

非局所非線形境界値問題の厳密解と大域的解構造
Exact solutions and the global solution structure of
nonlocal nonlinear boundary problems

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We are interested in the global structure of all solutions of several nonlocal nonlinear boundary problems arising in various fields. We show four examples.

The first problem is related with the Oseen's spiral flow [11].

Find a function $U(x)$ such that

$$(O) \quad \begin{cases} \{U_{xx} + AU - U^2\}_x = 0, & x \in (-\pi, \pi), \\ U(-\pi) = U(\pi), & U_x(-\pi) = U_x(\pi), \\ \int_{-\pi}^{\pi} U(x) dx = 0, \end{cases}$$

for arbitrarily fixed A .

It is easily seen that $U \equiv 0$ is the trivial solution of the above problem for any fixed A . Okamoto [10] started to investigate the global bifurcation structure of this problem. Moreover, Ikeda-Mimura-Okamoto [4] obtained the asymptotic shape of solutions as $A \rightarrow -\infty$.

Let us recall the standard notation of complete elliptic integrals:

$$K(k) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad k \in [0, 1),$$

$$E(k) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi} d\varphi, \quad k \in [0, 1).$$

Jacobi's elliptic functions $\operatorname{sn}(x, k)$ and $\operatorname{cn}(x, k)$ with the modulus k are defined as follows:

$$\operatorname{sn}^{-1}(z, k) := \int_0^z \frac{d\xi}{\sqrt{(1 - \xi^2)(1 - k^2 \xi^2)}}, \quad z \in [0, 1], \quad k \in [0, 1),$$

and

$$\operatorname{cn}^2(z, k) = 1 - \operatorname{sn}^2(z, k).$$

We note that

$$E(0) = K(0) = \frac{\pi}{2}, \quad E(1) = 1, \quad K(k) \sim \frac{1}{2} \log \left(\frac{16}{1 - k^2} \right) \quad \text{as } k \rightarrow 1.$$

Ikeda-Kondo-Okamoto-Yotsutani [3] have parameterized all solutions (A, U) of (O) in terms of the elliptic functions, and clarified the global bifurcation structure by the following Theorems 1 and 2.

Theorem 1 *All the solution (A, U) of (O) are parameterized by*

$$\left\{ \left(n^2 A(k), n^2 U(nx - x_0; A(k)) \right) : 0 < k < 1, -\pi < x_0 \leq \pi, n = 1, 2, 3 \dots \right\},$$

where

$$\begin{aligned} A(k) &:= \frac{4K(k)}{\pi^2} (3E(k) + (k^2 - 2)K(k)), \\ U(x; A(k)) &:= -\frac{6k^2 K(k)^2}{\pi^2} \operatorname{cn}^2 \left(\frac{K(k)}{\pi} x, k \right) \\ &\quad + \frac{6K(k)}{\pi^2} \{E(k) - (1 - k^2)K(k)\}. \end{aligned}$$

Theorem 2 *The function $A(k)$ is strictly monotone decreasing in $k \in (0, 1)$. It also satisfies $\lim_{k \rightarrow 0} A(k) = 1$ and $\lim_{k \rightarrow 1} A(k) = -\infty$.*

The second problem is related with structure of stationary solutions in S^1 of the Ginzburg-Landau equation.

Find a function $u(x)$ such that

$$(P) \begin{cases} u_{xx} - \frac{C^2}{u^3} + \lambda(1 - u^2)u = 0 & \text{in } [-\pi, \pi], \\ C := 2m\pi \left\{ \int_{-\pi}^{\pi} \frac{1}{u^2} dx \right\}^{-1}, \\ u(-\pi) = u(\pi), \quad u_x(-\pi) = u_x(\pi), \\ u > 0 & \text{in } [-\pi, \pi], \end{cases}$$

where m is a given integer and λ is a bifurcation parameter.

The structure of solutions is similar to that of Oseen's spiral flow, though the analysis is more difficult. Kosugi-Morita-Yotsutani [5] have clarified the global bifurcation structure of this problem.

We briefly explain about the original equation. Consider the following Ginzburg-Landau equation:

$$\begin{cases} \psi_{xx} + \lambda(1 - |\psi|^2)\psi = 0, & x \in (-\pi, \pi), \\ \psi(-\pi) = \psi(\pi), \quad \psi_x(-\pi) = \psi_x(\pi). \end{cases}$$

We here assume that $|\psi| > 0$ and ψ is written as the form

$$\psi = u(x) \exp(i\theta(x)),$$

where u and θ are both real-valued smooth functions. Clearly the equation is equivalent the following system:

$$\begin{cases} u_{xx} - (\theta_x)^2 u + \lambda(1 - u^2)u = 0, & x \in (-\pi, \pi), \\ (u^2 \theta_x)_x = 0, & x \in (-\pi, \pi), \\ u(-\pi) = u(\pi), \quad u_x(-\pi) = u_x(\pi), \\ \theta(\pi) - \theta(-\pi) = 2m\pi, \quad \theta_x(-\pi) = \theta_x(\pi), \end{cases}$$

where m is an integer. Thus, $\theta_x = C/u^2$ for a constant C and hence we obtain (P).

The third problem is related to find the minimum energy curve for given the length L and area M , which K.Watanabe [13] started to investigate.

For given $L > 0$ and $M > 0$ with $L^2 - 4\pi M > 0$, find a function $\kappa(s)$ such that

$$(E) \begin{cases} \left\{ \kappa_{ss} + \frac{1}{2} \kappa^3 + \mu \kappa \right\}_s = 0 & \text{in } [0, L], \\ \mu := \frac{1}{L^2 - 4\pi M} \left\{ M \int_0^L \kappa(s)^3 ds - \frac{L}{2} \int_0^L \kappa(s)^2 ds \right\}, \\ \kappa(0) = \kappa(L), \quad \kappa_s(0) = \kappa_s(L), \\ \int_0^L \kappa(s) ds = 2\pi. \end{cases}$$

Murai-Matsumoto-Yotsutani [9] have completely clarified the global bifurcation structure of this problem, though we need terribly complicated calculations and arguments. This result is written by Minoru Murai in this lecture note.

The final problem is a limiting equation for the Shigesada-Kawasaki-Teramoto model with cross-diffusion [12]. This problem is the hardest.

Find $(v(x), \tau)$ such that $\tau > 0$, and

$$(S) \begin{cases} \int_0^1 \frac{\tau}{v} (a_1 - b_1 \frac{\tau}{v} - c_1 v) dx = 0, \\ d_2 v_{xx} + v(a_2 - b_2 \frac{\tau}{v} - c_2 v) = 0 & \text{in } (0, 1), \\ v_x(0) = 0, \quad v_x(1) = 0, \\ v > 0 & \text{on } [0, 1] \end{cases}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_2$ are given positive constants.

We briefly explain the original equation. In 1979, Kawasaki-Shigesada-Teramoto proposed a cross-diffusion system

$$\begin{cases} u_t = \{(d_1 + \rho_{12}v)u\}_{xx} + u(a_1 - b_1u - c_1v) & (0 < x < 1, t > 0), \\ v_t = \{(d_2 + \rho_{21}u)v\}_{xx} + v(a_2 - b_2u - c_2v) & (0 < x < 1, t > 0), \\ v_x(0, t) = 0 \ (t > 0), \ v_x(1, t) = 0 \ (t > 0), \\ u(x, 0) = u_0(x) \geq 0 \ (0 < x < 1), \ v(x, 0) = v_0(x) \geq 0 \ (0 < x < 1), \end{cases}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are positive constants, ρ_{12} and ρ_{21} are nonnegative constants, and $u_0(x)$ and $v_0(x)$ are nonnegative initial data. This is a mathematical model to explain the segregation phenomena. Mathematical study of cross-diffusion equations was begun by M. Mimura in 1980 (see, e.g., [8]). There are various results concerning the existence of solutions to time-dependent problem (see, e.g., [1], [2] and references therein), and stationary problems. Sharp existence and non-existence results of stationary solutions are not known.

The limiting equation (S) was discovered by Lou-Ni [6] as a limiting equation when cross-diffusion effect $\rho_{12} \rightarrow \infty$. Actually, we see from the numerical computations that solutions of (S) approximate stable stationary solutions of the original time-dependent problem. Thus, it is important to know the structure of solutions of (S).

Lou-Ni-Yotsutani [7] have almost clarified the existence and the shape of solutions as follows. Let us put

$$A := \frac{a_1}{a_2}, \quad B := \frac{b_1}{b_2}, \quad C := \frac{c_1}{c_2}.$$

We concentrate on the case $B < C$ (strong competition case).

Theorem 3 (Existence) *Suppose that $B < C$. If*

$$\max\{0, \frac{B+C-2A}{C-B}\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2},$$

then there exists a solution $(v(x), \tau)$ of (S).

Theorem 4 (Nonexistence) *Suppose that $B < C$.*

(i) *If $d_2 \geq \frac{a_2}{\pi^2}$, then there exists no solution of (S).*

(ii) *If $A < \frac{B+C}{2}$, then there exists a $d_2^* = d_2^*(A, B, C, a_2) > 0$ such that there exists no solution of (S) for $d_2 \in (0, d_2^*]$.*

(iii) *If $A < B$, there exists no solution of (S).*

The following theorems give the shape of solutions.

Theorem 5 (Shape of solutions as $d_2 \rightarrow a_2/\pi^2$) Let $(v(x, d_2), \tau(d_2))$ be solutions of (S). If $A \geq B$, then

$$\begin{aligned} v(x; d_2) &\rightarrow 0, \\ \frac{v(x; d_2) - v(0; d_2)}{v(1; d_2) - v(0; d_2)} &\rightarrow \sin^2\left(\frac{\pi}{2}x\right), \\ \frac{\tau(d_2)}{v(x; d_2)} &\rightarrow \frac{a_2}{b_2} \cdot \frac{A/B + \sqrt{(A/B)^2 - A/B}}{1 + 2\{A/B - 1 + \sqrt{(A/B)^2 - A/B}\} \sin^2(\frac{\pi}{2}x)}, \end{aligned}$$

uniformly on $[0, 1]$ as $d_2 \rightarrow a_2/\pi^2$.

Theorem 6 (Shape of solutions as $d_2 \rightarrow 0$ for $A < \frac{B+3C}{4}$) Let $(v(x, d_2), \tau(d_2))$ be solutions of (S). If $A < \frac{B+3C}{4}$ and $B \neq C$, then

$$\begin{aligned} v(0; d_2) &\rightarrow 2 \cdot \frac{a_2}{c_2} \cdot \frac{\frac{B+3C}{4} - A}{C - B}, & v(x; d_2) &\rightarrow \frac{a_2}{c_2} \cdot \frac{A - B}{C - B} \quad \text{for } x > 0, \\ \frac{\tau(d_2)}{v(0; d_2)} &\rightarrow \frac{a_2}{2c_2} \cdot \frac{C - A}{C - B} \cdot \frac{A - B}{\frac{B+3C}{4} - A}, & \frac{\tau(d_2)}{v(x; d_2)} &\rightarrow \frac{a_2}{b_2} \cdot \frac{C - A}{C - B} \quad \text{for } x > 0, \end{aligned}$$

as $d_2 \rightarrow 0$.

Theorem 7 (Shape of solutions as $d_2 \rightarrow 0$ for $A \geq \frac{B+3C}{4}$) Let $(v(x, d_2), \tau(d_2))$ be solutions of (S). If $B < C$ and $A \geq \frac{B+3C}{4}$, then

$$\begin{aligned} v(0; d_2) &\rightarrow 0, & v(x; d_2) &\rightarrow \frac{3a_2}{4c_2} \quad \text{for } x > 0, \\ \frac{\tau(d_2)}{v(0; d_2)} &\rightarrow \infty, & \frac{\tau(d_2)}{v(x; d_2)} &\rightarrow \frac{a_2}{4c_2} \quad \text{for } x > 0, \end{aligned}$$

as $d_2 \rightarrow 0$.

Now, We will discuss about the uniqueness and non-uniquess. The following result is a part of joint projects with W.-M. Ni.

Theorem 8. Suppose that $B < C$. If d_2 is sufficiently small, the solution $(v(x), \tau)$ is unique for any given A .

Idea of a proof of Theorem 8. All solutions $(v(x), \tau)$ of

$$\begin{cases} d_2 v_{xx} + v(a_2 - b_2 \frac{\tau}{v} - c_2 v) = 0 & \text{in } (0, 1), \\ v_x(0) = 0, \quad v_x(1) = 0, \\ v > 0 & \text{on } [0, 1] \end{cases}$$

are represented by one parameter k . We denote it by $(v(x; k, d_2), \tau(k, d_2))$. Rewrite

$$\int_0^1 \frac{1}{v} (a_1 - b_1 \frac{\tau}{v} - c_1 v) dx = 0$$

as

$$\frac{b_1 \tau \int_0^1 \frac{1}{v^2} dx + c_1}{\int_0^1 \frac{1}{v} dx} = a_1.$$

We put the left hand side by $\check{a}_1(k, d_2)$. For given a_1 and sufficiently small d_2 , we show that there exists the unique solution k of $\check{a}_1(k, d_2) = a_1$ by using Theorems 6 and 7.

Theorem 9. Suppose that $C > 7B/3$. There exists an open set D such that (S) has at least two solutions $(v(x), \tau)$ for $d_2 \in D$.

Idea of a proof of Theorem 9. We investigate the function $\check{a}_1(k, d_2)$. We see that the Taylor expansion with respect to k at $k = 0$ of $\check{a}_1(k, d_2)$ becomes

$$\check{a}_1(k, d_2) = \text{constant} + \left\{ \left(13 \frac{C}{B} + 35 \right) \pi^4 d_2^2 - 14 \left(\frac{C}{B} - 1 \right) \pi^2 d_2 + \frac{C}{B} - 1 \right\} k^4 + \dots$$

The coefficient of k^4 becomes negative for some d_2 , if $C/B > 7/3$. This implies the non-uniqueness of the solutions.

It seems that the following conjecture holds in view of the above theorems and the numerical computation.

Conjecture 1: Suppose that $B < C$. For any d_2 with

$$\max\{0, \frac{B+C-2A}{C-B}\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2},$$

there exists the unique solution $(v(x), \tau)$ of (S).

Conjecture 2: Suppose that $B < C \leq 7B/3$. (S) has a solution $(v(x), \tau)$ if and only if d_2 satisfies

$$\max\{0, \frac{B+C-2A}{C-B}\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2}.$$

Moreover, the solution is unique.

Conjecture 3: Suppose that $C > 7B/3$. There exists the only one connected non-empty open set D such that (S) has exactly two solutions $(v(x), \tau)$ if and only if $d_2 \in D$.

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