

A Note on the Arithmeticity of the Figure-Eight Knot Orbifolds

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1. INTRODUCTION

Let K be the figure-eight knot, (K, n) the orbifold with underlying space $\mathbf{S}^3$, singular set K and isotropy group cyclic of order n .

Proposition 1 (Thurston [5], Hilden-Lozano-Montesinos [1]). *If $n > 3$, (K, n) is hyperbolic. Furthermore, (K, n) is arithmetic exactly for $n = 4, 5, 6, 8, 12$.*

In this paper, our aim is to describe concretely the arithmeticity of (K, n) for $n = 4, 5, 6, 8, 12$.

2. PRELIMINARIES

We can take a Kleinian model of (K, n) as follows ([1]):

$$\Gamma_n = \langle A, B \mid A^{-1}BAB^{-1}ABA^{-1}B^{-1}AB^{-1} = I, A^n = B^n = -I \rangle,$$

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, B = \begin{pmatrix} \mu & 1/\alpha \\ \alpha & \mu(\alpha - \mu) - 1 \end{pmatrix},$$

where

$$\alpha = 2 \cos \frac{\pi}{n}$$

$$\beta = \frac{1}{2}(1 + \alpha^2 + \sqrt{(\alpha^2 - 1)(\alpha^2 - 5)})$$

$$\lambda = \frac{1}{2}(\alpha + \sqrt{\alpha^2 - 4})$$

$$\mu = \frac{\lambda\beta - \alpha}{\lambda^2 - 1}.$$

Definition. Let Γ be a non-elementary Kleinian group and $\Gamma^{(2)}$ the subgroup generated by the squares of the elements of Γ . The invariant trace field of Γ is the field $\mathbb{Q}(\text{tr}\Gamma^{(2)})$, and denoted by $k\Gamma$. The invariant quaternion algebra is given by

$$\left\{ \sum a_i \gamma_i (\text{finite sum}) \mid a_i \in k\Gamma, \gamma_i \in \Gamma^{(2)} \right\},$$

and denoted by $A\Gamma$.

In fact, we see that $A\Gamma$ is a quaternion algebra over $k\Gamma$ from:

Lemma 2. *Let Γ be a Kleinian group of finite covolume. Then $A\Gamma$ is quaternion algebra over $k\Gamma$ if*

- (1) $k\Gamma$ is a number field with one complex place, and
- (2) $\text{tr}\Gamma^{(2)}$ consists of algebraic integers.

Furthermore, if we define

$$R_{k\Gamma} = \{a \in k\Gamma \mid a \text{ is an algebraic integer}\} \quad \text{and}$$

$$O\Gamma = \left\{ \sum b_i \gamma_i (\text{finite sum}) \mid b_i \in R_{k\Gamma}, \gamma_i \in \Gamma^{(2)} \right\},$$

then $O\Gamma$ is an order of $A\Gamma$.

The following lemma shows that if Γ is arithmetic, it is sufficient to take $k\Gamma$ and $A\Gamma$ as its algebraic tools (see [2] [6]).

Lemma 3. *Suppose that Γ is an arithmetic Kleinian group. Then*

$$\Gamma^{(2)} \subset P(O^1\Gamma)$$

where $P : SL(2, \mathbb{C}) \longrightarrow PSL(2, \mathbb{C})$ and $O^1\Gamma = \{x \in O\Gamma \mid \text{the norm of } x \text{ is } 1\}$.

We shall calculate $k\Gamma_n$ for $n = 4, 5, 6, 8, 12$. Since $k\Gamma_n = \mathbb{Q}(\alpha^2, \beta)$ by [1], we see that

$$k\Gamma_4 = \mathbb{Q}(\sqrt{-3}) \quad \left(\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}\right);$$

$$k\Gamma_5 = \mathbb{Q}\left(\sqrt{\frac{-1 - 3\sqrt{5}}{2}}\right) \quad \left(\cos \frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}\right);$$

$$k\Gamma_6 = \mathbb{Q}(\sqrt{-1}) \quad \left(\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}\right);$$

$$k\Gamma_8 = \mathbb{Q}(\sqrt{-1 - 2\sqrt{2}}) \quad \left(\cos \frac{\pi}{8} = \frac{\sqrt{2 + \sqrt{2}}}{2}\right);$$

$$k\Gamma_{12} = \mathbb{Q}(\sqrt{-2\sqrt{3}}) \quad \left(\cos \frac{\pi}{12} = \frac{\sqrt{2 + \sqrt{3}}}{2}\right).$$

All of them are the extension fields over $\mathbb{Q}$ of degree 2.

On the other hand, by [1],

$$A\Gamma_n = \left(\frac{\frac{1}{4}(\lambda^2 - \lambda^{-2})^2, \alpha^2\{\mu(\alpha - \mu) - 1\}}{k\Gamma_n} \right),$$

where

$$1 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} \frac{1}{2}(\lambda^2 - \lambda^{-2}) & 0 \\ 0 & -\frac{1}{2}(\lambda^2 - \lambda^{-2}) \end{pmatrix}$$

and

$$j = \begin{pmatrix} 0 & 1 \\ \alpha^2\{\mu(\alpha - \mu) - 1\} & 0 \end{pmatrix}.$$

In fact,

$$i^2 = \frac{1}{4}(\lambda^2 - \lambda^{-1})^2, \quad j^2 = \alpha^2\{\mu(\alpha - \mu) - 1\}, \quad ij = -ji.$$

3. MAIN THEOREM AND ITS PROOF

In the beginning of this section, we mention our main theorem.

Theorem 4. *For the Kleinian model Γ_n of (K, n) , the following are satisfied:*

(1) *If $n = 4, 6, 8, 12$, then*

$$\Gamma_n \cap P(O^1\Gamma_n) = \Gamma_n^{(2)} \quad \text{and}$$

$$[\Gamma_n : \Gamma_n \cap P(O^1\Gamma_n)] = 2.$$

(2) *For $n = 5$,*

$$\Gamma_5 \cap P(O^1\Gamma_5) = \Gamma_5, \quad \text{that is,}$$

$$[\Gamma_5 : \Gamma_5 \cap P(O^1\Gamma_5)] = 1.$$

To prove this theorem, we need the next lemma.

Lemma 5. *Let Γ be a finitely generated group, m the number of the generators of Γ . Then*

$$[\Gamma : \Gamma^{(2)}] \leq 2^m.$$

Proof. See [6].

Proof of Theorem 4. Lemma 3 and Lemma 5 show that

$$[\Gamma_n : \Gamma_n \cap P(O^1\Gamma_n)] \leq [\Gamma_n : \Gamma_n^{(2)}] \leq 4.$$

Furthermore by the relation

$$A^{-1}BAB^{-1}ABA^{-1}B^{-1}AB^{-1} = I,$$

we see that $AB \in \Gamma_n^{(2)}$. Hence $[\Gamma_n : \Gamma_n \cap P(O^1\Gamma_n)] \leq 2$, and it is sufficient to consider A (or B). We set $A = a_0 \cdot 1 + a_1 i + a_2 j + a_3 ij$. In this case, solving linear equations, we see that

$$a_0 = \alpha/2, \quad a_1 = 1/\alpha, \quad a_2 = a_3 = 0.$$

And since $\lambda^2 - \lambda^{-2} = \alpha\sqrt{\alpha^2 - 4}$, the norm of A equals to 1.

Now, we shall classify into two cases. In case $n = 4, 6, 8, 12$: Since $\alpha \notin k\Gamma_n$, we see $A \notin A\Gamma_n$. Hence $A \notin O\Gamma_n$. In case $n = 5$: By $A^5 = -I$, we have $-A = A^{-4} \in \Gamma_5^{(2)}$. On the other hand, since $\Gamma_5^{(2)} \subset \Gamma_5 \cap P(O^1\Gamma_5)$,

$$A = \sum (-b_i)\gamma_i$$

where $-b_i \in R_{k\Gamma_5}$, $\gamma_i \in \Gamma_5^{(2)}$ and the norm of A equals to 1. Therefore $A \in O^1\Gamma_5$.

The proof of Theorem 4 is now completed.

4. THE DIFFICULTY ABOUT THE COMPLEMENT

For $S^3 - K$, there is Riley's model Γ as its Kleinian model ([4]), so we know it is arithmetic. But it is difficult to calculate its arithmeticity same as the case of (K, n) . The difficulty comes from the lack of definite information about the order $O\Gamma$, but by relations in the fundamental group of $S^3 - K$ and experimental calculation in [6], we shall except the next problem.

Problem 6. *For the Kleinian model Γ of $S^3 - K$, is it satisfied that $\Gamma \cap P(O^1\Gamma) = \Gamma$? In other words,*

$$[\Gamma : \Gamma \cap P(O^1\Gamma)] = 1?$$

In future, our subject is to investigate geometrical properties of arithmetic hyperbolic 3-manifolds.

REFERENCES

1. H. M. Hilden, M. T. Lozano and J. M. Montesinos, *Arithmeticity of the figure-eight knot orbifolds*, in TOPOLOGY'90, Proceeding of the Research Semester in Low dimensional Topology at Ohio State University, De Gryter Verlag, 1992, pp 169–183.
2. C. Maclachlan and A. W. Reid, *Commensurability classes of arithmetic Kleinian groups and their Fuchsian subgroups*, Math. Proc. Camb. Phil. Soc. **102** (1987), 251–257.
3. A. W. Reid, *A non-Haken hyperbolic 3-manifold covered by a surface bundle*, preprint.
4. R. Riley, *A quadratic parabolic groups*, Math. Proc. Camb. Phil. Soc. **77** (1975), 281–288.
5. W. P. Thurston, *The geometry and topology of 3-manifolds*, Mineographed lecture notes at Princeton University, 1978/79.
6. A. Watanabe, Master thesis, Dept. of Math., Tokyo Inst. Tech. (1994).

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