

# A Support Problem for Superprocesses in Terms of Random Measure

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## §1. Introduction

The purpose of this expository article is to investigate the support problem for a special class of superprocesses in terms of random measure. In the theory of measure-valued stochastic processes, compact support problems have been discussed for many years. For instance, in the case of typical super-Brownian motion  $X = \{X_t; t \geq 0\}$ , Iscoe (1988) proved that if the initial measure  $X_0(dx)$  has a compact support, then for every  $t > 0$ ,  $X_t$  possesses a compact support. Let  $\mathcal{B}_+ \equiv \mathcal{B}_+(\mathbb{R}^n)$  be the totality of nonnegative Borel measurable functions on  $\mathbb{R}^n$ , and let  $L \equiv L(dx)$  be a locally finite random measure on  $\mathbb{R}^n$ . For  $\mathcal{B}_+ \ni f$ , we define  $\langle f, L \rangle := \int f(x)L(dx)$ . Furthermore,  $M_F(\mathbb{R}^n)$  denotes the totality of finite Borel measures on  $\mathbb{R}^n$  equipped with weak convergence topology. We define a differential operator  $P$  by

$$P := \frac{1}{2} \sum_{k=1}^n a_k(x) \frac{\partial^2}{\partial x_k^2} + \sum_{k=1}^n b_k(x) \frac{\partial}{\partial x_k} + c(x)(\cdot) \quad (1)$$

where we assume that  $a_k, b_k, c \in C_b^\infty(\mathbb{R}^n)$  satisfy  $\exists \delta > 0 : a_j > \delta > 0$ . As a matter of fact, our target process  $X = (\{X_t, t \geq 0\}, P_\mu)$  in terms of measure  $L$  is an  $M_F(\mathbb{R}^n)$ -valued Markov process, and its Laplace transition functional is given by

$$\mathbb{E}_\mu[e^{-\langle \varphi, X_t \rangle}] = e^{-\langle u(t), \mu \rangle}. \quad (2)$$

Here the function  $u(t) \equiv u(t, x)$  satisfies

$$\begin{cases} \partial_t u = Pu - \dot{L}(dx)u^2 \\ u(t, x)|_{t=0+} = \varphi(x) \end{cases} \quad (3)$$

where the symbol  $\dot{L}(dx)$  means  $\frac{L(dx)}{dx}$ . For brevity's sake, in what follows we shall proceed the argument simply for  $d = 1$ . Our discussion on construction of superprocesses can be extended up to multi-dimensional case. However, the argument on the compact support problem for superprocesses is restricted to one-dimensional case.

## §2. Main result

For  $\mu \in M_F(\mathbb{R})$ , the support of  $\mu$ , say,  $\text{supp}(\mu)$  is defined by

$$\text{supp}(\mu) := \{A \in \mathcal{B}(\mathbb{R}) : \mu(A^c) = 0\}. \quad (4)$$

While, the global support of superprocess  $X(\cdot)$ , say,  $\text{Gsupp}(X)$  is defined by

$$\text{Gsupp}(X) := \bigcup_{t \geq 0} \text{supp}(X_t(dx)). \quad (5)$$

It is a key point that we relate the support  $\text{Gsupp}(X)$  of superprocess  $X_t$  in terms of locally finite measure  $L = L(dx)$  on  $\mathbb{R}$  to a nonlinear singular elliptic boundary problem.

Let  $d = 1, a(x) > 0$ . We consider the associated boundary problem: for a differential operator  $P = \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$ ,

$$\begin{cases} Pv = v^2(x)\frac{L(dx)}{dx}, & a_1 < x < a_2 \\ v(a_1) = \beta_1, & v(a_2) = \beta_2. \end{cases} \quad (6)$$

When we denote the solution of (6) by  $v(x; \beta_1, \beta_2)$ , since  $\exists \{\beta_1^{(n)}\}_n \nearrow \infty, \exists \{\beta_2^{(n)}\}_n \nearrow \infty$ , the problem (6) possesses a unique solution  $v(x; \beta_1^{(n)}, \beta_2^{(n)})$ . Note that the solution  $v(x)$  is a continuous convex function defined on the interval  $I = [a_1, a_2]$ . Moreover, for  $\forall a_1 \leq x_0 \leq x \leq a_2$ ,  $v(x)$  satisfies

$$\begin{aligned} v(x) = & v(x_0) + \Phi_0(x_0)(x - x_0) + \int_{x_0}^x \Phi_1(y)v(y)dy \\ & + \int_{x_0}^x dy \int_{x_0}^y \Phi_2(z)v(z)dz + \int_{x_0}^x dy \int_{x_0}^y \frac{2v^2(z)}{a(z)}L(dz), \end{aligned} \quad (7)$$

where

$$\begin{aligned} \Phi_0(x) &= v'(x+) + \frac{2b(x)}{a(x)}, & \Phi_1(x) &= \frac{2b(x)}{a(x)}, \\ \Phi_2(x) &= \frac{2b(x)a'(x) - 2b'(x)a(x) + 2a(x)c(x)}{a(x)^2}. \end{aligned}$$

Then we can obtain an explicit expression of the approximate solution. For  $\psi \in C^+(\mathbb{R})$ ,  $\text{supp}(\psi) \subset (-K, K)$ ,  $\theta > 0$ , when we denote by  $v_K(t, x; \theta\psi)$  the solution of

$$u(t, x) = 0, \quad x \in (-K, K)^c$$

$$\begin{aligned} u(t, x) = & \theta \int_0^t \int_{-K}^K p_K(t-s, x, y)\psi(y)dyds \\ & - \int_0^t \int_{-K}^K p_K(t-s, x, y)u^2(s, y)L(dy)ds, \quad x \in (-K, K), \end{aligned} \quad (8)$$

then a simple fact  $v_K \geq 0$  yields concurrently to  $v_K \nearrow$  in  $t \nearrow \infty$   $v_K \nearrow$  in  $\psi$ , and furthermore it follows immediately that

$$v_K(t, x; \theta\psi) \leq \sup_{t,x} \int_0^t \int_{-K}^K p_K(t-s, x, y) \theta\psi(y) dy ds < \infty.$$

On the other hand,  $v_K(\theta, t, x; a_1, a_2)$  denote the solution of (8) with the test function replaced by  $\psi = 1_{[a_1, a_2]^c}$ .

For simplicity, we assume henceforth that  $\text{supp}(X_0) \subset [a_1, a_2] \subset (-K, K)$ ,  $b(x) = 0, c(x) > 0$ . We shall represent the positive support probability of superprocess  $X_t$  by the solution of (6). The argument of Iscoe (1988) for occupation time processes  $\int_0^t X_s^K ds$  or  $\int_0^t X_s ds$  implies that

$$\begin{aligned} E_{X_0}^L[\exp \left\{ -\theta \int_0^\infty X_s^K([a_1, a_2]^c) ds \right\}] \\ = \exp \left\{ - \int_{-\infty}^\infty v_K(\theta, x; a_1, a_2) X_0(dx) \right\} \end{aligned} \quad (9)$$

holds. And besides we have

$$v_K(\theta, x; a_1, a_2) = \lim_{t \rightarrow \infty} (\lim_{n \rightarrow \infty} v_K(\theta\psi_n; t, x)),$$

and we can deduce that  $v(x) \equiv v_K(\theta, x; a_1, a_2)$  satisfies that its second derivative  $v''$  is a signed measure, and also that for  $x \in (-K, K)$ ,

$$\begin{aligned} \frac{dv}{dx}(x \pm) &= \int_{x_0}^{x \pm} \frac{2c(y)v(y)}{a(y)} dy + \int_{x_0}^{x \pm} \frac{2v^2(y)}{a(y)} L(dy) \\ &\quad - 2\theta \int_{x_0}^{x \pm} 1_{[a_1, a_2]^c}(y) dy + (\text{Constant}). \end{aligned}$$

Thus the representation of probability for the support can be derived.

$$\begin{aligned} &P_{x_0}^L(\text{supp}(X_t) \cap [a_1, a_2]^c = \emptyset, \quad \forall t \geq 0) \\ &= \lim_{K \rightarrow \infty} P_{X_0}^L(\text{supp}(X_t^K) \cap [a_1, a_2]^c = \emptyset, \forall t \geq 0) \\ &\Leftarrow \text{by virtue of the right continuity of the path } X_t^K(\omega) \\ &= \lim_{K \rightarrow \infty} P_{X_0}^L \left( \int_0^\infty X_s^K([a_1, a_2]^c) ds = 0 \right) \\ &\Leftarrow \text{by the expression of the occupation time process (9)} \\ &= \lim_{K \rightarrow \infty} \lim_{\theta \rightarrow \infty} \exp \left\{ - \int_{-\infty}^\infty v_K(\theta, x; a_1, a_2) X_0(dx) \right\} \\ &= \lim_{n \rightarrow \infty} \exp \left\{ - \int_{a_1}^{a_2} v(x; \beta_1^{(n)}, \beta_2^{(n)}) X_0(dx) \right\} \end{aligned} \quad (10)$$

By virtue of the above-mentioned facts we can get the following principal result, the theorem for compact support.

**Theorem 1. (Main Result)** *Let  $\mu \in M_F(\mathbb{R})$  and  $\text{supp}(\mu) \subset [a_1, a_2]$ . Suppose that  $d = 1, a(x) > 0, b(x) = 0, c(x) > 0$ . For  $\forall \varepsilon > 0$  ( $\varepsilon \ll 1$  : sufficiently small), there exist proper real numbers  $\exists \underline{x} = \underline{x}(\varepsilon) < a_1, \exists \bar{x} = \bar{x}(\varepsilon) > a_2$  such that  $v$  is a nonnegative solution of (7) on the interval  $(\underline{x}, \bar{x})$ , i.e.  $v(x) \geq 0$  for  $x \in (\underline{x}, \bar{x})$ . If  $v$  satisfies the conditions*

$$\sup_{a_1 \leq x \leq a_2} v(x) \leq \varepsilon, \quad \lim_{x \rightarrow \underline{x}} v(x) = \lim_{x \rightarrow \bar{x}} v(x) = \infty, \quad (11)$$

*then the superprocess  $X = \{X_t, t \geq 0\}$  has the compact support.*

### §3. Formulation of superprocess by admissible functional

Let us denote by  $X = \{X_t, t \geq 0\}$  the measure-valued branching process corresponding to a locally finite random measure  $L$ , and  $P_\mu^L$  denotes the probability law of the measure-valued process  $X$ . Then a measure-valued process  $(X_t, P_\mu^L)$  in terms of random measure  $L$  is given by the following Laplace transition functional.

$$E_\mu^L[e^{-\langle \varphi, X_t \rangle}] = e^{-\langle u(t), \mu \rangle} \quad \text{with} \quad X_0 = \mu \in M_F(\mathbb{R}). \quad (12)$$

Here the function  $u(t, x)$  satisfies the following Cauchy problem.

$$\begin{cases} \partial_t u = Pu - \frac{L(dx)}{dx} u^2, \\ u(0, x) = \varphi \in C_b^+(\mathbb{R}). \end{cases} \quad (13)$$

Now, suggested by a formulation by Dawson-Fleischmann (1995), we shall consider the above initial value problem as an integral equation. As a matter of fact, when we write the fundamental solution to the aforementioned Cauchy problem by  $p$ , then we have

$$u(t, x) = \int p(t, x, y) \varphi(y) dy - \int_0^t \int p(t-s, x, y) u^2(s, y) L(dy) ds. \quad (14)$$

This means that we consider the mild solution to the above Cauchy problem. We shall assume henceforth:

[Assumption] For any  $c > 0$ ,

$$\int_{-\infty}^{\infty} e^{-cx^2} L(dx) < \infty, \quad \text{a.s.} \quad (15)$$

Recall a method to apply admissible Brownian functional in the studies on superprocesses by E.B. Dynkin (1994). Roughly speaking, it is nothing but a special

case that the branching rate term  $\gamma$  in the super-Brownian motion or the Dawson-Watanabe superprocess would be changed into a general additive functional which does not always possess its density. For a finite measure  $\tilde{L}$  on  $\mathbb{R}$  and a local time  $\ell_{t,x}(\omega)$  of Brownian motion  $B_s$ , we define the additive functional  $K_t^{[\tilde{L}]}(\omega)$  by

$$K_t^{[\tilde{L}]}(\omega) := \int \ell_{t,x}(\omega) \tilde{L}(dx). \quad (16)$$

We shall impose the following admissible conditions.

[Dynkin's Admissibility] For a Brownian motion  $(B_t, \Pi_{0,x})$ ,

- (i)  $\Pi_{r,x}[K^{[\tilde{L}]}(r, t)] < \infty$ , for  $\forall r < t, x$
- (ii)  $\Pi_{r,x}[K^{[\tilde{L}]}(r, t)] \rightarrow 0$  uniformly in  $x$  ( $r, t \rightarrow s$ )  $\forall s$

**Theorem 2.** (Dynkin, 1994) *If the transition function  $\mathcal{P}(r, \mu; t, C) = P_{r,\mu}(X_t \in C)$  satisfied the following two conditions, then the measure-valued Markov process named  $(\xi, K, \psi)$ -superprocess with parameters  $X = (X_t, P_{r,\mu})$  can be determined.*

$$\int \mathcal{P}(r, \mu; t, d\nu) e^{-\langle f, \nu \rangle} = \exp\{-\langle v(r), \mu \rangle\}, \quad (17)$$

$$v(r, x) + \Pi_{r,x} \int_r^t \psi(s, v(s))(\xi_s) dK_s = \Pi_{r,x} f(\xi_t). \quad (18)$$

#### §4. Construction of sequence of approximate measure-valued processes

In this section we shall construct a basic process as a limit of increasing sequence of finite measure  $M_F(\mathbb{R})$ -valued processes realized on the common basic probability space. This provides us with a proto-type in the construction of our target superprocess. For each  $K \in \mathbb{N}$ , we put

$$E_K := \bigcup_{n=1}^K \{n\} \times (-n, n), \quad (19)$$

and we denote by  $\tilde{X}_t^K \equiv \tilde{X}_t^K(dx)$  an  $M_F(E_K)$ -valued process. We shall first of all construct this measure-valued basic process  $\tilde{X}_t^K$  in what follows. For  $x \in (-n, n)$ , a Markov process  $w_K$  on  $E_K$  starting at a point  $(n, x)$  can be defined as

$$\begin{aligned} w_K(t) &:= (\{n\}, w(t)), & \text{for } 1 \leq t \leq \tau_n \\ w_K(\tau_n) &:= (\{n+1\}, w(\tau_n)), & \tau_n = \inf\{t > 0 : w(t) = \pm n\} \end{aligned}$$

where  $w$  is a  $P$ -diffusion starting at a point  $x$ . Notice that the stochastic process  $w_K$  dies out finally at time  $\tau_K$ . Next we consider a random measure  $L_K$ . In fact, we define

$$L_K(\{n\} \times (a, b)) := L((-n, n) \cap (a, b)), \quad \text{for } n \leq K.$$

On this account, we can define the admissible additive functional  $\mathcal{K}_t^{[L_K]}(w_K)$  by making use of this random measure  $L_K$ , i.e.

$$\mathcal{K}_t^{[L_K]}(w_K) := \int \tilde{\ell}_{t,y}(w_K) L_K(dy) \quad (20)$$

where  $\tilde{\ell}_{t,x}$  is a positive random variable given by

$$\tilde{\ell}_{t,x}(w) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(a-\varepsilon, a+\varepsilon)}(w(s)) ds. \quad (21)$$

Then an application of the previous Dynkin's existence theorem (Theorem 2) with this admissible additive functional  $\mathcal{K}_t^{[L_K]}$  gives us a superprocess, which we denote by  $\tilde{X}_t^K = \tilde{X}_t^K(dx)$ . That is to say,

$$E_{r,x}^{(L_K)} e^{-\langle \varphi, \tilde{X}_t^K \rangle} = \exp\{-\langle v(r), \mu \rangle\}, \quad (22)$$

$$v(r, x) + \tilde{\Pi}_{r,x}^P \int_r^t v(s, w_K(s))^2 d\mathcal{K}_t^{[L_K]} = \tilde{\Pi}_{r,x}^P \varphi(w_K(t)). \quad (23)$$

Next we shall construct a new approximate sequence of branching measure-valued processes by employing the above-mentioned process, and shall give its characterization. Before constructing the superprocess in question, we consider first the initial measure as its initial value. We choose a finite measure  $\mu \in M_F(\mathbb{R})$  as a candidate of the initial measure for our measure-valued process  $\tilde{X}_t^K$ . For  $n \geq 1$ , for each subset  $B \subset \mathbb{R}$  we define

$$\tilde{X}_0^K(\{n\} \times B) := \mu(B \cap \{[n-1, n) \cup (-n, -n+1]\}). \quad (24)$$

Then, if it is the case of the number  $M \in \mathbb{N}$  satisfying  $M > K$ , the law of the process  $\tilde{X}_t^M$  restricted to a set  $E_K = \cup_{n=1}^K \{n\} \times (-n, n)$  is equivalent to the law of the process  $\tilde{X}_t^K$ . In other words,

$$\mathcal{L}(\tilde{X}_t^M \upharpoonright E_K) = \mathcal{L}(\tilde{X}_t^K), \quad \text{for } \forall M > K.$$

Let us now denote by  $P_{X_0}^{L,K}$  the probability law of the measure-valued process  $\tilde{X}^K$ , and we put  $E_\infty := \bigcup_{n=1}^\infty \{n\} \times (-n, n)$  and  $\tilde{X}^\infty$  denotes an  $M(E_\infty)$ -valued process.

Then note that since the law  $\{P_{X_0}^{L,K}\}_K$  of  $\tilde{X}^K$  becomes a consistent family, its projective limit induces the law of  $M(E_\infty)$ -valued process  $\tilde{X}^\infty$ . Hence, if we define a new  $M_F((-K, K))$ -valued process  $X_t^K$  as

$$X_t^K(B) := \sum_{n=1}^K \tilde{X}_t^\infty(\{n\} \times B), \quad (25)$$

then an increasing sequence of stochastic processes  $\{X_t^K(B)\}_K \nearrow$  is obtained.

**Proposition 3.** (Characterization) *Let  $u_K(t, x)$  be a log-Laplace function of  $X_t^K$ . Then  $X_t^K$  satisfies the following*

$$E_{X_0^K}[e^{-\langle \varphi, X_t^K \rangle}] = e^{-\langle u_K(t, \mu), \mu \rangle}, \quad \text{with } X_0^K(dx) = \mu(dx). \quad (26)$$

Moreover, the function  $u_K(t, x)$  satisfies uniquely the following integral equation: for  $\varphi \in C_0(\mathbb{R})$ ,

$$\begin{aligned} u_K(t, x) = & \int_{-K}^K p_K(t, x, y) \varphi(y) dy \\ & - \int_0^t \int_{-K}^K p_K(t-s, x, y) u_K^2(s, y) L(dy) ds, \end{aligned} \quad (27)$$

$$E[X_t^K(B)] = \int_{-K}^K \int_B p_K(t, x, y) \mu(dx) dy, \quad (28)$$

where  $p_K(t, x, y)$  is the fundamental solution of the Dirichlet boundary value problem:

$$\partial_t u - Pu = 0, \quad u|_{\partial(-K, K)} = 0 \quad (29)$$

## §5. Existence of superprocess in terms of finite measure

Therefore  $M_F(\mathbb{R})$ -valued process  $X = \{X_t, t \geq 0\}$  with the initial measure  $\mu \in M_F(\mathbb{R})$  can be defined by the following limit

$$X_t(dx) := \lim_{K \rightarrow \infty} X_t^K(dx). \quad (30)$$

We call this stochastic process  $X_t$  a superprocess in terms of random measure  $L$  which represents a random media. Next we shall extend  $p_K(t, \cdot, \cdot)$  onto  $\mathbb{R} \times \mathbb{R}$ . Namely,

$$p_K(t, x, y) = 0 \quad \text{if } x \text{ or } y \notin (-K, K).$$

Then, since  $p_K(t, \cdot, \cdot) \nearrow p(t, \cdot, \cdot)$ , we may apply the monotone convergence theorem to obtain

$$E[X_t(B)] = \int_{-\infty}^{\infty} \int_B p(t, x, y) \mu(dx) dy \quad \forall B \in \mathcal{B}(\mathbb{R}). \quad (31)$$

On the other hand, since we have  $\{X_t^K(\cdot)\}_K \nearrow$  in  $K$ , the sequence of log-Laplace functions  $\{u_K(t, \cdot)\}_K$  associated with the sequence of those measure-valued processes is also increasing  $\nearrow$  in  $K$ . As a consequence, by using the monotone convergence theorem again, the log-Laplace function  $u(t, x)$  of the above-mentioned limit process  $X_t(dx)$  can also be obtained by

$$u(t, x) = \lim_{K \rightarrow \infty} u_K(t, x). \quad (32)$$

Finally, an application of the monotone convergence theorem again leads to the following :

$$\begin{aligned}
 u(t, x) &= \lim_{K \rightarrow \infty} u_K(t, x) \\
 &= \lim_{K \rightarrow \infty} \int_{-K}^K p_K(t, x, y) \varphi(y) dy \\
 &\quad - \lim_{K \rightarrow \infty} \int_0^t \int_{-K}^K p_K(t-s, x, y) u_K^2(s, y) L(dy) ds \\
 &= \int_{-\infty}^{\infty} p(t, x, y) \varphi(y) dy - \int_0^t \int_{-\infty}^{\infty} p(t-s, x, y) u^2(s, y) L(dy) ds. \quad (33)
 \end{aligned}$$

*Remark.* It is interesting to note that the above construction requires us only local finiteness of the random measure  $L(dx)$ .

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## References.

- [1] Dawson, D. and Fleischmann, K. : Super-Brownian motions in higher dimensions with absolutely continuous measure states. *J. Theoret. Probab.* **8** (1995), 179–206.
- [2] Dôku, I. : A certain class of immigration superprocesses and its limit theorem. *Adv. Appl. Stat.* **6** (2006), 145–205.
- [3] Dôku, I. : A limit theorem of superprocesses with non-vanishing deterministic immigration. *Sci. Math. Japn.* **64** (2006), 563–579.
- [4] Dôku, I. : A limit theorem of homogeneous superprocesses with spatially dependent parameters. *Far East J. Math. Sci.* **38** (2010), 1–38.
- [5] Dôku, I. : Tumour immunoreaction and environment-dependent models. *Trans. Japn. Soc. Indu. Appl. Math.* **26** (2016), 213–252.
- [6] Dynkin, E. B. : *An Introduction to Branching Measure-Valued Processes*. Amer. Math. Soc., Providence, RI, 1994.
- [7] Iscoe, I. : On the supports of measure-valued critical branching Brownian motion. *Ann. Probab.* **16** (1988), 200–221.

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