

Torelli theorem for certain rational surfaces and root system of type A

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For an integer $n \geq 2$, let Σ_n be the n -th Hirzebruch surface defined by

$$(0.1) \quad \{(\zeta_0 : \zeta_1 : \zeta_2)(s : t) \in \mathbf{P}^2 \times \mathbf{P}^1 \mid s^n \zeta_0 = t^n \zeta_1\},$$

where $\mathbf{P}^k$ is k -dimensional complex projective space. Let X_n be a surface obtained by blowing up $n + 1$ points of Σ_n and D be an anti-canonical divisor on X_n such that D consists of four nonsingular rational curves and its intersection diagram is a circle (thus D forms a square).

We study the isomorphism classes of the pairs (X_n, D) . The isomorphism classes can be characterized in terms of the root system of type A. E. Looijenga investigated the isomorphism classes of rational surfaces with anti-canonical divisors [L]. We deal with another class of rational surfaces. The method and formulation are very similar to those of Looijenga's.

1. HIRZEBRUCH SURFACES

We assume $n \geq 3$. Σ_n is a subvariety of $\mathbf{P}^2 \times \mathbf{P}^1$ (cf (0.1)). Let $\pi : \Sigma_n \rightarrow \mathbf{P}^1$ be the second projection. Σ_n is a $\mathbf{P}^1$ -bundle over $\mathbf{P}^1$. Let F be a fiber of the projection $\pi : \Sigma_n \rightarrow \mathbf{P}^1$ and S be the section defined by $\zeta_0 = \zeta_1 = 0$.

DEFINITION. we say that $n + 1$ points $P_1, \dots, P_{n+1}$ of Σ_n are in 'general position' if they satisfy the following conditions: (1) $P_i \neq P_j$ for $i \neq j$ and (2) there exists a nonsingular curve in the complete linear system $|nF + S|$ passing through $P_1, \dots, P_{n+1}$.

REMARK. If $P_1, \dots, P_{n+1}$ are in general position, then $P_i \notin S$ and no two of P_i are on a fiber.

Let $p : X_n \rightarrow \Sigma_n$ be the morphism obtained by blowing up $n + 1$ points $P_1, \dots, P_{n+1}$ in general position.

LEMMA 1.1. If D is an anti-canonical divisor on X_n and satisfies the following conditions:

- (1) D is the strict transform of an anti-canonical divisor D' on Σ_n ,

- (2) D' consists of four irreducible components and its intersection diagram is a circle,
 (3) $P_1, \dots, P_{n+1}$ are on only one component of D' and not on other components,

then

$$D = F_1 + F_2 + S + C,$$

where F_i is a strict transform of a fiber of the projection $\pi : \Sigma_n \rightarrow \mathbf{P}^1$, S is the strict transform of the $(-n)$ -section of Σ_n and C is the strict transform of the unique nonsingular curve of $|nF + S|$ passing through $P_1, \dots, P_{n+1}$.

NOTATION. We say that an anti-canonical divisor D on X_n is of ' $\#$ -type' if it satisfies the condition of lemma 1.1. We denote by F_0 and F_∞ the components of D which are the strict transforms of the fibers of π .

2. HOMOLOGY AND ROOT SYSTEM

Let X_n and $D = F_0 + F_\infty + S + C$ be as in §1. Consider the homology exact sequence:

$$\begin{array}{ccccccc} \dots & & \rightarrow & H_3(X_n; \mathbb{Z}) & \rightarrow & H_3(X_n, X_n - D; \mathbb{Z}) & \\ & & & \parallel & & & \\ & & & 0 & & & \\ \partial_* & H_2(X_n - D; \mathbb{Z}) & \xrightarrow{i_*} & H_2(X_n; \mathbb{Z}) & \xrightarrow{j_*} & H_2(X_n, X_n - D; \mathbb{Z}) & \\ \rightarrow & \dots & & & & & \end{array}$$

We extend the intersection form in $H_2(X_n; \mathbb{Z})$ to $H_2(X_n; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Let

$$Q = \ker j_* \subset H_2(X_n; \mathbb{Z})$$

and

$$R = \{\alpha \in Q \mid \alpha \cdot \alpha = -2\}.$$

LEMMA 2.1. R is a root system of type A_n in $Q \otimes_{\mathbb{Z}} \mathbb{R}$ and Q is generated by R . The set $\{e_i - e_{i-1} \mid 1 \leq i \leq n\}$ is the basis of R , where e_i is the class of the exceptional curve $E_i = p^{-1}(P_i)$.

We now have the short exact sequence:

$$(2.1) \quad 0 \rightarrow H_3(X_n, X_n - D; \mathbb{Z}) \xrightarrow{\partial_*} H_2(X_n - D; \mathbb{Z}) \xrightarrow{i_*} Q \rightarrow 0.$$

LEMMA 2.2 (K. IRIE).

$$H_3(X_n, X_n - D; \mathbb{Z}) \simeq \mathbb{Z}$$

Let ε be the generator of $H_3(X_n, X_n - D; \mathbb{Z})$. We next consider a meromorphic 2-form on X_n which has poles only along D .

LEMMA 2.3. *There exists a unique meromorphic 2-form ω on X_n such that*

- (1) ω has poles only along D ,
- (2) $\omega(\partial_*(\varepsilon)) = 1$.

Furthermore, we can choose an affine coordinate z on $C(\subset D)$ such that $F_0 \cap C = 0$, $F_\infty \cap C = \infty$ and

$$\text{Res}_C \omega = \frac{1}{(2\pi i)^2} \frac{dz}{z}.$$

It follows from this lemma, we can define a character $\chi : Q \rightarrow \mathbb{C}^*$ by

$$\chi(i_*[\Gamma]) = \exp 2\pi i \int_\Gamma \omega,$$

where $\Gamma \in H_2(X_n - D; \mathbb{Z})$.

$$\begin{array}{ccc} H_2(X_n - D; \mathbb{Z}) & \xrightarrow{\exp 2\pi i \int_{[\cdot]} \omega} & \mathbb{C}^* \\ & \searrow i_* & \uparrow \chi \\ & & Q \end{array}$$

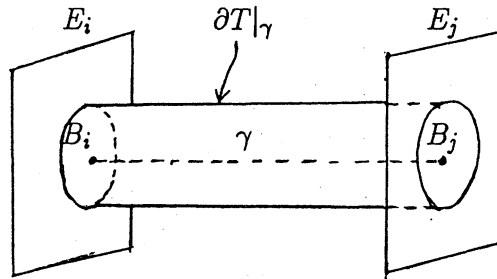
3. TORELLI THEOREM FOR THE PAIR (X_n, D)

We first consider the value of χ at the class $e_i - e_j \in Q$, where e_i and e_j are the homology classes of the exceptional curves $E_i = p^{-1}(P_i)$ and $E_j = p^{-1}(P_j)$ respectively. Let $B_i = E_i \cap C$ and let T be a closed tubular neighborhood of C in X_n such that $T \cap E_i$ and $T \cap E_j$ are fibers. Let γ be an injective path in C from B_i to B_j and let

$$\Gamma_{i,j} = (E_i \setminus (E_i \cap T)) \cup \partial T|_\gamma \cup (E_j \setminus (E_j \cap T)).$$

We can take the orientation such that $\Gamma_{i,j}$ is homologous to $E_i - E_j$ in X_n . Hence we have

$$i_*([\Gamma_{i,j}]) = e_i - e_j.$$



Since E_i and E_j are the inverse image of the points P_i and P_j respectively, we have

$$\int_{E_i \setminus (E_i \cap T)} \omega = \int_{E_j \setminus (E_j \cap T)} \omega = 0.$$

Therefore

$$\int_{\Gamma_{i,j}} \omega = \int_{\partial T|_\gamma} \omega.$$

By the residue formula, we have

$$\begin{aligned} \int_{\partial T|_\gamma} \omega &= 2\pi i \int_\gamma \text{Res}_C \omega \\ &= \frac{1}{2\pi i} \int_\gamma \frac{dz}{z} \\ &= \frac{1}{2\pi i} \int_{t_i}^{t_j} \frac{dz}{z} \\ &= \frac{1}{2\pi i} \log \frac{t_j}{t_i} \pmod{\mathbb{Z}}, \end{aligned}$$

where t_i and t_j are the affine coordinates of the points B_i and B_j respectively. Then we now have

$$\begin{aligned} \chi(e_i - e_j) &= \exp 2\pi i \int_{\Gamma_{i,j}} \omega \\ &= \frac{t_j}{t_i} \end{aligned}$$

The important point is that this is the cross ratio of $C \cap F_0, C \cap F_\infty, B_j$ and B_i . Thus we have the theorem of Torelli type.

THEOREM. Let X_n and X'_n be the surfaces defined in §1 and let D and D' be anti-canonical divisors of $\#$ -type on X_n and X'_n respectively (cf. notation in §1). Let denote root lattices by Q and Q' , root systems by R and R' , and characters by χ and χ' defined as in §2 for X_n and X'_n respectively. If $\varphi : H_2(X_n; \mathbb{Z}) \rightarrow H_2(X'_n; \mathbb{Z})$ is an isometry such that

- (1) $\varphi([F_i]) = [F'_i]$,
- (2) $\varphi([C]) = [C']$,
- (3) $\varphi(R) = R'$,
- (4) $\varphi^*(\chi') = \chi$,

then there exists a unique isomorphism $\Phi : X_n \rightarrow X'_n$ which maps F_i to F'_i and C to C' and induces φ .

REFERENCE

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