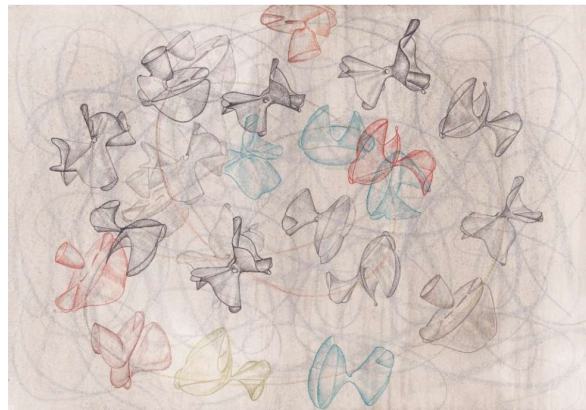


On a 3-dimensional K-moduli space of Fano 3-folds

Anne-Sophie Kaloghiros

Brunel University London

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Goal of the talk: K-moduli of pure states of four qubits

Joint work with I. Cheltsov, M. Fedorchuk and K. Fujita

Describe component of K-moduli space of Fano 3-folds of anticanonical degree 24 associated to family MM_{4-1}

An element X in family MM_{4-1} is $(1, 1, 1, 1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Theorem (CFFK'24)

The component of $M_{3,24}^{Kps}$ that parametrises K-polystable limits of Fano 3-folds in family MM_{4-1} is the blow up of $(2 : 2 : 0 : 0) \in \mathbb{P}(1, 3, 4, 6)$ with weights $(1, 2, 3)$.

- 1 Introduction - K-moduli spaces of Fano 3-folds
- 2 Geometry of divisors $(1, 1, 1, 1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ - first GIT
- 3 Another family of degenerations
- 4 K-moduli and the right GIT

Kähler Einstein manifolds and the Calabi problem

Calabi problem

Which compact complex manifolds admit a Kähler-Einstein metric?

- Canonical structure on manifold: Ricci class proportional to Kähler class
Hence canonical class of the manifold has a definite sign
- Yau, Aubin/Yau: positive solution when canonical class ample or trivial
- Fano case more subtle: some Fano manifolds are KE, others aren't
(Matsushima obstruction)

Yau-Tian-Donaldson conjecture/ Chen-Donaldson-Sun theorem/ Liu-Xu-Zhuang

Let X be a Fano manifold (resp. (X, Δ) a log Fano pair), then:

$$\begin{cases} X \text{ has a KE metric} \\ (X, \Delta) \text{ has a weak KE metric} \end{cases} \iff \begin{cases} X \\ (X, \Delta) \end{cases} \text{ is K-polystable}$$

K-polystability

K-polystability introduced to characterise KE Fano manifolds

- original definition in terms of Donaldson-Futaki invariants
- recent valuative formulations of K-polystability

More surprisingly:

K-moduli stacks and spaces [Xu and collaborators]

There is a projective good moduli space $M_{n,V}^{\text{Kps}}$ whose points parametrise K-polystable $\mathbb{Q}$ -Fano varieties of dimension n and volume V .

The $\mathbb{C}$ -points of $M_{n,V}^{\text{Kps}}$ parametrise Kähler-Einstein $\mathbb{Q}$ -Fano varieties.

The Calabi problem for Fano surfaces and 3-folds

In any dimension, there are finitely many families of smooth Fano manifolds.

- 10 deformation families of smooth del Pezzo surfaces
- 105 deformation families of smooth Fano 3-folds

For each deformation family of smooth Fano manifolds, we ask 3 versions of the Calabi problem:

- ① Is the general member K-polystable?
Does the family correspond to a non-empty component of $M_{n,V}^{\text{Kps}}$?
- ② Which members are K-polystable?
- ③ What is the compactification of the associated component of $M_{n,V}^{\text{Kps}}$?
What are the K-polystable limits of Fano manifolds in the family?

Calabi problem for del Pezzo surfaces

A smooth del Pezzo surface S_d of degree d is:

- (d=9) $\mathbb{P}^2$
- (d=8) $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow up of $\mathbb{P}^2$ in a point
- (d=7) the blow up of $\mathbb{P}^2$ in two points
- (d=6) a divisor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree (1, 1, 1)
- (d=5) a section of $\text{Gr}(2, 5) \subset \mathbb{P}^9$ by a linear subspace of codimension 4
- (d=4) a complete intersection of two quadrics in $\mathbb{P}^4$.
- (d=3) a cubic surface in $\mathbb{P}^3$
- (d=2) a quartic surface in $\mathbb{P}(1, 1, 1, 2)$
- (d=1) a sextic surface in $\mathbb{P}(1, 1, 2, 3)$

Calabi problem for del Pezzo surfaces

There are two smooth del Pezzo surfaces with non-reductive automorphisms:

- If $S_8 = \text{Bl}_p \mathbb{P}^2$, then $\text{Aut}(S_8) \cong \mathbb{G}_a^2 \rtimes \text{PGL}_2(\mathbb{C})$
- If $S_7 = \text{Bl}_{p,q} \mathbb{P}^2$, then $\text{Aut}(S_7) \cong (\mathbb{B}_2 \times \mathbb{B}_2) \rtimes \mu_2$, where $\mathbb{B}_2 = \text{Borel subgroup of } \text{PGL}_2(\mathbb{C})$

Corollary (Matsushima obstruction)

If S is $\text{Bl}_p(\mathbb{P}^2)$ or $\text{Bl}_{p,q}(\mathbb{P}^2)$, $\text{Aut}(S)$ is not reductive and S is not Kähler-Einstein (K -polystable).

Theorem (Tian)

A smooth del Pezzo surface that is not the blow up of $\mathbb{P}^2$ at one or two points is Kähler-Einstein (K -polystable).

This settles the first two versions of the Calabi problem for Fano surfaces.

Calabi problem for del Pezzo surfaces

- Families of dP surfaces of degree $d \geq 5$ have 0-dimensional moduli
- Associated components of $M_{2,d}^{\text{Kps}}$ for $d = 5, 6, 8$ and 9 are points
- For $d = 3, 4$, K and GIT-moduli spaces coincide

Theorem (Mabuchi-Mukai, Odaka-Spotti-Sun)

A K -polystable limit S of smooth del Pezzo surfaces of degree 4 is the intersection $Q \cap Q' \subset \mathbb{P}^4$ of simultaneously diagonalisable quadrics

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \\ \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 = 0 \end{cases}$$

where no three of the λ_i s are equal.

The K -moduli $M_{2,4}^{\text{Kps}} \simeq \mathbb{P}(1, 2, 3)$, and the locus of singular surfaces is cut out by the divisor $D = \{z_1^2 = 128z_2\}$.

If $[S] \in M_{2,4}^{\text{Kps}}$, S is smooth, or has 2 or 4 A_1 singularities.

Calabi problem for del Pezzo surfaces

Theorem (Odaka-Spotti-Sun)

A K -polystable limit S of smooth del Pezzo surfaces of degree 3 is either stable and has no worse than A1 singularities or is $X_3^T = \{x_0^3 - x_1x_2x_3 = 0\}$ and has 3 A2 singularities.

The K -moduli $M_{2,3}^{Kps} \simeq \mathbb{P}(1, 2, 3, 4, 5)$, and the locus of singular surfaces is cut out by $D = \{(z_1^2 - 64z_2)^2 = 2^{11}(8z_4 + z_1z_3)\}$.

Cases of degrees $d = 1$ or 2 are more subtle, as the natural GIT and K -moduli spaces are different.

The associated components $M_{2,2}^{Kps}$ and $M_{2,1}^{Kps}$ are 6 and 8-dimensional respectively and are explicitly described by Odaka-Spotti-Sun. Boundary points have worse than canonical singularities.

Calabi problem for degree 2 del Pezzo surfaces

- A smooth dP2 is a double cover $S \rightarrow \mathbb{P}^2$ branched over $F_4 \subset \mathbb{P}^2$
- If $[F_4] \in \overline{P}_4^{GIT}$, then F_4 is
 - stable if it has no worse than A1 or A2 singularities
 - strictly polystable if it is (a) a double conic $2C$ or (b) $C_1 \cup C_2$ where C_1, C_2 are tangent at 2 points and at least one of them is smooth.
- The $2 : 1$ cover $S \rightarrow \mathbb{P}^2$ branched along F_4 with $[F_4] \in \overline{P}_4^{GIT}$ is
 - non-normal hence non K -polystable if $F_4 = 2C$
 - K -polystable if F_4 is stable (S has no worse than A1 or A2 singularities) or $F_4 = C_1 \cup C_2$ (S has no worse than A3 singularities).

Calabi problem for degree 2 del Pezzo surfaces

Theorem (Odaka-Spotti-Sun, Ascher-DeVleming-Liu)

A K -polystable limit of smooth del Pezzo surfaces of degree 2 is

- a double cover $S \rightarrow \mathbb{P}^2$ branched along $(F_4 = 0) \subset \mathbb{P}^2$ with $[F_4] \in \overline{P}_4^{GIT} \setminus [2C]$

$$S_\infty = \{x_4^2 = f_4(x_1, x_2, x_3)\} \subset \mathbb{P}(1, 1, 1, 2)$$

- a double cover $S \rightarrow \mathbb{P}(1, 1, 4)$ branched along $\{z_3^2 = f_8(z_1, z_2)\}$ where f_8 is a binary octic.

$$S_\infty = \{x_3^2 + x_4^2 = f_8(x_1, x_2)\} \subset \mathbb{P}(1, 1, 4, 4)$$

The K -moduli space $M_{2,2}^{Kps}$ is a weighted blowup of $[2C] \in \overline{P}_4^{GIT}$.

- (Odaka) A K -polystable Fano variety is always normal.

Methods of proof

Valuative criterion (Fujita-Li)

X is K -polystable $\iff \beta(E) > 0$ for all E over X

For E/X , $\beta(E) = A_X(E) - \frac{1}{(-K_X)^n} \int_0^\infty \text{vol}(-K_X - tE) dt$

- the **stability threshold** $\delta_X = \inf_{E/X} \frac{A_X(E)}{S_X(E)}$ is estimated with Abban-Zhuang's theory of admissible flags
- there are equivariant versions of the Fujita-Li criterion (Zhuang)
- X is **divisorially unstable** if there is a divisor $E \subset X$ such that $\beta(E) < 0$

Theorem (Fujita)

The general member of 26 families of smooth Fano 3-folds is divisorially unstable.

The Calabi problem for smooth Fano 3-folds

Have a complete answer to the first formulation of the Calabi problem:

Theorem (Fano project 21 (previously known ~ 65 cases))

Let X be the general member of one of the 105 deformation families of Fano 3-folds. Then one of:

- *X belongs to family MM_{2-26} , or*
 - *X belongs to one of the 26 deformation families of K -divisorially unstable Fano 3-folds, or*
 - *X is K -polystable*
- 78 families with smooth K -polystable members
 - Both K -polystable and non- K -polystable smooth Fano 3-folds in several families
 - Have a complete answer for 58 out of 78 families with K -polystable members

K -moduli spaces of Fano 3-folds

Standard deformation theoretic argument: when non-empty, the dimension of the component of $M_{3,V}^{Kps}$ associated to the deformation family $MM_{\rho-N}$ is

$$-\chi(X, T_X) \text{ for any } X \text{ in the family}$$

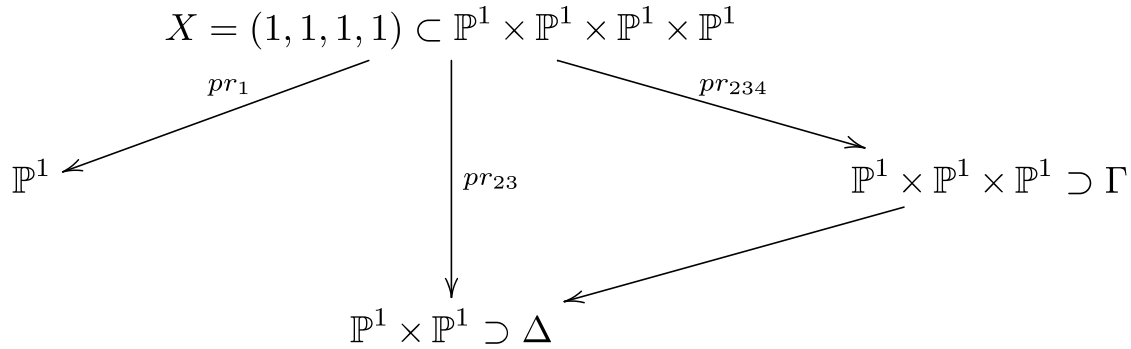
- As in the case of del Pezzo surfaces, birational classification is not modular - no details on isomorphic Fano varieties
- **Parametrisation** is a starting point for description of K -moduli spaces
- Many (44/105) families have 0-dimensional moduli
 - 20 of these consist of one smooth K -polystable object
 - 1 additional family with non-empty K -moduli space
 - Other 23 families have empty K -moduli
- 8 families have 1-dimensional moduli
 - 6 families with K -polystable general member
 - 2 families with empty K -moduli space

Geometry of family MM_{4-1}

Members of family MM_{4-1} are divisors $(1, 1, 1, 1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by a linear combination of monomials of the form

$$x_1^i y_1^{1-i} x_2^j y_2^{1-j} x_3^k y_3^{1-k} x_4^l y_4^{1-l} \text{ for } i, j, k, l \in \{0, 1\}$$

The birational geometry of X is illustrated by



Geometry of family MM_{4-1} - earlier results

- (Belousov-Loginov) Smooth members of family MM_{4-1} are K-polystable
- (Quantum Computing) Natural GIT picture associated to the action of

$$\Gamma_0 = (\text{SL}_2(\mathbb{C}))^4 \text{ and } \Gamma = \Gamma_0 \rtimes \mathfrak{S}_4 \text{ on } V = H^0((\mathbb{P}^1)^4, \mathcal{O}(1, 1, 1, 1))$$

Theorem

Up to Γ action, the equation $f \in V$ can be put in a normal form that is of one of 9 types.

Equations in these normal forms form sublinear systems, for example $\mathcal{G} \subset V$

$$\mathcal{G} = \left\{ f = \frac{a+d}{2}(x_1 x_2 x_3 x_4 + y_1 y_2 y_3 y_4) + \frac{a-d}{2}(x_1 x_2 y_3 y_4 + x_3 x_4 y_1 y_2) + \frac{b+c}{2}(x_1 x_3 y_2 y_4 + x_2 x_4 y_1 y_3) + \frac{b-c}{2}(x_1 x_4 y_2 y_3 + x_2 x_3 y_1 y_4) \right\}$$

Geometry of family MM_{4-1}

Let $f \in V$, then

- ① If $f \in V$ and $\{f = 0\}$ smooth, then $\Gamma \cdot f \cap \mathcal{G} \neq \emptyset$
- ② If $f \in \mathcal{G}$, then $\text{Aut}(X)$ contains a subgroup μ_2^4 of $i^*(\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1))$,
- ③ If $f \in \mathcal{G}$, then $\{f = 0\}$ is reducible $\iff f \in \Gamma \cdot \{(0 : 0 : 0 : 1)\}$
- ④ If $f, f' \in \mathcal{G}$ and $X = \{f = 0\}, X' = \{f' = 0\}$ are irreducible, then $X \simeq X' \iff \Gamma \cdot f \cap \Gamma \cdot f' \neq \emptyset$

Theorem (CFFK'24)

- The GIT quotient $\mathbb{P}V^{\text{ss}} // \Gamma \simeq \mathbb{P}(1, 3, 4, 6)$
- The orbit $\Gamma \cdot \mathbb{P}\mathcal{G}$ is the set of GIT polystable points
- The subgroup $\Gamma^{\mathcal{G}} = \{\sigma \in \Gamma \mid \sigma(\mathcal{G}) \subset \mathcal{G}\} \simeq W(F_4)$, so

$$\mathbb{P}V^{\text{ss}} // \Gamma \simeq \mathbb{P}(1, 3, 4, 6) \simeq \mathbb{P}^3 // W(F_4)$$

- Explicit description of $\mathbb{P}\mathcal{G} \rightarrow \mathbb{P}(1, 3, 4, 6)$

Boundary of the GIT-moduli space

There is a stratification

$$\begin{aligned} (\mathbb{P}V^{\text{ss}} // \Gamma)^{\text{Sing}} &= (\mathbb{P}V^{\text{ss}} // \Gamma)^{\text{Red}} \sqcup (\mathbb{P}V^{\text{ss}} // \Gamma)^{\text{Curv}} \sqcup (\mathbb{P}V^{\text{ss}} // \Gamma)^{6A_1} \\ &\sqcup (\mathbb{P}V^{\text{ss}} // \Gamma)^{4A_1} \sqcup (\mathbb{P}V^{\text{ss}} // \Gamma)^{2A_1} \end{aligned}$$

and if X_p is determined by $p = (a : b : c : d) \in \mathbb{P}\mathcal{G}$, then

- X_p is singular $\iff (a^2 - b^2)(a^2 - c^2)(a^2 - d^2)(b^2 - c^2)(b^2 - d^2)(c^2 - d^2) = 0$
- X_p is reducible $\iff p$ maps to $[2 : 2 : 0 : 0] \in \mathbb{P}(1, 3, 4, 6)$

Theorem (CFFK'24)

X_p is K-polystable whenever it is irreducible

Another family of degenerations of MM_{4-1}

As in the case of del Pezzo surfaces of degree 2 want a **K-polystable replacement** for the reducible GIT polystable degeneration

$$(1, 1, 0, 0) + (0, 0, 1, 1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

View this as a product of pairs

$$(C = (1, 1) \subset \mathbb{P}^1 \times \mathbb{P}^1) \rightsquigarrow (C_\infty \subset \mathbb{P}(1, 1, 2))$$

where C_∞ is the section at infinity in the cone over C .

Consider degenerations of the form $(2, 2) \subset \mathbb{P}(112) \times \mathbb{P}(112)$ with equation

$$w_1 w_2 + \alpha s_1 t_1 s_2 t_2 + \beta(s_1^2 s_2^2 + t_1^2 t_2^2) + \gamma(s_1^2 t_2^2 + t_1^2 s_2^2) = 0\}$$

parametrised by $\mathbb{P}^2(\alpha, \beta, \gamma)$

Theorem

X_p is K-polystable for all $p \in E$

K-polystable degenerations of MM_{4-1}

Theorem (CFFK'24)

A K-polystable limit of members of MM_{4-1} is one of

- an irreducible $(1, 1, 1, 1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of the form

$$X = \{a(x_1 x_2 x_3 x_4 + y_1 y_2 y_3 y_4) + b(x_1 x_2 y_3 y_4 + x_3 x_4 y_1 y_2) + c(x_1 x_3 y_2 y_4 + x_2 x_4 y_1 y_3) + d(x_1 x_4 y_2 y_3 + x_2 x_3 y_1 y_4) = 0\} \quad (*)$$

for $(a : b : c : d) \in \mathbb{P}^3$.

- an irreducible $(2, 2) \subset \mathbb{P}(1, 1, 2) \times \mathbb{P}(1, 1, 2)$ of the form

$$X = \{w_1 w_2 + \alpha s_1 t_1 s_2 t_2 + \beta(s_1^2 s_2^2 + t_1^2 t_2^2) + \gamma(s_1^2 t_2^2 + t_1^2 s_2^2) = 0\} \quad (**)$$

for $(\alpha : \beta : \gamma) \in \mathbb{P}^2$.

K-moduli space

Theorem

The component of K-moduli space $M_{3,24}^{\text{Kps}}$ associated to family MM_{4-1} is the blowup of $\mathbb{P}(1, 3, 4, 6)$ at the smooth point $\{[2 : 2 : 0 : 0]\}$ with weights $(1, 2, 3)$.

How to determine the weights?

- Direct study of 3-folds of 3-folds of the form (**)
- Gives rise to a second GIT quotient
- Show that the two GIT quotients “glue” to form the K-moduli space.