [7]). Moreover, in [12], [13] we realized $\overline{\mathcal{T}}/W$ and $\overline{\mathcal{T}}/W$ as target spaces for an adjoint quotient of G .

During a stay at MSRI (1984), D. Peterson announced a proof of the above conjecture including a number of structural properties of $\overline{G}$ ([9], $\overline{G}$ being considered as the continuous spectrum with respect to some topology). In connection with his infinite-dimensional algebraic-geometric approach to the flag manifolds of Kac-Moody groups, M. Kashiwara also studied the abstract maximal spectrum of $\mathbb{C}[G]$ (without topology on $\mathbb{C}[G]$), cf. [5]. Finally, C. Mokler ([8]) made a quite thorough study of $\overline{G}$ in the context of some infinite-dimensional algebraic geometry based on suitably topologized coordinate rings. In particular, he gave a detailed proof of our conjecture. This is what we want to report upon.

3. A topology on the algebra of strongly regular functions

Let V be a complex vector space. Then we may view the symmetric algebra $S(V^*)$ of its dual space V^* as the coordinate ring of the variety V . If $\dim_{\mathbb{C}} V < \infty$ we have

$$\mathrm{Hom}_{k\text{-alg}}(S(V^*), \mathbb{C}) = \mathrm{Hom}(V^*, \mathbb{C}) = V^{**} = V.$$

However, if $\dim_{\mathbb{C}} V = \infty$ we have $V \subset V^{**}$, $V \neq V^{**}$, and $\mathrm{Specmax} S(V^*)$ is strictly larger than V . To remedy this defect we put the following topology on the algebra $S(V^*)$:

A basis of neighborhoods of $0 \in S^*(V^*)$ is given by the "cofinite" ideals

$$\{J(V') \mid V' \subset V \text{ a finite-dimensional subspace}\},$$

$$J(V') = \{f \in S(V^*) \mid f|_{V'} \equiv 0\}.$$

Now, the continuous maximal spectrum

$$\mathrm{Specm}^{\circ} S(V^*) = \mathrm{Hom}_{\mathrm{cont}\text{-}k\text{-alg}}(S(V^*), \mathbb{C})$$

is easily identified with V (i.e. Hilbert's Nullstellensatz gives $V' = S(V^*)/J(V')$ for all finite-dimensional $V' \subset V$).

To put a topology on $\mathbb{C}[G]$ we embed G , and finally $\overline{G}$, into a larger space M constructed as follows:

We fix contravariant forms $\langle \cdot, \cdot \rangle$ on all modules $L(\Lambda), \Lambda \in P^+$, and extend them to a form, also denoted by $\langle \cdot, \cdot \rangle$, on the direct sum

$$L := \bigoplus_{\Lambda \in P^+} L(\Lambda)$$

by requiring $L(\Lambda)$ and $L(\Lambda')$ to be orthogonal for $\Lambda \neq \Lambda'$. Let M denote the subalgebra of $\text{End}(L)$ satisfying

- $\varphi(L(\Lambda)) \subset L(\Lambda)$ for all $\Lambda \in P^+$,
- the adjoint φ^* of φ with respect to $\langle \cdot, \cdot \rangle$ exists.

We let $\mathbb{C}[M]$ denote the $\mathbb{C}$ -algebra generated by all matrix coefficients $c_{v,w} : M \rightarrow \mathbb{C}, v, w \in L, c_{v,w}(\varphi) = \langle v, \varphi w \rangle$, and consider the "cofinite" topology on $\mathbb{C}[M]$ given by the neighborhood basis of 0

$$\{J(M')|M' \subset M \text{ a subspace of finite dimension}\},$$

$J(M')$ being the vanishing ideal of M' .

Then we have

- $\text{Specm}^\circ \mathbb{C}[M] = M$
- M is a "weak" algebraic monoid (i.e. right and left multiplication on M by given elements of M are "morphisms" of M ; note that there is no comultiplication on $\mathbb{C}[M]$).

By the definition of the contravariant forms on the $L(\Lambda)$ and L we have a natural embedding $G \hookrightarrow M$. Moreover, $\mathbb{C}[G]$ is the image of $\mathbb{C}[M]$ under the restriction from M to G . We now put the quotient topology with respect to $\mathbb{C}[M] \rightarrow \mathbb{C}[G]$ on $\mathbb{C}[G]$ and we obtain

- $\text{Specm}^\circ \mathbb{C}[G] = \overline{G}$ = Zariski-closure of G in M ,
- $\overline{G}$ is a "weak" algebraic monoid (in the sense above).

4. The Tits cone and the closure of the maximal torus

Let $V = P \otimes_{\mathbb{Z}} \mathbb{R}$ be the "real" character group, $\overline{C} = \{\omega \in V \mid \omega(h_i) \geq 0 \text{ for all } i = 1, \dots, l\}$ a *fundamental Weyl chamber*, and $I = W \cdot \overline{C}$ the union of all W -translates of $\overline{C}$. Then I is a convex solid cone, called the *Tits cone*. The interior I° of I is a domain of discontinuity of W . (For details, cf. [2]).

Example: Let A be the "hyperbolic" matrix

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \text{ corresponding to } \begin{array}{ccc} \bullet & \bullet & \bullet \\ \alpha_1 & \alpha_2 & \alpha_3 \end{array}$$

Now, the matrix A defines a symmetric bilinear form on $V \cong \mathbb{R}^3$ of signature $(+, +, -)$, and with respect to some convention I° may be identified with the interior of the positive light cone. The Weyl group W is isomorphic to $PGL_2(\mathbb{Z})$ acting as a group of hyperbolic motions on the unit disc $\cong \mathbb{P}(I^\circ) \subset \mathbb{P}(V)$.

The boundary of I is of particular interest for us. A subset $I' \subset I$ is called a (*rational*) *boundary component* of I if there is a $\gamma \in V^* = H \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. a $\gamma \in H$) such that

- $\omega(\gamma) \geq 0$ for all $\omega \in I$
- $\omega(\gamma) = 0$ for $\omega \in I$ implies $\omega \in I'$.

It is possible to classify all boundary components of I in terms of a special subset of them:

A subset $\Theta \subset \Pi$ is called *pure* if either $\Theta = \emptyset$ or if all connected components of Θ (in an obvious sense) are of infinite type.

To any pure subset $\Theta \subset \Pi$ we may associate the following subset $I(\Theta)$ of I :

$$I(\Theta) = \{\omega \in I \mid \omega(h_i) = 0 \text{ for all } i \text{ such that } \alpha_i \in \Theta\}.$$

We now have the following result, essentially due to Looijenga ([6]):

Theorem:

- i) Let $\Theta \subset \Pi$ be pure. Then $I(\Theta)$ is a rational boundary component of I .
- ii) Let $I' \subset I$ be a boundary component. Then there is a unique pure $\Theta \subset \Pi$ and a $w \in W$ such that $I' = w.I(\Theta)$. In particular, all boundary components of I are rational.

Example: We take up the previous example. There are 3 pure subsets of Π :

$$\emptyset, \quad \Theta = \{\alpha_1, \alpha_2\}, \quad \Pi = \{\alpha_1, \alpha_2, \alpha_3\}.$$

The corresponding boundary components are

$$I, \quad \begin{array}{l} \text{all rational half-lines on} \\ \text{the positive light cone} \end{array}, \quad \{0\}.$$

To determine the closure $\overline{T}$ of T in $\overline{G}$ we first have to describe the restriction of $\mathbb{C}[G]$ to T . Since all weights of a module $L(\Lambda)$, $\Lambda \in P^+$, are contained in $I \cap P$, and since $\overline{C} \cap P = P^+$ we obtain

$$\mathbb{C}[G]|_T = \mathbb{C}[P \cap I],$$

the semigroup algebra of $P \cap I$. It is easily seen that the induced topology on $\mathbb{C}[P \cap I]$ is discrete, thus

$$\overline{T} = \text{Specm}^\circ \mathbb{C}[P \cap I] = \text{Specm} \mathbb{C}[P \cap I].$$

Through $\mathbb{C}[P \cap I]$ is not finitely generated its maximal spectrum can be determined similarly as in the usual "finite type" theory of torus embeddings (cf. e.g. [1]), i.e. one has

$$\overline{T} = \bigcup_{I'} T/\text{Ann}(I') = \bigcup_{\Theta} \bigcup_{w \in W} T/w\text{Ann}(I(\Theta))w^{-1},$$

where $\text{Ann}(I') = \{t \in T | \omega(t) = 1 \text{ for all } \omega \in I'\}$ and where I' (resp. Θ) runs through all rational boundary components of I (resp. all pure subsets of Π).

As a subset of M , the completion $\overline{T}$ has a quite natural representation theoretic realization:

Let $\Theta \subset \Pi$ be a pure subset. We define the projection operator $e(\Theta) \in M$ by

$$e(\Theta)v = \begin{cases} v & \text{if } v \in L(\Lambda)_\mu \text{ and } \mu \in I(\Theta) \\ 0 & \text{if } v \in L(\Lambda)_\mu \text{ and } \mu \notin I(\Theta). \end{cases}$$

Then the boundary stratum $T/\text{Ann}(I(\Theta))$ is realized as the T -orbit $T.e(\Theta)$ of $e(\Theta)$ under left multiplication by T . To realize $e(\Theta)$ as a boundary point of $\overline{T}$ choose a one-parameter subgroup $\gamma \in H = \text{Hom}(\mathbb{C}^*, T)$ such that $\omega(\gamma) \geq 0$ for all $\omega \in I$ and $\omega(\gamma) = 0$ exactly when $\omega \in I(\Theta)$. Since for all $s \in \mathbb{C}^*$, $v \in L(\Lambda)_\omega$, we have

$$\gamma(s)v = s^{\omega(\gamma)}v,$$

we clearly obtain (in M)

$$\lim_{s \rightarrow 0} \gamma(s) = e(\Theta).$$

5. Unipotent subgroups

To study the unipotent radicals U^+, U^- of B^+, B^- as well as those of general parabolic subgroups we have to take a closer look at the action of G on $L(\Lambda)$, $\Lambda \in P^+$. We consider $L(\Lambda)$ as a variety with the coordinate ring $\mathbb{C}[L(\Lambda)]$ generated by the functions $c_w : L(\Lambda) \rightarrow \mathbb{C}$, $c_w(v) = \langle v, w \rangle$, and equipped with the appropriate "cofinite" topology. Then, for any fixed $v \in L(\Lambda)$, the orbit map

$$\begin{array}{ccc} M & \longrightarrow & L(\Lambda) \\ m & \longmapsto & mv \end{array}$$

is a morphism of varieties (with continuous comorphism $\mathbb{C}[L(\Lambda)] \rightarrow \mathbb{C}[M]$). We shall make use of the following results of Kac and Peterson ([10],[3] Lemma 4.3)

- The Kostant cone $\mathcal{V}(\Lambda) = (Gv_0) \cup \{0\}$, with $v_0 \in L(\Lambda)_\Lambda \setminus \{0\}$, is Zariski closed in $L(\Lambda)$.
- If Λ is a regular dominant weight, $\Lambda \in P^{++}$ (i.e. $\Lambda(h_i) > 0$ for $i = 1, \dots, l$), then $\mathbb{C}[G]|_{U^-}$ is generated by the matrix coefficients c_{xv_0, v_0} , x running through all elements in $\mathfrak{g}$ (in fact, $x \in \mathfrak{g}^- = \bigoplus_{\alpha \in \Sigma^-} \mathfrak{g}_\alpha$, where Σ^- is the system of all negative roots, is sufficient).

Theorem ([8] Satz 5.6,1)): The groups U^+ and U^- are Zariski closed in M .

Proof: Because of the existence of the anti-involution $*$: $G \rightarrow G$ it is sufficient to consider U^- . Assume $v_0 \in L(\Lambda)_\Lambda \setminus \{0\}$ ($\Lambda \in P^{++}$) chosen such that $\langle v_0, v_0 \rangle = 1$. This implies

$$\begin{aligned} c_{v_0, v_0}(u) &= 1 \quad \text{for all } u \in U^-, \text{ and} \\ c_{v_0, v_0}(\varphi) &= 1 \quad \text{for all } \varphi \in \overline{U^-}. \end{aligned}$$

Let $\varphi \in \overline{U^-}$. Then $\langle v_0, \varphi v_0 \rangle = 1$ implies $\varphi v_0 \neq 0$. Since $M \rightarrow L(\Lambda), m \mapsto mv_0$, is continuous and $\mathcal{V}(\Lambda)$ is closed in $L(\Lambda)$ we get $\varphi v_0 \in Gv_0 \subset \mathcal{V}(\Lambda)$. Thus, using the Birkhoff decomposition of G , we find $u \in U^-, n \in N$ such that

$$\varphi v_0 = u^- n v_0.$$

Because of $(u^-)^* \in U^+$ we have

$$1 = \langle v_0, \varphi v_0 \rangle = \langle (u^-)^* v_0, n v_0 \rangle = \langle v_0, n v_0 \rangle$$

and thus $n = 1$, or $\varphi v_0 = u^- v_0$. This implies $c_{xv_0, v_0}(\varphi) = c_{xv_0, v_0}(u^-)$ for all $x \in \mathfrak{g}$, or $\varphi = u^- \in U^-$, q.e.d.

Recall that any subset $\Psi \in \Pi$ gives rise to a Weyl subgroup

$$W_\Psi = \langle s_{\alpha_i} | \alpha_i \in \Psi \rangle$$

and parabolic subgroups

$$P_\Psi^\pm = \langle B^\pm, W_\Psi \rangle$$

with unipotent radicals

$$U_\Psi^\pm = \bigcap_{w \in W_\Psi} w U^\pm w^{-1}.$$

It is obvious that $U_\Psi^\pm$ are Zariski closed in M , as well.

6. The main result

For any $i \in \{1, \dots, l\}$ we fix a highest weight vector $v_i \in L(\Lambda_i)_{\Lambda_i} \setminus \{0\}$ and define the principal open subset $D_i \subset \overline{G}$ by

$$D_i = \{\varphi \in \overline{G} \mid c_{v_i, v_i}(\varphi) \neq 0\}.$$

We can almost cover $\overline{G}$ by these sets. Let $\Pi_\infty \subset \Pi$ the maximal pure subset of Π , i.e. Π is the "orthogonal" union of the set Π_∞ and a subset $\Pi \setminus \Pi_\infty$ of finite type.

Proposition A ([M], Satz 5.16): We have

$$\bigcup_{i=1}^l \bigcup_{g, h \in G} gD_ih = \overline{G} \setminus T.e(\Pi_\infty).$$

Proof: To simplify our presentation, we shall assume $\Pi = \Pi_\infty$ and $P^\circ = \{0\}$. Then $e(\Pi) = e(\Pi_\infty) \in M$ is characterized by the property $e(\Pi)v = 0$, for all $v \in L(\Lambda), \Lambda \in P^+ \setminus \{0\}$. Consider $\varphi \in \overline{G}$ and assume $\varphi \notin gD_ih$ for all $i \in \{1, \dots, l\}, g, h \in G$. Then

$$\langle gv_i, \varphi hv_i \rangle = 0, \text{ for all } i, g, h.$$

Since $L(\Lambda_i)$ is spanned by all $gv_i, g \in G$, we obtain $\varphi|_{L(\Lambda_i)} = 0$. Since any $L(\Lambda), \Lambda \in P^+ \setminus \{0\}$ is made up from tensor products of the $L(\Lambda_i)$ and subsequent reduction, we get

$$\varphi|_{L(\Lambda)} = 0 \quad \text{for all } \Lambda \in P^+ \setminus \{0\}$$

or $\varphi = e(\Pi)$.

(This proof can be easily adopted to the general case.)

As a next step, we shall determine the structure of the open sets $D_i \subset \overline{G}$. For that recall the parabolic subgroups

$$P_i^\pm = P_{\Pi \setminus \{\alpha_i\}}^\pm$$

with unipotent radicals

$$U_i^\pm = U_{\Pi \setminus \{\alpha_i\}}^\pm,$$

Levi subgroup $G_i = P_i^+ \cap P_i^-$ and Weyl group $W_i = W_{\Pi \setminus \{\alpha_i\}}$. Then G_i is the Kac–Moody group attached to the realization $(H, \Pi \setminus \{\alpha_i\}, \Pi \setminus \{h_i\})$. Let $\mathbb{C}[G_i]$ denote the algebra of strongly regular functions on G_i and let $\mathbb{C}[G]_i$ denote the algebra of restricted functions from $\mathbb{C}[G]$ to the subgroup G_i . Then the function c_{v_i, v_i} restricts to the character Λ_i on G_i , and representation theoretic arguments quickly show (cf. [8], section 5.1.2):

Lemma: The inclusion $\mathbb{C}[G]_i \subset \mathbb{C}[G_i]$ induces an isomorphism from the localization of $\mathbb{C}[G]_i$ with respect to Λ_i to $\mathbb{C}[G_i]$:

$$(\mathbb{C}[G]_i)_{\Lambda_i} \xrightarrow{\sim} \mathbb{C}[G_i] .$$

Proposition B: For any $i \in \{1, \dots, l\}$ we have an isomorphism of infinite-dimensional varieties

$$D_i = U_i^- \times \text{Specm}^\circ \mathbb{C}[G_i] \times U_i^+ .$$

Proof: Let us first look at $D_i \cap G$. Then the Birkhoff–decomposition

$$G = \bigcup_{w \in W} U^- w T U^+$$

gives

$$D_i \cap G = \bigcup_{w \in W_i} U^- w T U^+ = U_i^- \cdot G_i \cdot U_i^+ \text{ (direct product)} .$$

Recall that the $U_i^\pm$ are closed in M , therefore in $\overline{G}$ and in D_i . By the Lemma, the closure of G_i in D_i can be identified with $\text{Specm}^\circ \mathbb{C}[G_i]$. This gives the claim.

Applying downward induction to Propositions A and B we arrive at our main result.

Theorem([8], Satz 5.18): We have

$$\overline{G} = \{ge(\Theta)h \mid \Theta \subset \Pi \text{ pure } g, h \in G\} = G \cdot \overline{T} \cdot G .$$

Remarks: Proposition B for the case of the minimal parabolic B^+ may already be found in [3], Lemma 4.4. Its general version for arbitrary parabolics is due to Kashiwara ([5], Proposition 5.3.5), who has also given a form of Proposition A in a somewhat different context ([5], Proposition 6.3.1).

7. An application

In [8] one finds many more results on the structure of $\overline{G}$. Here, we want to conclude with an application to the adjoint quotient of G studied in [12], [13], [14] (details are forthcoming). Recall that G admits a "parabolic" partition

$$G = \bigcup_{\substack{\Theta \subset \Pi \\ \text{pure}}} G(\Theta)$$

parallel to a stratification of $\overline{T}/W$

$$\overline{T}/W = \bigcup_{\Theta \subset \Pi} (\overline{T}/W)(\Theta)$$

$((\overline{T}/W)(\Theta))$ the image of $T/\text{Ann}(I(\Theta))$ in $\overline{T}/W$.

The adjoint quotient defined in [12], [13] is a conjugation invariant map

$$\chi : G \rightarrow \overline{T}/W$$

mapping $G(\Theta)$ to $(\overline{T}/W)(\Theta)$ for any pure $\Theta \subset \Pi$. With the help of a theory of "optimal one-parameter semisubgroups" in $\overline{G}$ the partition and the map χ can be extended to a conjugation invariant map $\overline{\chi} : \overline{G} \rightarrow \overline{T}/W$ with the following properties, basic in geometric invariant theory:

- Every fibre of $\overline{\chi}$ contains a unique closed conjugacy class,
- two elements $\varphi, \psi \in \overline{G}$ are mapped to the same point in $\overline{T}/W$ if and only if the closures of their conjugacy classes meet,

$$\overline{(\text{Ad}(G)\varphi)} \cap \overline{(\text{Ad}(G)\psi)} \neq \emptyset.$$

Remarks: 1) If one considers $\chi : G \rightarrow \overline{T}/W$ these statements hold only for the "classical" part $G(\emptyset)$ mapping onto T/W .

2) The closed (= minimal = semisimple) orbits in all fibres of $\overline{\chi}$ are given as the orbits of the elements $t.e(\Theta)$, $\Theta \subset \Pi$ pure, $t \in T$.

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