数理解析研究所講究録
第 1872 巻 2014 年 158-164

On quotients of Hom-functors

D.Tambara

Department of Mathematical Sciences, Hirosaki University

1. Introduction

A hom-functor on a category C is the functor $\mathrm{Hom}(-, X)$ for an object X of C. We consider the quotient functor $\mathrm{Hom}(-, X)/G$ by a subgroup G of $\mathrm{Aut}\,X$. We are interested in replacing hom-functors in the definitions of limit and adjoint by quotients of hom-functors.

2. limit

We recall the definition of limit in terms of hom-functor. **Set** denotes the category of sets. For a small category C, $[C^{\mathrm{op}}, \mathbf{Set}]$ denotes the category of contravariant functors $C \to \mathbf{Set}$. $[C^{\mathrm{op}}, \mathbf{Set}]$ has limits. For instance, the product $F \times G$ of F and G in $[C^{\mathrm{op}}, \mathbf{Set}]$ is given by

$$(F \times G)(A) = F(A) \times G(A) \quad \text{for } A \in C.$$

And the final object $\mathbf{1}$ of $[C^{\mathrm{op}}, \mathbf{Set}]$ is given by

$$\mathbf{1}(A) = \{1\} \quad \text{for } A \in C.$$

For $X \in C$, the hom-functor h_X is defined by

$$h_X(A) = \mathrm{Hom}(A, X).$$

A functor $F \colon C^{\mathrm{op}} \to \mathbf{Set}$ is said to be representable if $F \cong h_X$ for some X.

For $X_1, X_2, Z \in C$ we have

$$Z \text{ is a product of } X_1 \text{ and } X_2 \iff h_Z \cong h_{X_1} \times h_{X_2}.$$

Therefore

$$\begin{aligned} &\text{product of two objects exists in } C \\ \iff &\text{product of two representable functors is representable.} \end{aligned}$$

And similarly

$$\text{a final object exists in } C \iff \mathbf{1} \text{ is representable.}$$

The existence of a limit in C is thus expressed as the representability of a limit of hom-functors. We first aim to replace representability by familial representability.

3. Sum of hom-functors

A functor $F\colon C^{\mathrm{op}} \to \mathbf{Set}$ is said to be familially representable if

$$F \cong \coprod h_{X_i}$$

for some family X_i of objects in C ([Carboni and Johnstone]).

Theorem 1. Let C be a finite category. The following conditions are equivalent to each other.

(i) $h_X \times h_Y$ and $\mathbf{1}$ are familially representable $(\forall X, Y \in C)$.
(ii) Finite limits of hom-functors are familially representable.
(iii) Pushouts and coequalizers exist in C.
(iv) Finite connected limits exist in C.
Moreover these conditions imply that all morphisms of C are epimorphisms.

Remark. "(iii) $\implies$ (iv)" is generally true.

For the proof of the theorem we may follow the proof of the general representability theorem in [Freyd and Scedrov]. It simplifies owing to our finiteness assumption. We may also use the characterization of familially representable functors ([Leinster]).

An interest with such categories comes from an attempt to define general Burnside rings. Suppose that C satisfies (i) of Theorem 1. For any $X, Y \in C$ we take isomorphisms

$$h_X \times h_Y \cong \coprod h_{Z_i}$$

and

$$\mathbf{1} \cong \coprod h_{W_j}.$$

Then the free abelian group based on the isomorphism classes of objects of C becomes a ring by setting

$$[X][Y] = \sum [Z_i],$$
$$1 = \sum [W_j].$$

Here $[X]$ stands for the isomorphism class of an object X. This ring may be called the Burnside ring of C.

4. The Burnside ring of a finite category

Let C be a finite category. Assume that C satisfies the following conditions.
(B1) For every $X, Y \in C$ there exists a unique family of integers c_Z^{XY} such that

$$|\operatorname{Hom}(A, X)||\operatorname{Hom}(A, Y)| = \sum_Z c_Z^{XY} |\operatorname{Hom}(A, Z)| \quad (\forall A \in C).$$

(Here $|S|$ stands for the cardinality of a set S.)

(B2) There exists a unique family of integers d_Z such that

$$1 = \sum_Z d_Z |\operatorname{Hom}(A, Z)| \quad (\forall A \in C).$$

Then the free abelian group based on the isomorphism classes of objects of C becomes a ring:

$$[X][Y] = \sum_Z c_Z^{XY}[Z],$$
$$1 = \sum_Z d_Z[Z].$$

Theorem. ([Yoshida]) Assume that a finite category C satisfies the following conditions.

(Y1) C has the unique epi-mono factorization property.

(Y2) C has the coequalizer

$$\operatorname{Coeq}(X \underset{\alpha}{\overset{1}{\rightrightarrows}} X)$$

for any $\alpha \in \operatorname{Aut} X$.

Then C satisfies (B1) and (B2).

The following diagram shows the relationship between Theorem 1 and Yoshida's theorem:

$$\begin{array}{ccc}
 & & [X][Y] = \sum c_Z^{XY}[Z], \\
\text{pushout, coequalizer exist} & \Longrightarrow & 1 = \sum d_Z[Z], \\
 & & c_Z^{XY}, d_Z \in \mathbb{N} \\
\Downarrow & & \Downarrow \\
 & & [X][Y] = \sum c_Z^{XY}[Z], \\
\text{epi-mono factorization,} \ \operatorname{Coeq}(X \rightrightarrows X) \ \text{exist} & \Longrightarrow & 1 = \sum d_Z[Z], \\
 & & c_Z^{XY}, d_Z \in \mathbb{Z}
\end{array}$$

A problem will be to characterize categories satisfying (B1) and (B2).

Here are examples of generalized Burnside rings. Let G be a finite group.

(1) Let C be the category whose objects are G-sets G/H for all subgroups H, and whose morphisms are G-maps. Then C satisfies the condition of Theorem 1. The resulting ring is the ordinary Burnside ring of G.

(2) Let $\mathcal{F}$ be a family of subgroups of G which is closed under conjugation and intersection. Let C be the category whose objects are G-sets G/H for $H \in \mathcal{F}$. Then C satisfies the condition of Theorem 1.

(3) Let $\mathcal{F}$ be the set of all p-centric subgroups of G. Let C be the category whose objects are G-sets G/H for $H \in \mathcal{F}$. Then C satisfies the condition that $h_X \times h_Y$ are familially representable ([Diaz and Libman], [Oda]). Further examples of $\mathcal{F}$ are found in [Oda and Sawabe].

(4) For a fusion system $\mathcal{F}$ a certain category $\mathcal{O}(\mathcal{F}^c)$ is defined. Then $C = \mathcal{O}(\mathcal{F}^c)$ satisfies the condition that $h_X \times h_Y$ are familially representable ([Puig], [Diaz and Libman]).

5. Finiteness of connected components of powers of a functor

FinSet denotes the category of finite sets. Let K be a finite category. We say $G \in [K, \mathbf{FinSet}]$ is connected if G is nonempty and never expressed as a sum of nonempty objects. Every $F \in [K, \mathbf{FinSet}]$ is a sum of connected objects, each of which we call a connected component of F. For $F \in [K, \mathbf{FinSet}]$ and $n \geq 0$ we have

$$F^n = F \times \cdots \times F$$

in $[K, \mathbf{FinSet}]$.

Theorem 2. For $F \in [K, \mathbf{FinSet}]$, the following are equivalent.

(i) Connected components of F^n for all n have only finitely many isomorphism classes.

(ii) $F(\alpha)$ is injective for every morphism α of K.

This theorem relates to Theorem 1 as follows: Let $F\colon K \to \mathbf{FinSet}$ satisfy (ii) of Theorem 2. Let C be a representative system of isomorphism classes of connected components of F^n for all n. Then C is finite. View C as a category (a full subcategory of $[K, \mathbf{FinSet}]$). For $X, Y \in C$, $X \times Y$ is a sum of objects of C and $\mathbf{1}$ is a sum of objects of C. So C satisfies condition (i) of Theorem 1.

Conversely every finite category satisfying condition (i) of Theorem 1 arises this way.

6. Quotient of hom-functor

Let C be a category. Let X be an object of C and G a subgroup of $\operatorname{Aut} X$. We define the functor $h_X/G\colon C^{\mathrm{op}} \to \mathbf{Set}$ by

$$(h_X/G)(A) = \operatorname{Hom}(A, X)/G.$$

Here $\operatorname{Hom}(A, X)/G$ is the quotient set relative to the natural action of G on $\operatorname{Hom}(A, X)$.

Theorem 3. Let C be a finite category. The following conditions are equivalent to each other.

(i) $h_X \times h_Y$ and $\mathbf{1}$ are isomorphic to sums of quotients of hom-functors ($\forall X, Y$).

(ii) Finite limits of hom-functors are isomorphic to sums of quotients of hom-functors.

(iii) Pushouts exist in C.

(iv) Finite simply connected limits exist in C.

These conditions imply that all morphisms of C are epimorphisms.

Remark. "(iii) $\Longrightarrow$ (iv)" is true for a general C ([Paré]).

7. Category with pushouts

We here give an example of a category with pushouts.

Let P be a partially ordered set. Suppose that a group G acts on P:

$$\sigma \in G, x \in P \rightsquigarrow x^\sigma \in P.$$

The category PG is defined as follows.

(object) Objects of PG are elements of P.

(morphism) For $x, y \in P$

$$\mathrm{Hom}_{PG}(x, y) = \{\sigma \mid \sigma \in G, x \leq y^\sigma\}.$$

(composition) Composition is given by multiplication in G.

Proposition. If P has pushouts, then so does PG.

That P has pushouts means that if $z \leq x, z \leq y$, then there exists $\sup(x, y)$.

Suppose that for each $x \in P$ a subgroup K_x of G is given. Assume the following conditions hold.

(i) $\sigma \in K_x \Longrightarrow x^\sigma = x$

(ii) $x \leq y \Longrightarrow K_x \leq K_y$

(iii) ${K_x}^\sigma = K_{x^\sigma}$

We then define the category D as follows.

(object) Objects of D are elements of P.

(morphism) For $x, y \in P$ we set

$$\mathrm{Hom}_D(x, y) = \mathrm{Hom}_{PG}(x, y)/K_y.$$

Here K_y acts on $\mathrm{Hom}_{PG}(x, y)$ by multiplication in G.

(composition) The composition of D is induced by that of PG.

Proposition. If P has pushouts, then so does D.

8. Adjoint

We recall the definition of adjoint in terms of hom-functor. Let $F\colon B \to C$ and $G\colon C \to B$ be functors. "G is a right adjoint of F" means

$$\mathrm{Hom}_C(F(X), Y) \cong \mathrm{Hom}_B(X, G(Y))$$
$$\text{(naturally in } X,\ Y).$$

This isomorphism, X viewed a variable, is written as

$$\mathrm{Hom}_C(F(-), Y) \cong h_{G(Y)}$$
$$\text{(naturally in } Y).$$

$\mathrm{Hom}_C(F(-), Y) = h_Y \circ F$ denoted by $F^*(h_Y)$, this is written as

$$F^*(h_Y) \cong h_{G(Y)}.$$

Thus

$$\begin{aligned} &F \text{ has a right adjoint} \\ \iff &F^*(h_Y) \text{ are representable for all } Y \in C. \end{aligned}$$

We next aim to replace representability in the right-hand side by familial representability.

9. Discrete fibration

Recall that a functor $F\colon B \to C$ is called a discrete fibration if the following condition holds.

$$\begin{gathered} \forall\, g\colon F(X) \to Y' \quad \text{morphism of } C, \\ \exists!\, f\colon X \to X' \quad \text{morphism of } B, \\ F(f) = g. \end{gathered}$$

If $F\colon B \to C$ is a discrete fibration, then

$$F^*(h_Y) \cong \coprod_{X \in F^{-1}(Y)} h_X$$

for every $Y \in C$.

Proposition. Let $F\colon B \to C$ be a functor. The following are equivalent.

(i) $F^*(h_Y)$ are familially representable for all $Y \in C$.

(ii) There exists a factorization

$$\begin{array}{ccc} & & C' \\ & {\scriptstyle F'}\nearrow & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[F]{} & C \end{array}$$

such that F' has a right adjoint and π is a discrete fibration.

10. Condition (G)

Here we aim to replace representability in the definition of adjoint by being isomorphic to a sum of quotients of hom-functors.

Let $F\colon B \to C$ be a functor. We introduce the condition (G) for F. It consists of the following:

(i)

$$\begin{aligned} &g\colon F(X) \to Y' \\ &\implies \exists f\colon X \to X',\ F(f) = g. \end{aligned}$$

(ii)

$$f_1 \colon X \to X_1',\ f_2 \colon X \to X_2',\ F(f_1) = F(f_2)$$
$$\Longrightarrow \exists u \colon X_1' \to X_2',\ F(u) = 1,\ f_2 = uf_1.$$

If condition (G) holds, then $F^*(h_Y)$ is isomorphic to a sum of quotients of hom-functors for every $Y \in C$.

Theorem 4. Let $F \colon B \to C$ be a functor. Assume that C is finite. The following are equivalent.

(i) $F^*(h_Y)$ are isomorphic to sums of quotients of hom-functors for all $Y \in C$.

(ii) There exists a commutative diagram

$$\begin{tikzcd} B' \arrow[r, "F'"] \arrow[d, "\nu"'] & C' \arrow[d, "\pi"] \\ B \arrow[r, "F"'] & C \end{tikzcd}$$

such that F' has a right adjoint, ν is full and dense, and π satisfies condition (G).

References

[1] A.Carboni and P.Johnstone, Connected limits, familial representability and Artin glueing, Math.Struct.Comp.Science 5 (1995), 441–459.

[2] A.Diaz and A.Libman, The Burnside ring of fusion systems, Advances in Math. 222 (2009), 1943–1963.

[3] P.J.Freyd and A.Scedrov, "Categories, Allegories", North-Holland, Amsterdam, 1990.

[4] T.Leinster, The Euler characteristic of a category, Documenta Math. 13 (2008), 21–49.

[5] R.Paré, Simply connected limits, Can.J.Math. 42 (1990), 731–746.

[6] L.Puig, Frobenius categories, J.Algebra 303 (2006), 309–357.

[7] F.Oda, The generalized Burnside ring with respect to p-centric subgroups, J.Algebra 320 (2008), 3726–3732.

[8] F.Oda and M.Sawabe, A collection of subgroups for the generalized Burnside rings, Advances in Math. 222 (2009), 307–317.

[9] T.Yoshida, On the Burnside rings of finite groups and finite categories, Advanced Studies in Pure Mathematics 11 (1987), 337–353.