

A numerical semigroup whose genus described by its minimum odd is large ¹

Jiryo Komeda

Center for Basic Education and Integrated Learning
Kanagawa Institute of Technology

Abstract

The main theorem in this article is a generalization of the result of [2]. We consider Weierstrass semigroups H attained by ramification points of double coverings of trigonal curves. Especially, we characterize H whose genus described by its minimum odd is large. We give a necessary and sufficient condition for such a numerical semigroup H to be non-Weierstrass.

1 Terminologies and introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if its complement $\mathbb{N}_0 \setminus H$ is finite. The cardinality of $\mathbb{N}_0 \setminus H$ is called the *genus* of H , denoted by $g(H)$. We denote by $m(H)$ the minimum positive integer in H , which is called the *multiplicity* of H . In this paper H always stands for a numerical semigroup. We set

$$d_2(H) = \{h \in \mathbb{N}_0 \mid 2h \in H\},$$

which is a numerical semigroup. Let n be the minimum odd integer in H . Then we get

$$g(H) \leq 2g(d_2(H)) + \frac{n-1}{2}.$$

If the above equality holds, then we get

$$H = 2d_2(H) + \langle n \rangle,$$

where $\langle a_1, \dots, a_t \rangle := a_1\mathbb{N}_0 + \dots + a_t\mathbb{N}_0$ for positive integers $a_1, \dots, a_t$. We are interested in H satisfying one of the following:

$$(1) \quad g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1,$$

$$(2) \quad g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 2.$$

¹This paper is an extended abstract and the details will be published (see [3])

2 On numerical semigroups H with $H = 2d_2(H) + \langle n, n+2 \rangle$

We set

$$c(H) = \min\{h \in H \mid h + \mathbb{N}_0 \subseteq H\},$$

which is called the *conductor* of H . The integer $c(H) - 1$ is called the *Frobenius number* of H . An element $\gamma \in \mathbb{N}_0 \setminus H$ is said to be a *pseudo-Frobenius number* of H if $\gamma + h \in H$ for any positive integer $h \in H$. We denote by $PF(H)$ the set of the pseudo-Frobenius numbers of H .

We are interested in whether a numerical semigroup is gained by algebraic geometry objects. A *curve* means a projective non-singular irreducible algebraic curve over an algebraically closed field k of characteristic 0. For a pointed curve (C, P) we set

$$H(P) = \{\alpha \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = \alpha P\},$$

where $k(C)$ is the field of rational functions on C . $H(P)$ is a numerical semigroup of genus $g(C)$ where $g(C)$ is the genus of C , which is called the *Weierstrass semigroup* of P .

Theorem 2.1 *Let n be the minimum odd integer in H . Assume that*

$$n \geq c(d_2(H)) + m(d_2(H)) - 1 \text{ and } H = 2d_2(H) + \langle n, n+2 \rangle.$$

Then the following are equivalent:

$$(i) \ g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1$$

$$(ii) \ 1 \in PF(d_2(H))$$

(iii) $d_2(H)$ is ordinary, that is to say, $d_2(H) = \langle g+1, g+1+1, \dots, g+1+g \rangle$, which is the Weierstrass semigroup $H(P)$ of any ordinary point P of a curve of genus g .

Theorem 2.2 *Let n be the minimum odd integer in H . Assume that*

$$n \geq c(d_2(H)) + m(d_2(H)) - 1 \text{ and } H = 2d_2(H) + \langle n, n+2 \rangle.$$

Then the following are equivalent:

$$(i) \ g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 2$$

(ii) $d_2(H) = \{0, m \rightarrow m+s-g, s+2 \rightarrow\}$ where $m := m(d_2(H))$, $g := g(d_2(H))$ and $s \in \mathbb{N}_0$ with $g \leq s \leq 2g-2$ and $\left\lfloor \frac{s+1}{2} \right\rfloor + 1 \leq m \leq g$. Here, $[]$ means the Gauss symbol.

Remark 2.3 *The following result is due to Kim [1]:*

Let C be a trigonal curve, that is to say, a three sheeted covering of the projective line $\mathbb{P}^1$, of genus $g \geq 5$. Let $P \in C$ be a non-ramification point, but a non-ordinary point on C .

Then the Weierstrass semigroup $H(P)$ is $\{0, m \longrightarrow m + s - g, s + 2 \longrightarrow\}$ for some m such that $\left\lfloor \frac{s+1}{2} \right\rfloor + 1 \leq m \leq g$ with $g \leq s \leq 2g - 2$. Moreover, each sequence actually occurs as the Weierstrass semigroup of some non-ramification point on a trigonal curve of genus g . This semigroup coincides with $d_2(H)$ in Theorem 2.2

3 On numerical semigroups H with $H = 2d_2(H) + \langle n, n + 4 \rangle$

Theorem 3.1 Assume that n is an odd integer satisfying $n \geq c(d_2(H)) + m(d_2(H)) - 1$ and $H = 2d_2(H) + \langle n, n + 4 \rangle$. Assume that $m := m(d_2(H))$ is odd. Then the following are equivalent:

- (i) $g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1$.
- (ii) $d_2(H) = \langle m, m + 1, \dots, m + m - 1 \rangle$ or
 $\langle m, m + 2, m + 4, \dots, m + m - 1, m + 2j + 1, m + 2(j + 1) + 1, \dots, m + 2((m - 3)/2) + 1, 2m + 1, 2m + 3, \dots, 2m + 2(j - 1) + 1 \rangle$ for some j with $1 \leq j \leq (m - 3)/2$ or
 $\langle m, m + 2, m + 4, \dots, m + m - 1, 2m + 1, 2m + 3, \dots, 2m + m - 2 \rangle$.

Remark 3.2 The following result is due to Kim [1]:

Let C be a trigonal curve of genus $g \geq 5$. Let $P \in C$ be a ramification point that is not total. Assume that $m(H(P))$ is odd. Then the set $\mathbb{N}_0 \setminus H(P)$ is

$$\{1, 2, 3, \dots, 2n - 1, 2n, 2n + 2, 2n + 4, \dots, 2g - 2n\}$$

with $\frac{g-1}{3} < n \leq \frac{g}{2}$ where g is the genus of C . Moreover, each sequence actually occurs as the Weierstrass semigroup of some non-total ramification point on a trigonal curve of genus g .

We set $n = \frac{m(d_2(H)) - 1}{2}$. We set $g = m(d_2(H)) - 1$ in the first case in Theorem 3.1 (ii)

and $g = m(d_2(H)) - 1 + j$ in the second case in Theorem 3.1 (ii). We set $g = \frac{3m(d_2(H)) - 3}{2}$ in the third case in Theorem 3.1 (ii). Then $d_2(H)$ in Theorem 3.1 (ii) coincides with $H(P)$ in the above Known result.

Theorem 3.3 Let n be the minimum odd integer in H . Assume that $n \geq c(d_2(H)) + m(d_2(H)) - 1$. Let $H = 2d_2(H) + \langle n, n + 4 \rangle$. Assume that $m := m(d_2(H))$ is even. Then the following are equivalent:

- (i) $g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1$.
- (ii) $d_2(H) = \{m, m + 1, \dots\}$ or
 $\{m, m + 2, m + 4, \dots, m + 2(g - m), m + 2(g - m + 1), m + 2(g - m + 1) + 1, \dots\}$
 where we set $m = m(d_2(H))$ and $g = g(d_2(H))$.

Remark 3.4 The following result is also due to Kim [1]:

Let C be a trigonal curve of genus $g \geq 5$. Let $P \in C$ be a ramification point that is not total. Assume that $m(H(P))$ is even. Then the set $\mathbb{N}_0 \setminus H(P)$ is $\{1, 2, 3, \dots, 2n-1, 2n, 2n+1, 2n+3, \dots, 2g-2n-1\}$

with $\frac{g-1}{3} \leq n \leq \frac{g}{2}$. Moreover, each sequence actually occurs as the Weierstrass semigroup of some non-total ramification point on a trigonal curve of genus g .

The following is the first Key Lemma in this article:

Lemma 3.5 Assume that $m(d_2(H))$ is even. We set $n = \frac{m(d_2(H)) - 2}{2}$ and $g = g(d_2(H))$.

- (i) If $m(d_2(H)) \geq \frac{2g(d_2(H)) + 4}{3}$, then $d_2(H)$ in Theorem 3.3 (ii) coincides with $H(P)$ in Remark 3.4.
- (ii) If $m(d_2(H)) < \frac{2g(d_2(H)) + 4}{3}$, then $d_2(H)$ in Theorem 3.3 (ii) can not be attained by any non-total ramification point of a trigonal curve of genus g .

4 Non-Weierstrass numerical semigroups

A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) with $H(P) = H$. A numerical semigroup H is said to be of *double covering type*, which is abbreviated to *DC*, if there exists a double covering of a curve with a ramification point P with $H(P) = H$. Hence, if H is DC, then it is Weierstrass.

Theorem 4.1 Let n be the minimum odd integer in H . Assume that n is an odd integer with $n \geq c(d_2(H)) + m(d_2(H)) - 1$. Let $H = 2d_2(H) + \langle n, n+4 \rangle$ and assume that $g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1$. Moreover, assume that one of the following holds:

- (i) $m := m(d_2(H))$ is odd.
- (ii) $m := m(d_2(H))$ is even and $m \geq \frac{2g(d_2(H)) + 4}{3}$.

If $n \geq 2g(d_2(H)) + 3$, then H is DC, hence it is Weierstrass.

Outline of the proof: By the assumption which is either (i) or (ii) it follows from Remark 3.1 and (i) in Lemma 3.5 that there is a pointed trigonal curve (C, P) with a non-total ramification point P satisfying $H(P) = H$. We can construct a double covering $\pi : \tilde{C} \rightarrow C$ with a ramification point $\tilde{P}$ over P satisfying $H(\tilde{P}) = H$. Hence the numerical semigroup H is DC

Remark 4.2 By a way similar to the proof of Theorem 4.1 the numerical semigroups $H = 2d_2(H) + \langle n, n+2 \rangle$ in Theorem 2.1 with $n \geq 2g(d_2(H)) + 3$ and in Theorem 2.2 with $n \geq 2g(d_2(H)) + 5$ are Weierstrass.

Remark 4.3 The following result is due to Torres [4]. Assume that $g(H) \geq 6g(d_2(H)) + 4$. If H is a Weierstrass numerical semigroup, then it is DC.

We use the contrapositive of the above, which is the second Key Lemma in this article.

Lemma 4.4 Assume that $g(H) \geq 6g(d_2(H)) + 4$. If H is not DC, then it is non-Weierstrass.

Theorem 4.5 Let n be the minimum odd integer in H . Assume that $n \geq c(d_2(H)) + m(d_2(H)) - 1$. Let $H = 2d_2(H) + \langle n, n + 4 \rangle$ and assume that $g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1$.

Moreover, assume that $m := m(d_2(H))$ is even and $m < \frac{2g(d_2(H)) + 4}{3}$. Then the following hold:

- (i) Then H is not DC.
- (ii) If $n \geq 8g(d_2(H)) + 11$, then it is non-Weierstrass.

Outline of the Proof: (i) Assume that H is DC. Then there exists a double covering of a pointed curve (C, P) with a ramification point $\tilde{P}$ over P satisfying that

$$H(\tilde{P}) = H = 2d_2(H) + \langle n, n + 4 \rangle = 2H(P) + \langle n, n + 4 \rangle,$$

which implies that we must have

$$\dim (\{f \in k(C) \setminus \{0\} \mid (f) \geq -2P - Q\} \cup \{0\}) = 2$$

for some point Q of C distinct from P . But by the assumptions that m is even and $m < (2g(d_2(H)) + 4)/3$ it follows from Lemma 3.5 that such a pointed curve (C, P) does not exist. Hence, H is not DC.

- (ii) By Lemma 4.4 the numerical semigroup H is non-Weierstrass.

The following is the main theorem in this article, which is deduced from Theorems 4.1 and 4.5.

Theorem 4.6 Assume that $2 \in PF(d_2(H))$ and $H = 2d_2(H) + \langle n, n + 4 \rangle$. Let n be an odd integer with $n \geq 8g(d_2(H)) + 11$. Then the following are equivalent:

- (i) $m(d_2(H))$ is even and we have $m(d_2(H)) < \frac{2g(d_2(H)) + 4}{3}$.
- (ii) H is non-Weierstrass.

The following are known examples, which are due to [1].

Remark 4.7 Let $l \geq 2$ and n be an odd integer with $n \geq 16l + 19$ (resp. $16l + 27$). We set $H = 2\langle 4, 6, 4l + 1, 4l + 3 \rangle + \langle n, n + 4 \rangle$

(resp. $H = 2\langle 4, 6, 4l + 3, 4(l + 1) + 1 \rangle + \langle n, n + 4 \rangle$). Then H are non-Weierstrass numerical semigroups, which are the first examples of non-Weierstrass numerical semigroups whose multiplicities are 8, because

$$m(d_2(H)) = 4 < \frac{2g(d_2(H)) + 4}{3} = \frac{2(2l + 1) + 4}{3} \quad (\text{resp. } \frac{2(2l + 2) + 4}{3}).$$

Hence, Theorem 4.6 is a generalization of the above important example. In fact, we get many examples of non-Weierstrass numerical semigroups.

Remark 4.8 Let s be a non-negative integer. We set

$$H_s = \langle 2s + 4, 2s + 4 + 2, \dots, 2s + 4 + 2(s + 1), 2(2s + 4) + 1, \\ 2(2s + 4) + 3, \dots, 2(2s + 4) + 2s + 3 \rangle.$$

Then we have the following:

- i) $g(H_s) = 3s + 5$, $PF(H_s) \ni 2$ and $m(H_s) < \frac{2g(H_s) + 4}{3}$.
- ii) Let n be an odd integer with $n \geq 24s + 51$. Then $H = 2H_s + \langle n, n + 4 \rangle$ is a non-Weierstrass numerical semigroup whose multiplicity is $4s + 8$.

References

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