

# Bijectivity of straightenings for families of renormalizable cubic polynomials

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## Abstract

We give a description of combinatorics of renormalizable polynomials in terms of rational lamination, and study the set  $\mathcal{R}_d(f_0)$  of renormalizable polynomials of degree  $d$  with a given center  $f_0$ , and prove that if  $f_0$  is non-contiguous, then  $\mathcal{R}_d(f_0)$  is compact.

Moreover, in the case of capture (or disjoint) renormalizations of cubic polynomials, if  $\mathcal{R}_3(f_0)$  is compact, then the straightening map on it is bijective (or homeomorphism). We also prove that we can always characterize the combinatorics of a renormalizable cubic polynomial of capture (or disjoint) type in such a way.

## 1 Introduction

### 1.1 The case of quadratic polynomials

It is well-known that the Mandelbrot set  $\mathcal{M} = \{c \in \mathbb{C}; J(z^2 + c) \text{ is connected}\}$  contains infinitely many homeomorphic copies of itself, called *baby Mandelbrot sets*, proved by Douady and Hubbard [Ha]. A baby Mandelbrot set corresponds to a set of renormalizable parameters of given combinatorics. More precisely, consider a quadratic polynomial  $z^2 + c_0$  such that the critical point 0 is periodic. Then there exists a homeomorphism  $T_{c_0} : \mathcal{M} \rightarrow \mathcal{M}(c_0) \subset \mathcal{M}$  such that  $T_{c_0}(0) = c_0$  and for  $c' = T_{c_0}(c)$ ,  $z^2 + c'$  has a renormalization hybrid equivalent to  $z^2 + c$  except when  $c = \frac{1}{4}$  and  $\mathcal{M}(c_0)$  is of *satellite type*, that is,  $T_{c_0}(c)$  is on the boundary of some hyperbolic component other than the one containing  $c_0$ .

**Definition.** Let  $z^2 + c_0$ ,  $\mathcal{M}(c_0)$  and  $T_{c_0}$  be as above. We call  $c_0$  the *center* and  $T_{c_0}(\frac{1}{4})$  the *root* of  $\mathcal{M}(c_0)$ . A homeomorphism  $T_{c_0}$  is the *tuning map* and its inverse  $S_{c_0} = T_{c_0}^{-1}$  is the *straightening map* for  $\mathcal{M}(c_0)$ .

If  $\mathcal{M}(c_0)$  is not of satellite type, then it is of *primitive type*.

Here we consider another description of a baby Mandelbrot set, which is based on the notion of rational laminations introduced by Thurston [Th] (see also [Ki]).

**Definition.** Let  $f$  be a polynomial of degree  $d \geq 2$  such that  $J(f)$  is connected. Consider an equivalence relation  $\lambda_f$  on  $\mathbb{Q}/\mathbb{Z}$  such that  $\theta \sim_{\lambda_{c_0}} \theta'$  if and only if the external rays  $R_f(\theta)$  and  $R_f(\theta')$  for  $f$  land at the same point. It is called the *rational lamination* of  $f$ .

We can characterize  $\mathcal{M}(c_0)$  combinatorially as follows:

$$\mathcal{M}(c_0) = \{c \in \mathbb{C}; \lambda_{z^2+c} \supset \lambda_{z^2+c_0}\}.$$

Here rational laminations are considered as subsets of  $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$ . In other words,  $\lambda \subset \lambda'$  if  $\lambda'$  is stronger relation than  $\lambda$ ;

$$\theta \sim_{\lambda} \theta' \Rightarrow \theta \sim'_{\lambda} \theta'.$$

Let  $s$  be the period of 0 for  $z^2 + c_0$ . Define

$$\mathcal{R}(c_0) = \{c \in \mathcal{M}(c_0); z^2 + c \text{ has a renormalization of period } s\}.$$

Then

$$\mathcal{M}(c_0) \setminus \mathcal{R}(c_0) = \begin{cases} \{\text{root}\} & \text{for satellite case,} \\ \emptyset & \text{for primitive case.} \end{cases}$$

In particular,  $\mathcal{M}(c_0) = \mathcal{R}(c_0)$  if and only if it is of primitive case and it is equivalent to  $\mathcal{R}(c_0)$  is compact.

## 1.2 The case of cubic polynomials

For  $d \geq 2$ , let  $\text{Poly}_d$  be the family of monic centered polynomials of degree  $d$ . Let  $\mathcal{C}_d = \{f \in \text{Poly}_d; J(f) \text{ is connected}\}$  be the *connectedness locus* of  $\text{Poly}_d$ . We notice that  $\text{Poly}_d$  can be considered as the parameter space of affine conjugacy class of polynomial of degree  $d$  with marked invariant external ray (which determines that of “angle 0”). In this paper, we always consider an polynomials (or a polynomial-like map) has marked invariant external ray.

Now we consider the case  $d = 3$ . Milnor [Mi] classified hyperbolic cubic polynomials into four cases:

- Adjacent case** One periodic Fatou component contains two critical points (with multiplicity).
- Bitransitive case** One periodic Fatou component  $U$  contains a critical point and its forward image  $f^n(U) (\neq U)$  contains the other.
- Capture case** One periodic Fatou component  $U$  contains a critical point and the other critical point is contained in a strictly preperiodic Fatou component  $U'$ , hence  $f^N(U') = U$  for some  $N \geq 1$ .

**Disjoint case** There exist two disjoint periodic cycle of Fatou components. Each contains one critical point.

We apply this classification also to renormalizations of cubic polynomials.

**Definition.** For  $f \in \mathcal{C}_3$ , a *renormalization* of  $f$  is a polynomial-like restriction  $f^s : U' \rightarrow U$  of some iterate of  $f$  with connected filled Julia set  $K(f^s; U', U) = \bigcap_{n \geq 0} (f^s|_{U'})^{-n}(U')$  such that every critical point  $\omega$  of  $f$  is contained in at most one of  $U', f(U'), \dots, f^{s-1}(U')$ . We identify renormalizations of  $f$  if they have the same filled Julia set.

**Adjacent case** We say  $f$  has an *adjacent renormalization* if  $f$  has a renormalization of degree 3.

**Bitransitive case** We say  $f$  has an *bitransitive renormalization* if  $f$  has a renormalization of degree 4.

**Capture case** We say  $f$  has an *capture renormalization* if  $f$  has a renormalization  $f^s : U' \rightarrow U$  of degree two such that a critical point  $\omega_0 \in U'$  and the other critical point  $\omega$  (called the *captured critical point*) satisfied that  $f^N(\omega) \in K(f^s; U', U)$  for some  $N > 0$ . The smallest such  $N$  is called the *capture time*.

**Disjoint case** We say  $f$  has *disjoint renormalizations* if  $f$  has two renormalizations  $f^{s_i} : U'_i \rightarrow U_i$  of degree two such that critical points  $\omega_1, \omega_2$  of  $f$  satisfies  $\omega_i \in U'_i$ .

When  $d \geq 4$ , we can similarly classify renormalizable polynomials whose renormalizations contain or capture all the critical points.

As in the case of quadratic polynomials, we would like to consider combinatorial characterization of renormalizations.

**Definition.** We say  $f_0 \in \mathcal{C}_d$  is a *center* if  $f_0$  is hyperbolic and postcritically finite (every critical point is eventually periodic).

Note that even if a critical point of a center  $f_0$  is preperiodic, its periodic orbit is superattracting.

For a center  $f_0 \in \mathcal{C}_d$ , define

$$\begin{aligned} \mathcal{C}_d(f_0) &= \{f \in \mathcal{C}_d; \lambda_f \supset \lambda_{f_0}\}, \\ \mathcal{R}_d(f_0) &= \left\{ f \in \mathcal{C}_d(f_0); \begin{array}{l} f \text{ has a renormalization (renormalizations)} \\ \text{of the same type and period(s) as } f_0 \end{array} \right\} \end{aligned}$$

By definition,  $\mathcal{C}_2(z^2 + c_0) = \mathcal{M}(c_0)$  and  $\mathcal{R}_2(z^2 + c_0) = \mathcal{R}(c_0)$  when  $d = 2$ .

**Definition.** Let  $f_0 \in \mathcal{C}_d$  be a center. We say  $f_0$  is *contiguous* if there exist Fatou components  $U_1, U_2$  such that  $\overline{U_1}$  intersects  $\overline{U_2}$ .

It is equivalent that there exist *periodic* Fatou components  $U_1, U_2$  such that  $\overline{U_1}$  intersects  $\overline{U_2}$ .

The first theorem is valid for any degree:

**Main Theorem 1.** *Let  $f_0 \in \mathcal{C}_d$  be a center. If  $f_0$  is contiguous, then  $\mathcal{R}_d(f_0) = \mathcal{C}_d(f_0)$*

and it is compact.

This theorem is a generalization of the results in [In2].

*Remark 1.* When  $d \geq 3$ ,  $\mathcal{C}_d(f_0)$  is not compact in general. In fact, let  $f_0(z) = z^3 - \frac{3}{2}z$ . The critical points of  $f_0$  are  $\omega_{\pm} = \pm \frac{1}{\sqrt{2}}i$  and  $f_0(\omega_{\pm}) = -2\omega_{\pm}^3 = \omega_{\pm}$ . Therefore,  $f_0$  has two invariant Fatou components  $U_{\pm}$ . It is easy to see that  $\overline{U_+} \cap \overline{U_-} = \{0\}$ . Since  $f_0$  is real, the external rays of angle 0 and  $\frac{1}{2}$  land at 0. In other words, 0 and  $\frac{1}{2}$  are  $\lambda_{f_0}$ -equivalent.

Now consider  $f(z) = z^3 + \frac{11}{12}z + \frac{1}{108}$ . It is affinely conjugate to  $g(z) = z^3 + \frac{1}{2}z^2 + z$ . It is easy to see that  $f$  and  $g$  are real and  $g$  (hence  $f$ ) has a real parabolic fixed point of multiplier one and a real repelling fixed point. Therefore,  $f \notin \mathcal{C}_3(f_0)$ . However, for small  $\varepsilon > 0$ ,  $f + \varepsilon \in \mathcal{C}_3(f_0)$ . Indeed, since  $f$  is parabolic-attracting (following Adam Epstein),  $f + \varepsilon$  must have attracting fixed point. However,  $f + \varepsilon$  has only one real fixed point, which is repelling. Hence the other fixed points  $x$  and  $\bar{x}$  are both attracting. This implies that their attractive basins both contain the real fixed point. Thus  $\lambda_{f+\varepsilon} = \lambda_{f_0}$  and  $f + \varepsilon \in \mathcal{C}_3(f_0)$ .

Therefore,  $\mathcal{C}_3(f_0) \ni f + \varepsilon \rightarrow f \notin \mathcal{C}_3(f_0)$  and  $\mathcal{C}_3(f_0)$  is not compact.

In the following, we assume  $d = 3$  and consider only quadratic renormalizations (i.e., capture or disjoint renormalizations).

**Main Theorem 2.** *Consider a capture renormalization (or disjoint renormalizations) of  $f \in \mathcal{C}_3$ . Then there exists a center  $f_0$  such that  $f \in \mathcal{R}_d(f_0)$  and the given renormalization(s) of  $f$  are the renormalization(s) in the definition of  $\mathcal{R}_d(f_0)$ .*

The Julia set of  $f$  can be non-locally connected (such as the case  $f$  has a Cremer periodic point). However, we can still find enough landing relations which characterize the renormalization(s).

Now we define the straightening map, which is based on the straightening theorem by Douady and Hubbard [DH2].

**Theorem 1.1** (Straightening theorem). *Every polynomial-like map  $f : U' \rightarrow U$  of degree  $d \geq 2$  is hybrid equivalent to a polynomial  $g$  of the same degree.*

*Furthermore, if  $K(f; U', U)$  is connected, then  $g$  is unique up to affine conjugacy.*

We say a polynomial-like map  $f : U' \rightarrow U$  is hybrid equivalent to another polynomial-like map  $g : V' \rightarrow V$  (or polynomial  $g$ ) if there exists a quasiconformal map  $\psi$  defined near  $K(f; U', U)$  such that  $\bar{\partial}\psi = 0$  a.e. on  $K(f; U', U)$  and  $g \circ \psi = \psi \circ f$ . Such  $\psi$  is called a *hybrid conjugacy* between  $f$  and  $g$ .

As noticed before, we always consider polynomials and polynomial-like maps with marked invariant external rays. Hence if  $K(f; U', U)$  is connected,  $g \in \text{Poly}_d$  is uniquely determined. Furthermore, although  $\psi$  is not uniquely determined,  $\psi|_{K(f; U', U)}$  is uniquely determined.

**Definition.** Let  $f_0 \in \mathcal{C}_3$  be a center of capture or disjoint type. Define the *straightening map*  $S_{f_0}$  as follows:

**Capture case** For  $f \in \mathcal{R}_3(f_0)$ , let  $f^s : U' \rightarrow U$  be the corresponding renormalization and  $\omega$  be the captured critical point and  $N$  be the capture time. By the straightening theorem,  $f^s : U' \rightarrow U$  is hybrid equivalent to a unique quadratic polynomial of the form  $g(z) = z^2 + c$  by a hybrid conjugacy  $\psi$ . Denote  $z = \psi(f^N(\omega))$  (it is well-defined).

Define  $S_{f_0}(f) = (c, z)$ . Then  $S_{f_0}$  is defined as a map  $S_{f_0} : \mathcal{R}_3(f_0) \rightarrow \mathcal{MK} = \bigcup_{c \in \mathcal{M}} \{c\} \times K(z^2 + c)$ .

**Disjoint case** For  $f \in \mathcal{R}_3(f_0)$ , let  $f^{s_i} : U'_i \rightarrow U_i$  ( $i = 1, 2$ ) be the corresponding renormalizations. (We fix the order of renormalizations.) By the straightening theorem, there exists a polynomial  $z^2 + c_i$  hybrid equivalent to  $f^{s_i} : U'_i \rightarrow U_i$ . Define  $S_{f_0}(f) = (c_1, c_2) : \mathcal{R}_3(f_0) \rightarrow \mathcal{M}^2$ .

**Remark 2.** By the upper semi-continuity of  $K(z^2 + c)$  on  $c$ ,  $\mathcal{MK}$  is a compact set in  $\mathbb{C}^2$ .

**Remark 3.** For a quadratic-like family  $(f_\lambda : U'_\lambda \rightarrow U_\lambda)_{\lambda \in \Lambda}$ , we can take a straightening map  $S : \Lambda \rightarrow \text{Poly}_2$  ( $f_\lambda \sim_{hb} S(f_\lambda)$ ) to be continuous [DH2].

Hence if  $f_0 \in \mathcal{C}_3$  is of capture type,  $f \mapsto c$  is continuous. It is not known whether  $f \mapsto z$  is continuous or not.

If  $f_0$  is of disjoint type,  $S_{f_0}$  is continuous.

**Main Theorem 3.** *Let  $f_0 \in \mathcal{C}_3$  be a center of capture or disjoint type. If  $\mathcal{R}_d(f_0)$  is compact, then the straightening map  $S_{f_0}$  is bijective.*

*In particular, if  $f_0$  is of disjoint type, then  $S_{f_0} : \mathcal{R}_d(f_0) \rightarrow \mathcal{M}^2$  is a homeomorphism.*

**Remark 4.** Our proofs of Main Theorem 2 and Main Theorem 3 depends on the continuous dependence of straightening maps of quadratic-like families (Remark 3). Hence our proofs can be generalized only to quadratic renormalizations (of higher degree polynomials), and we cannot apply them to renormalizations of higher degree.

## 2 Proof of Main Theorem 1

The proof of Main Theorem 1 depends on [In2, Theorem 1.1], which is the following in our context.

**Theorem 2.1.** *Let  $f_0 \in \mathcal{C}_d$  be a center and  $f_l \rightarrow f$  be a convergent sequence with  $f_l \in \mathcal{C}_d(f_0)$ .*

*If there exist more than  $d - 1$  disjoint sets  $O_k$  ( $k = 1, \dots, N \geq d - 1$ ) which consist of periodic orbits and separate periodic Fatou components of  $f_0$ , then  $f \in \mathcal{C}_d(f_0)$ .*

Here, we say a union  $O$  of (repelling or parabolic) periodic orbits of  $f_0$  separates periodic Fatou components if there exists  $n \geq 0$  such that different periodic Fatou components lie in different components of

$$\mathbb{C} \setminus \overline{\bigcup_{\theta \in \Theta} R_{f_0}(\theta)}$$

where  $\Theta$  be the set of landing angles of  $f^{-n}(O)$ .

Hence it is sufficient to show the following two lemmas to prove Main Theorem 1:

**Lemma 2.2.** *If  $f_0 \in \mathcal{C}_d$  is a center of non-contiguous type, then there exist infinitely many sets consisting of periodic orbits which separate periodic Fatou components of  $f_0$ .*

**Lemma 2.3.** *If  $f_0 \in \mathcal{C}_d$  is a center of non-contiguous type, then  $\mathcal{R}_d(f_0) = \mathcal{C}_d(f_0)$ .*

The proof of Lemma 2.3 is a standard argument using Yoccoz puzzles. Compare [In2, Proposition 5.6] (see also [Mc, §8.2] and [In1, Lemma 4.1]).

To prove Lemma 2.2, we need the result of Poirier [Po]. First we introduce the notion of Hubbard trees [DH1].

**Definition.** Let  $f_0$  be a postcritically finite polynomial. A Jordan arc in the closure of a bounded Fatou component of  $f_0$  is *regulated* if it consists of (at most two) segments of internal rays. More generally, a Jordan arc in  $K(f_0)$  is *regulated* if its intersection with the closure of any Fatou component is regulated.

Any two points  $z_1, z_2 \in K(f_0)$  can be joined uniquely by a regulated arc, which we denote by  $[z_1, z_2]_{f_0}$ .

**Definition.** A set  $X \subset K(f_0)$  is *regulated connected* if for any  $z_1, z_2 \in X$ , we have  $[z_1, z_2]_{f_0} \subset X$ . A *regulated hull*  $[X]_{f_0}$  of  $X \subset K(f_0)$  is the minimal closed regulated connected subset of  $K(f_0)$  containing  $X$ .

The *Hubbard tree*  $T_{f_0}$  of  $f_0$  is the regulated hull of the postcritical set of  $f_0$ .

The following result is proved by Poirier [Po, Theorem B].

**Theorem 2.4.** *Let  $f_0$  be a postcritically finite polynomial. Let  $v \in T_{f_0} \cap J(f_0)$  be a periodic point. Then The number of rays land at  $v$  is equal to the number of incident edges of  $T_{f_0}$  at  $v$ . Furthermore, there is exactly one ray landing between each pair of consecutive edges.*

*Proof of Lemma 2.2.* Let  $f_0$  is a center of non-contiguous type. Consider an edge  $e$  of  $T_{f_0}$ . Then there exist some  $n \geq 0$ , and  $p > 0$  such that  $f_0^{n+p}(e) \supset f_0^n(e)$ .

Since  $f_0$  is non-contiguous,  $e$  is not contained in a union of the closure of periodic Fatou components. This implies that  $f_0^n(e)$  contains infinitely many repelling periodic points.

Therefore we can take  $O$  by choosing one periodic point for each edge  $e$  and taking the union of their orbits. Clearly, there exist infinitely many such choices.  $\square$

### 3 Proofs of Main Theorem 2 and Main Theorem 3

For simplicity, we only treat capture renormalizations in this section. The case of disjoint renormalizations is similar.

### 3.1 Proof of Main Theorem 3, part 1: injectivity

Here we prove the injectivity of the straightening map. Let  $f_0$  be a center of capture type and let  $f_1, f_2 \in \mathcal{C}_d$ . Assume  $S(f_1) = S(f_2) = (c_0, z_0)$ . Take a set  $O$  consists of periodic orbits of  $f_0$  which separates periodic Fatou components of  $f_0$  and let  $\Theta$  be the set of landing angles of  $O$ .

Consider a sequence  $\Lambda = (\lambda_k)_{k \geq 0}$  of equivalence relations on  $\mathbb{Q}/\mathbb{Z}$  by  $\lambda_k = \lambda_{f_0}|_{f_0^{-k}(O)}$ . For each  $k \geq 0$ , we say  $\theta, \theta' \in \mathbb{R}/\mathbb{Z}$  are  $\lambda_k$ -unlinked if for any  $\theta_1 \sim_{\lambda_k} \theta_2$ ,  $\theta$  and  $\theta'$  lie in the same component of  $\mathbb{R}/\mathbb{Z} \setminus \{\theta_1, \theta_2\}$ . It is an equivalence relation and let  $\mathcal{P}_k(\Lambda)$  be the set of  $\lambda_k$ -unlinked classes. For each  $L \in \mathcal{P}_k(\Lambda)$ , we can define the corresponding Yoccoz puzzle  $P_{f_i}(L)$  of depth  $k$  for  $f_i$  (see [In2] for more details). Denote  $\mathcal{P}_k(f_i, \Lambda)$  the set of Yoccoz puzzles of depth  $k$  for  $f_i$ .

Then  $\mathcal{P}_k(f_i, \Lambda)$  has the following properties:

- (i)  $\mathcal{P}_k(f_i, \Lambda)$  is a partition of  $K(f_i)$ ; the interiors of each puzzle piece of depth  $k$  are disjoint and  $\bigcup_{P \in \mathcal{P}_k(f_i, \Lambda)} P \supset K(f_i)$ .
- (ii) For any  $P \in \mathcal{P}_{k+1}(f_i, \Lambda)$ , there exists some  $P' \in \mathcal{P}_k(f_i, \Lambda)$  such that  $P' \supset P$ .
- (iii) For any  $P \in \mathcal{P}_{k+1}(f_i, \Lambda)$ ,  $f(P) \in \mathcal{P}_k(f_i, \Lambda)$ .
- (iv)  $\theta \in L \in \mathcal{P}_k(\Lambda)$  is equivalent that  $R_{f_i}(\theta)$  intersects the interior of  $P_{f_i}(L)$ .

Define a quasiconformal homeomorphism  $\varphi_0 : \mathbb{C} \rightarrow \mathbb{C}$  such that for each  $P_{f_1}(L) \in \mathcal{P}_0(f_1, \Lambda)$ ,  $\varphi_0(P_{f_1}(L)) = P_{f_2}(L)$  and  $\varphi_0 \circ f_1 = f_2 \circ \varphi_0$  on  $\partial P$  (for any  $P \in \mathcal{P}_0(f_1, \Lambda)$ ),  $\partial U'$  and  $\bigcup_{n=0}^{s-1} f_1^n(K(f_1^s; U'_1, U_1))$ . Since  $f_1^s : U'_1 \rightarrow U_1$  and  $f_2^s : U'_2 \rightarrow U_2$  are hybrid equivalent, such a homeomorphism  $\varphi_0$  exists.

For each  $k > 0$ , define  $\varphi_k : U_1 \rightarrow U_2$  inductively as follows. Let  $\varphi_k = \varphi_{k-1}$  outside  $\bigcup_{P \in \mathcal{P}_k(f_1, \Lambda)} P$ . For  $P \in \mathcal{P}_k(f_1, \Lambda)$ , take the corresponding puzzle piece  $P' \in \mathcal{P}_k(f_2, \Lambda)$ . We can define a quasiconformal homeomorphism  $\varphi_k : P \rightarrow P'$  such that  $\varphi_{k-1} \circ f_1 = f_2 \circ \varphi_k$ . Indeed, it is trivial if  $f_1 : P \rightarrow f_1(P)$  is conformal. Assume  $P$  contains a critical point. Then it is  $\omega_0 \in K(f_1^s; U'_1, U_1)$  or the captured critical point  $\omega$ . If  $\omega_0 \in P$ , then since  $\omega_0 \in K(f_1^s; U'_1, U_1)$ , define  $\varphi_k = \varphi_{k-1}$  on  $K(f_1^s; U'_1, U_1)$  and it can be extend quasiconformally on  $P$ . If  $\omega \in P$ , then since  $\varphi_0(f_1^N(\omega)) = f_2^N(\omega')$ , where  $\omega' \in P'$  is the captured critical point of  $f_2$ . Therefore, we can lift  $\varphi_{k-1}$  by  $f_1$  and  $f_2$  and define  $\varphi_k$  on  $P$ .

By construction,  $\varphi_k$  is a quasiconformal homeomorphism and the maximal dilatation of  $\varphi_k$  does not depend on  $k$ . Therefore, passing to a subsequence,  $\varphi_k$  converges to a quasiconformal homeomorphism  $\varphi$ . It is a conjugacy between  $f_1$  and  $f_2$  and since  $\bigcup_{k \geq 0} f^{-k}(K(f_i^s; U'_i, U_i)) = K(f_i)$  a.e., it is a hybrid conjugacy. Since  $f_1$  and  $f_2$  are cubic polynomials with connected Julia set, they are affinely conjugate.

Therefore, the straightening map  $S : \mathcal{C}_3(f_0) \rightarrow \mathcal{MK}$  is injective.

### 3.2 Proof of Main Theorem 2

A basic tool for the proofs of Main Theorem 2 and the surjective part of Main Theorem 3 is the following (cf. [DH2, Proposition 12]):

**Lemma 3.1.** *Let  $f_n : U'_n \rightarrow U_n$  be a sequence of quadratic-like maps and let  $z_n \in K(f_n; U'_n, U_n)$ . Assume*

- (i)  $f_n$  converges to a quadratic-like map  $f : U' \rightarrow U$ , i.e.,  $f_n \rightarrow f$  uniformly on some neighborhood of  $K(f; U', U)$ .
- (ii)  $z_n \rightarrow z \in K(f; U', U)$ .
- (iii)  $f_n$  is hybrid equivalent to  $g_n(z) = z^2 + c_n$  and  $f$  is hybrid equivalent to  $g(z) = z^2 + c$ .
- (iv)  $c_n \rightarrow c$ .
- (v)  $K(f_n) \rightarrow K(f)$  in the Hausdorff topology.

Then  $g_n$  converges to  $g$  and passing to a subsequence,  $\psi_n$  converges to a hybrid conjugacy  $\psi$  between  $f$  and  $g$ .

In particular,  $\psi_n(z_n)$  converges to  $\psi(z)$ .

*Proof of Main Theorem 2.* Assume  $f \in \mathcal{C}_3$  has a renormalization  $f^s : U' \rightarrow U$  of capture type hybrid equivalent to  $Q(z) = z^2 + c$ . Let  $\omega$  be the captured critical point and let  $N > 0$  be the capture time.

The case when  $f$  is locally connected (in particular, when  $f$  is hyperbolic) is easy because  $J(f)$  is homeomorphic to  $S^1/\lambda_f$ .

Consider the case  $c \in \partial\mathcal{M}$ . There exists an open set  $\mathcal{U} \subset \text{Poly}_3$  such that there exists a holomorphic family of quadratic-like restriction  $(g^s : U'_g \rightarrow U_g)_{g \in \mathcal{U}}$  and  $U'_f = U'$  and  $U_f = U$ . Then  $E = \{g \in \mathcal{U}; g^s : U'_g \rightarrow U_g \text{ is hybrid equivalent to } Q\}$  is an analytic set of  $\mathcal{U}$  [DH2, §II.6, Corollary 2].

**Lemma 3.2.**  *$E$  has dimension one.*

Before we prove the lemma, we introduce the notion of active critical points for a family of polynomials with marked critical points.

**Definition.** For an analytic family of polynomial  $(g_\lambda, \omega_\lambda)_{\lambda \in \Lambda}$  with marked critical points, we say  $\omega_\lambda$  is *active* at  $\lambda_0$  (or simply,  $\omega_{\lambda_0}$  is *active*) if  $(\lambda \mapsto g_\lambda^n(\omega_\lambda))$  does not form a normal family at  $\lambda_0$ .

*Proof.* If  $E$  has dimension two ( $E$  is an open set in  $\mathcal{U}$ ), then  $S : E \rightarrow \mathcal{MK}$  is continuous by Lemma 3.1 and  $S(g) = (Q, z(g))$  for any  $g \in E$ . Let  $g^s : U'_g \rightarrow U_g$  be the corresponding renormalization for  $g$ ,  $\omega(g)$  be the captured critical point and  $\omega_0(g) \in U'_g$  be the other critical point. Then  $\omega_0$  is not active because  $g^n(\omega_0(g)) \in K(g^s; U'_g, U_g)$ . On the contrary, if  $\omega$  is also not active, then  $f$  is structurally stable in  $E$  and this implies  $f$  carries an invariant line field on  $J(f)$ . But since  $J(f) \setminus \bigcup_{n \geq 0} f^{-n}(J(f^s; U', U))$  has measure zero, it follows that  $Q$  also carries an invariant

line field, which contradicts the assumption  $c \in \partial M$ . Therefore,  $\omega$  is active. Hence there exists some  $g_1$  arbitrarily close to  $f$  such that  $\omega(g_1)$  is preperiodic. Take an analytic subset  $A$  of  $E$  containing  $g_1$  where  $\omega(g)$  is preperiodic. Then  $\omega(g)$  is not active on  $A$  and repeating the above argument,  $Q$  carries an invariant line field and it is a contradiction.  $\square$

Similar argument shows that  $\omega_0$  is not active and  $\omega$  is active on  $E$ . Take a sequence  $x_n \rightarrow f^N(\omega)$  such that  $x_n \in K(f^s; U', U)$  is periodic. For  $g \in E$ , let  $x_n(g) \in K(g^s; U'_g, U_g)$  be the continuation of  $x_n$ . Since for any  $g \in E$ ,  $g^s : U'_g \rightarrow U_g$  is a quadratic-like map hybrid equivalent to  $Q$ ,  $x_n$  is defined a neighborhood of  $E$  (independent of  $n$ ). If  $g^N(\omega(g)) - x_n(g)$  converges uniformly to zero as  $n \rightarrow \infty$ , then this implies that  $g^N(\omega(g)) \in K(g^s; U'_g, U_g)$  for all  $g \in E$  near  $f$  and  $n \geq N$ , which contradicts that  $\omega$  is active on  $E$ . Hence  $g^N(\omega(g)) - x_n(g)$  converges to a non-constant holomorphic function  $h$  with  $h(f) = 0$ . This implies that for sufficiently large  $n$ , there exists some  $g_2$  such that  $g_2^N(\omega(g_2)) - x_n(g_2) = 0$ . In other words, there exists some  $g_2$  arbitrarily close to  $f$  such that it has a capture renormalization of the same period and capture time as  $f$  such that the captured critical point is preperiodic.

Let  $F$  be an one-dimensional analytic set containing  $g_2$  such that the marked critical point  $\omega(g)$  is preperiodic. Then, the straightening map  $S$  satisfies that  $\pi_1 \circ S$  is not constant, where  $\pi_1 : \mathbb{C}^2 \rightarrow \mathbb{C}$  is the projection to the first coordinate. Hence there exists some  $g_3$  arbitrarily close to  $g_2$  such that  $\pi_1 \circ S(g_3)$  is hyperbolic. Hence there exists some  $f_0$  such that  $g_3 \in C_3(f_0)$ . Since for a given  $s, N > 0$ , there exist only finitely many center  $f_0$  of capture type of period  $s$  and capture time  $N$ , hence we may assume there exists a center  $f_0$  such that we choose such  $g_3$  arbitrarily close to  $f$  and  $g_3 \in C_3(f_0)$ . Taking a limit  $g_3 \rightarrow f$ , we can verify that  $f$  also lies in  $C_3(f_0)$ , this implies that  $f \in \mathcal{R}_3(f_0)$ .

Finally, consider the case  $c$  lies in a queer component (non-hyperbolic component of the interior of  $\mathcal{M}$ ). Then  $f$  is infinitely renormalizable. In particular, all periodic points are repelling. Therefore, we can find many biaccessible points (e.g., there exists exactly  $d$  fixed points and  $d - 1$  invariant landing angles. Hence there exists a fixed point with non-invariant landing angles, which implies that it is biaccessible), and we can construct  $f_0$  by using the result of Kiwi [Ki].  $\square$

### 3.3 Proof of Main Theorem 3, part 2: surjectivity

Let  $(c, z) \in \mathcal{MK}$ . We want to find  $f \in C_3(f_0)$  such that  $S(f) = (c, z)$ .

If  $z^2 + c$  is hyperbolic and  $z$  lies in the interior of  $K(z^2 + c)$ , then we can construct  $f$  by constructing a rational lamination ("combinatorial tuning")  $\lambda$  from  $\lambda_{f_0}$  and  $\lambda_{z^2+c}$ , realizing it by a cubic polynomial [Ki] and perturbing it by the standard quasiconformal deformation technique.

Taking a limit on  $z$ , we can also construct  $f$  for the case  $z^2 + c$  hyperbolic and  $z \in J(z^2 + c)$  by Lemma 3.1.

Consider the case  $c \in \partial\mathcal{M}$ . If  $z^2 + c$  does not have a Siegel disk, then  $K(z^2 + c) = J(z^2 + c)$  has no interior and  $c \mapsto K(z^2 + c)$  is continuous at  $c$ . Since  $z^2 + c$  can be approximated by hyperbolic polynomials, we can similarly find  $f$  by Lemma 3.1. If  $z^2 + c$  has a Siegel disk, then taking a radial limit from the hyperbolic component  $H$  with  $c \in \partial H$ , we can obtain a sequence  $c_n \rightarrow c$  with  $K(z^2 + c_n) \rightarrow K(z^2 + c)$ . Hence this case is also similar.

The last case is when  $c$  lies in a queer component  $A$ . First take  $c' \in \partial A$  and (eventually) periodic  $z' \in K(z^2 + c')$ . Then there exists  $f' \in \mathcal{C}_3(f_0)$  such that  $S(f') = (c', z')$ . The captured critical point  $\omega'$  for  $f'$  is preperiodic. Hence it is contained in a one-dimensional subspace  $E$  of  $\text{Poly}_3$ , where one marked critical point is preperiodic. The straightening map  $S$  is defined on a neighborhood  $\mathcal{U}$  of  $z$  in  $E$ . Since  $S$  is injective,  $S$  is non-constant. Hence  $S$  is an open map [DH2, §III.3]. This implies that there exists a map  $f'' \in E$  such that  $S(f'') = (c'', z'')$  with  $c'' \in A$  and  $z''$  is (eventually) periodic for  $z^2 + c''$ .

Since  $z^2 + c''$  carries an invariant line field on its Julia set, we can deform quasiconformally  $z^2 + c''$  and  $f''$  at the same time, and obtain an open set  $\mathcal{U}' \subset E \cap \mathcal{C}_3(f_0)$  such that  $\pi_1 \circ S(\mathcal{U}') = A$ .

This proves that for any  $(c, z)$  with  $c \in A$  and (eventually) periodic  $z$ , there exists some  $f \in \mathcal{C}_3(f_0)$  such that  $S(f) = (c, z)$ . Since such points are dense in  $\mathcal{MK} \cap (A \times \mathbb{C})$ , this holds for any  $(c, z) \in \mathcal{MK} \cap (A \times \mathbb{C})$  (note that the interiors of  $K(f)$  and  $K(z^2 + c)$  are empty).

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