

Connection Between the State Feedback L_2/L_1 and H_2 Optimal Control Problems for Continuous-Time Linear Periodically Time-Varying Systems

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Abstract—This paper gives an insightful perspective on the optimal control problems for continuous-time linear periodically time-varying (LPTV) systems, newly formulating/solving the problem of finding the optimal periodic state feedback controller minimizing the $L_{2,2}/L_{1,p'}$ induced norm of the closed-loop system (where the parameter $p' \in [1, \infty]$ is associated with the choice of the spatial norm in evaluating the disturbance at each time). Exploring further on our recent analysis on the $L_{2,2}/L_{1,p'}$ induced norm, the solution to this problem is revealed to be independent of p' and also optimal in the sense of the classical H_2 norm. This result, which is new even in the linear time-invariant (LTI) case, is established by exploiting the key feature that the $L_{2,2}/L_{1,p'}$ induced norm is also related to the observability Gramian and the corresponding periodic Lyapunov equation (PLE). A practical advantage of establishing this fact is further discussed, including the role of p' .

Index Terms—Periodic Systems, H_2 Norm, Induced Norm, Periodic Riccati Equation, Duality.

I. INTRODUCTION

THIS paper is concerned with the control of continuous-time linear periodically time-varying (LPTV) systems, which are quite common and contain a number of practical systems such as spacecraft [1], rotary wings [2], power systems [3], and so on [4], [5].

The norms defined for dynamical systems play a fundamental role in robust control theory [6]. Regarding the *analysis* of performance measures for *linear time-invariant* (LTI) systems, the L_2/L_1 induced norm is equal to the H_2 norm in the single-input case (and the L_∞/L_2 induced norm equals the H_2 norm in the single-output case) [7]. This property on the L_2/L_1 induced norm no longer holds for general (i.e.,

multi-input) LTI systems as well as LPTV systems, and thus the relationship between the controller *synthesis* problems of minimizing the L_2/L_1 induced norm and the H_2 norm is nontrivial, in general.

This paper considers the state feedback case for h -periodic systems (meaning that their coefficient matrices in the state equations have period h), and establishes a crucial fact that a class of optimal control problems share the same optimal h -periodic state feedback gain, provided that the class consists of problems whose associated performance measure can be translated to a representation through the observability Gramian of the closed-loop system in a monotonic fashion defined in an appropriate sense. In particular, this main result implies that minimizations of the L_2/L_1 induced norm and the H_2 norm are achieved by a common optimal state feedback gain. In this regard, it should be noted that taking an appropriate performance measure to be reduced is generally quite important; if an inadequate measure is considered in controller synthesis, the resulting closed-loop system could exhibit unsatisfactory performance. The aforementioned main result is hence practically significant, as well as theoretically, because it greatly alleviates the puzzles in determining which specific performance measure to be minimized in achieving satisfactory controller synthesis.

In addition, we also consider the output estimation problems for LPTV systems, in which the L_∞/L_2 induced norm and H_2 norm of the associated error system are to be minimized. In the face of the fact that the dual space of L_∞ is *not* L_1 [8], however, laying a foundation for a duality-like viewpoint with the L_2/L_1 induced norm treatment is an important and challenging issue. In this regard, exploiting the relevance of the L_2/L_1 induced norm to the observability Gramian, rather than directly working on the L_1 and L_∞ spaces, plays a key role because it then can be related with the controllability Gramian in a dual fashion, to which the L_∞/L_2 induced norm can also be related, as the recent result in [9] shows. Through such elaborated arguments, we eventually clarify that a duality-like viewpoint can indeed be established, leading to our second main result showing that the L_∞/L_2 induced norm and H_2 norm minimization problems share the same optimal h -periodic estimator gain for h -periodic systems.

The notation used in this paper is as follows. We use $\text{sq}(M)$ to mean $M^T M$ for a real matrix M , and for $h > 0$, the symbol $\mathbb{S}_+^n[0, h)$ denotes the set of h -periodic piecewise continuous matrix-valued functions $M(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ such that $M(t)$ is positive semidefinite all the time. We denote by $\|\cdot\|_{L_{p,p'}}$

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This is the author's accepted manuscript of an article published in *IEEE Control Systems Letters*, vol. 10, pp. 1231–1236, 2026, doi: 10.1109/LCSYS.2026.3704413. Aside from the correction of a few minor typographical errors, this manuscript is identical in content to the version of record.

This work was funded in part by the JSPS KAKENHI Grant 25K07793 and 26KJ1523.

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the $L_{p,p'}$ norm of a vector-valued function $f : [0, \infty) \rightarrow \mathbb{R}^N$ defined as

$$\|f\|_{L_{p,p'}} := \begin{cases} \left(\int_0^\infty |f(t)|_{p'}^p dt \right)^{1/p} & (1 \leq p < \infty), \\ \text{ess sup}_{t \in [0, \infty)} |f(t)|_{p'} & (p = \infty), \end{cases} \quad (1)$$

where $|\cdot|_{p'}$ is the p' -norm on $\mathbb{R}^N$ given by

$$|v|_{p'} := \begin{cases} \left(\sum_{i=1}^N |v_i|^{p'} \right)^{1/p'} & (1 \leq p' < \infty), \\ \max_{i=1, \dots, N} |v_i| & (p' = \infty). \end{cases} \quad (2)$$

The subscripts p and p' can represent practical concerns in controller synthesis, where determining what sort of signals should be regarded small or large as a design strategy is important. Taking $p = \infty$ corresponds to determining the size of the signal by its absolute peak value in time, and as p reduces, averaged behavior of the signal rather than the time-specific peak becomes more prominent until the size is determined in the other extreme by the time-integrated absolute value for $p = 1$. The role of p' is similar for vector-valued signals, where we are concerned with the *spatial* peak or average of the elementwise values at each time, instead of the *temporal* ones relevant to p . We denote by $L_{p,p'}$ the normed space of f such that $\|f\|_{L_{p,p'}}$ is well-defined. We sometimes drop p' and denote it simply by L_p . For a positive semidefinite matrix M and $p' \in [1, \infty]$, we define

$$\mu_{p'}(M) := \max_{|v|_{p'} \leq 1} v^T M v. \quad (3)$$

II. L_2/L_1 AND H_2 OPTIMAL STATE FEEDBACK CONTROL PROBLEMS FOR LPTV SYSTEMS

A. Problem Formulations

This section deals with the LPTV generalized plant $\mathcal{P}$ with the disturbance input w , control input u , and performance output z described by

$$\mathcal{P} : \begin{cases} \dot{x}(t) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t), \\ z(t) = C_1(t)x(t) + D_{12}(t)u(t), \end{cases} \quad (4)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $w(t) \in \mathbb{R}^{n_w}$, and $z(t) \in \mathbb{R}^{n_z}$. We assume that each of the coefficient matrices is piecewise continuous and h -periodic, the state x is accessible for feedback control, and the following assumptions are satisfied.

Assumption 2.1: The pair $(A(t), B_2(t))$ is stabilizable.

Assumption 2.2: $\forall t \in \mathbb{R}$, $R(t) := D_{12}(t)^T D_{12}(t) > 0$.

Assumption 2.3: For $\hat{A}(t)$ and $\hat{C}(t)$ defined as

$$\hat{A}(t) := A(t) - B_2(t)R(t)^{-1}D_{12}(t)^T C_1(t), \quad (5)$$

$$\hat{C}(t) := C_1(t) - D_{12}(t)R(t)^{-1}D_{12}(t)^T C_1(t), \quad (6)$$

each of the characteristic multipliers (i.e., the eigenvalues of the monodromy matrix [4]) of $\hat{A}(t)$ on the unit circle, if one exists, is $(\hat{A}(t), \hat{C}(t))$ -observable.

Roughly speaking, Assumption 2.3 corresponds to the situation in discrete-time LTI systems where the so-called PBH

test is applied only for the eigenvalues of the matrix A on the unit circle. This is weaker than assuming that $(\hat{A}(t), \hat{C}(t))$ is detectable. See [10], [11] for more details on the relevant notions in LPTV systems.

Let us apply the h -periodic static state feedback controller $u(t) = K(t)x(t)$ to the generalized plant $\mathcal{P}$. Then, the h -periodic closed-loop system $\mathcal{G}$ with the disturbance input w and the performance output z is described by

$$\mathcal{G} : \begin{cases} \dot{x}(t) = \tilde{A}(t)x(t) + B_1(t)w(t), \\ z(t) = \tilde{C}(t)x(t), \end{cases} \quad (7)$$

where

$$\tilde{A}(t) := A(t) + B_2(t)K(t), \quad (8)$$

$$\tilde{C}(t) := C_1(t) + D_{12}(t)K(t). \quad (9)$$

Assumption 2.1 is equivalent to assuming the existence of a matrix-valued h -periodic function $K(t)$ such that $A(t) + B_2(t)K(t)$ is stable, i.e., the associated monodromy matrix of $\mathcal{G}$ is Schur stable. Under such $K(t)$, the closed-loop system $\mathcal{G}$ (and $x(0) = 0$) determines for each $p' \in [1, \infty]$ the bounded linear operator mapping the input $w \in L_{1,p'}$ to the output $z \in L_{2,2}$. The $L_{2,2}/L_{1,p'}$ induced norm of the system $\mathcal{G}$ is defined as the norm of this operator:

$$\|\mathcal{G}\|_{L_{2,2}/L_{1,p'}} := \sup_{\|w\|_{L_{1,p'}} \leq 1} \|z\|_{L_{2,2}}. \quad (10)$$

It is also simply referred to as the L_2/L_1 induced norm when the underlying p' is not of particular importance, and is known [12] to be given by

$$\|\mathcal{G}\|_{L_{2,2}/L_{1,p'}} = \sup_{\tau \in [0, h)} \mu_{p'}^{1/2}(F(\tau)). \quad (11)$$

Here, $F(\tau) \in \mathbb{R}^{n_w \times n_w}$ is defined as

$$F(\tau) := B_1(\tau)^T W_o(\tau) B_1(\tau), \quad (12)$$

where $W_o(\tau)$ is the observability Gramian of the system $\mathcal{G}$, which admits the representation

$$W_o(\tau) = \int_\tau^\infty \text{sq} \left(\tilde{C}(\theta) \tilde{\Phi}(\theta; \tau) \right) d\theta \quad (13)$$

with the state transition matrix $\tilde{\Phi}(\theta; \tau) \in \mathbb{R}^{n \times n}$ corresponding to $\tilde{A}(t)$. The observability Gramian $W_o(\tau)$ can also be characterized as the symmetric and h -periodic solution of the following periodic Lyapunov equation (PLE), where the solution uniquely exists [10], [13] for $\mathcal{G}$.

$$-\dot{P}(t) = \tilde{A}(t)^T P(t) + P(t) \tilde{A}(t) + \tilde{C}(t)^T \tilde{C}(t) \quad (14)$$

On the other hand, the time-domain definition of the H_2 norm [13], [14] of the system $\mathcal{G}$ is given as

$$\|\mathcal{G}\|_{H_2} = \left(\frac{1}{h} \int_0^h \text{tr}(F(\tau)) d\tau \right)^{1/2}, \quad (15)$$

where $\text{tr}(\cdot)$ denotes the trace of a square matrix.

Problem 2.4 (The L_2/L_1 optimal control problem): Given $p' \in [1, \infty]$, find an h -periodic static controller

$u(t) = K(t)x(t)$ stabilizing the closed-loop system $\mathcal{G}$ and minimizing the $L_{2,2}/L_{1,p'}$ induced norm (11).

Problem 2.5 (The H_2 optimal control problem [15]): Find an h -periodic static controller $u(t) = K(t)x(t)$ stabilizing the closed-loop system $\mathcal{G}$ and minimizing the H_2 norm (15).

The relationship among the optimal state feedback gains for these control problems would not be obvious even when the underlying generalized plant $\mathcal{P}$ is time-invariant, for which $F(\tau)$ becomes independent of τ . More specifically, given a (constant) stabilizing state feedback gain K , the $L_{2,2}/L_{1,p'}$ induced norm (11) depends on p' , and no p' corresponds to the H_2 norm (15), in general, except for the special case where $\mathcal{P}$ is a single-input system [7]. The significant contribution of the present paper lies in newly formulating/solving Problem 2.4 and employing our recent result [12] to establish the crucial fact that Problems 2.4 and 2.5 share the same optimal control law, despite the fact that the L_2/L_1 induced norm and H_2 norm of a given LPTV systems are not directly related with each other. The main result is summarized formally as follows.

Theorem 2.6: The h -periodic state feedback gain

$$K_+(t) := -R(t)^{-1}(B_2(t)^T P_+(t) + D_{12}(t)^T C_1(t)) \quad (16)$$

stabilizes the closed-loop system $\mathcal{G}$ and minimizes the $L_{2,2}/L_{1,p'}$ induced norm regardless of $p' \in [1, \infty]$, where the matrix-valued function $P_+(t)$ is given as the unique stabilizing solution of the periodic Riccati equation (PRE)

$$\begin{aligned} -\dot{P}(t) &= \hat{A}(t)^T P(t) + P(t) \hat{A}(t) \\ &\quad - P(t) \hat{B}(t) \hat{B}(t)^T P(t) + \hat{C}(t)^T \hat{C}(t) \end{aligned} \quad (17)$$

with $\hat{A}(t)$ and $\hat{C}(t)$ defined in (5) and (6), respectively, and $\hat{B}(t)$ defined as

$$\hat{B}(t) := B_2(t)R(t)^{-1/2}. \quad (18)$$

Furthermore, the controller $u(t) = K_+(t)x(t)$ also minimizes the H_2 norm of the closed-loop system $\mathcal{G}$; in other words, Problems 2.4 and 2.5 share the same optimal state feedback gain (16).

The following Section II-B is devoted to the proof, together with describing what is meant by the stabilizing solution of the PRE in the statement of the above theorem.

B. Proof of the Main Result and Its Corollary

The proof of Theorem 2.6 is completed by deriving the optimal control law for each of the control problems, which is carried out in a deeply related fashion to the arguments in the celebrated review article [14] on LPTV systems.

In the minimization of the norms given by (11) and (15), the feedback gain $K(t)$ affects only the observability Gramian $W_o(\tau)$, through which $F(\tau)$ is defined in (12). We thus focus on the PLE (14) giving an alternative characterization of the Gramian $W_o(\tau)$. With $\hat{A}(t)$ and $\hat{C}(t)$ defined in (5) and (6), respectively, together with the stabilizing state feedback gain $K(t)$, the PLE (14) for the closed-loop system $\mathcal{G}$ can be rewritten as

$$\begin{aligned} -\dot{P}(t) &= \hat{A}(t)^T P(t) + P(t) \hat{A}(t) \\ &\quad - P(t) \hat{B}(t) \hat{B}(t)^T P(t) + \hat{C}(t)^T \hat{C}(t) + Q_{K(t),P(t)}, \end{aligned} \quad (19)$$

where (the only $K(t)$ -dependent) $Q_{K(t),P(t)}$ is defined as

$$Q_{K(t),P(t)} := \text{sq}(D_{12}(t)(K(t) + R(t)^{-1}(B_2(t)^T P(t) + D_{12}(t)^T C_1(t))). \quad (20)$$

The key role for our further arguments is played by the monotonicity property in the periodic Riccati equation (PRE) described in Theorem 6.10 in [10], whose central assertion used in our arguments can be stated in the following fashion.

Lemma 2.7 (Monotonicity property in PRE [10]):

Suppose that the stabilizable pair $(\hat{A}(t), \hat{B}(t))$ with the period h is given. Then, the PRE

$$\begin{aligned} -\dot{P}(t) &= \hat{A}(t)^T P(t) + P(t) \hat{A}(t) \\ &\quad - P(t) \hat{B}(t) \hat{B}(t)^T P(t) + Q(t) \end{aligned} \quad (21)$$

has a unique strong solution $P^s(t)$ for each given $Q(\cdot) \in \mathbb{S}_+^n[0, h)$, where a strong solution $P^s(t)$ is defined as the symmetric and h -periodic solution such that every characteristic multipliers of the periodic function $\hat{A}(t) - \hat{B}(t) \hat{B}(t)^T P^s(t)$ lies in the closed unit disk. Let $P_1^s(t)$ and $P_2^s(t)$ be the strong solutions of the PRE with the h -periodic $Q(t) = Q_1(t)$ and $Q(t) = Q_2(t)$, respectively. If $Q_1(t) \geq Q_2(t)$ for each t (i.e., $Q_1 - Q_2 \in \mathbb{S}_+^n[0, h)$), then $P_1^s(t) \geq P_2^s(t)$ for each t (i.e., $P_1^s - P_2^s \in \mathbb{S}_+^n[0, h)$).

To exploit the above monotonicity property in our arguments, we begin by considering the following PRE obtained by eliminating $Q_{K(t),P(t)}$ in (19):

$$\begin{aligned} -\dot{P}(t) &= \hat{A}(t)^T P(t) + P(t) \hat{A}(t) \\ &\quad - P(t) \hat{B}(t) \hat{B}(t)^T P(t) + \hat{C}(t)^T \hat{C}(t) \end{aligned} \quad (22)$$

By Assumption 2.1, $(\hat{A}(t), \hat{B}(t))$ is also stabilizable, and thus the PRE (22) has a unique strong solution $P^s(t)$. In addition, by Assumption 2.3, the strong solution $P^s(t)$ of (22) is a stabilizing solution [10], where a strong solution $P^s(t)$ is called a stabilizing solution if the periodic function $\hat{A}(t) - \hat{B}(t) \hat{B}(t)^T P^s(t)$ is stable. Let us denote the unique (strong and) stabilizing solution of (22) by $P_+(t)$, and let

$$K_+(t) := -R(t)^{-1}(B_2(t)^T P_+(t) + D_{12}(t)^T C_1(t)), \quad (23)$$

with which we have

$$Q_{K_+(t),P_+(t)} = 0 \quad (24)$$

by the definition (20). This implies that the differential equation (19) with $K(t)$ set to $K_+(t)$, shown below, has a unique stabilizing solution $P_+(t)$.

$$\begin{aligned} -\dot{P}(t) &= \hat{A}(t)^T P(t) + P(t) \hat{A}(t) \\ &\quad - P(t) \hat{B}(t) \hat{B}(t)^T P(t) + \hat{C}(t)^T \hat{C}(t) + Q_{K_+(t),P(t)} \end{aligned} \quad (25)$$

In particular, the associated consequence that the periodic function $\hat{A}(t) - \hat{B}(t) \hat{B}(t)^T P_+(t)$ is stable immediately implies the stability of the closed-loop system $\mathcal{G}$ with the state feedback $u(t) = K_+(t)x(t)$ (i.e., $K_+(t)$ is a stabilizing state feedback gain) because $A(t) + B_2(t)K_+(t) = \hat{A}(t) - \hat{B}(t) \hat{B}(t)^T P_+(t)$ follows from direct computations. In addition, since (19) is equivalent to the PLE (14), the unique stabilizing solution $P_+(t)$ of (25) (i.e., (19) with $K(t) =$

$K_+(t)$ is the unique periodic solution of the PLE (14), i.e., the observability Gramian of the closed-loop system $\mathcal{G}$ with the state feedback $u(t) = K_+(t)x(t)$. This together with (12) implies that with $K(t) = K_+(t)$, the L_2/L_1 induced norm and H_2 norm of $\mathcal{G}$ described by (11) and (15), respectively, are given by

$$\|\mathcal{G}\|_{L_2,2/L_1,p'} = \sup_{\tau \in [0,h)} \mu_{p'}^{1/2}(B_1(\tau)^T P_+(\tau) B_1(\tau)), \quad (26)$$

$$\|\mathcal{G}\|_{H_2} = \left(\frac{1}{h} \int_0^h \text{tr}(B_1(\tau)^T P_+(\tau) B_1(\tau)) d\tau \right)^{1/2}. \quad (27)$$

This implies the optimality of $K_+(t)$ in the L_2/L_1 (regardless of $p' \in [1, \infty]$) as well as the H_2 sense. To see this, we first note that for each stabilizing state feedback gain $K(t) = K'(t)$ and the associated unique periodic solution $P'(t)$ to the PLE (14), the definition (20) for $Q_{K(t),P(t)}$ implies that $Q_{K'(\cdot),P'(\cdot)} \in \mathbb{S}_+^n[0, h)$. This together with (24) and Lemma 2.7 implies that $P'(t) \geq P_+(t)$ and thus $B_1(t)^T P'(t) B_1(t) \geq B_1(t)^T P_+(t) B_1(t)$ for every $t \in [0, h)$. This completes the proof of Theorem 2.6.

The last part of the proof can readily establish a stronger assertion for a more general class of control problems.

Corollary 2.8: If the map $\nu : \mathbb{S}_+^n[0, h) \rightarrow \mathbb{R}$ is such that

$$\nu(M_1) \leq \nu(M_2) \text{ whenever } M_2 - M_1 \in \mathbb{S}_+^n[0, h), \quad (28)$$

then, for whatever state feedback control problem in which the cost function to be minimized is either directly or equivalently described as $\nu(W_o(\cdot))$ through the observability Gramian of the closed-loop system, the optimal state feedback gain is once again $K_+(t)$ given by (16).

Remark 2.9: In addition to the L_2/L_1 induced norm and H_2 norm, the worst-timing-type H_2 norm defined in [16] as

$$\|\mathcal{G}\|_{H_2^*} = \sup_{\tau \in [0, h)} \text{tr}^{1/2}(F(\tau)) \quad (29)$$

is also equivalently described as $\nu(W_o(\cdot))$ with a map ν satisfying (28), and thus the same optimal gain is shared.

III. L_∞/L_2 AND H_2 OPTIMAL OUTPUT ESTIMATION PROBLEMS FOR LPTV SYSTEMS

This section studies the output estimation problems of LPTV systems, where the latent and unmeasurable output must be estimated from the available information on the input and the measurement output. The estimation error must then be evaluated and minimized, where the choice of the norm arises and a different choice could lead to different estimator performance evaluation for a fixed estimator gain. Yet, this section is devoted to showing that the optimal estimator gain does not depend on the choice of the measure (i.e., the norm) for evaluating the error, as long as it is directly or equivalently described through the controllability Gramian W_c as $\nu(W_c)$ with a map ν satisfying (28).

Although such arguments are indeed motivated from the preceding arguments on state feedback and through the viewpoint of duality, it should be stressed that the arguments never proceed without employing our recent result on the analysis of

the L_2/L_1 induced norm [12] and L_∞/L_2 induced norm [9]. In fact, the following arguments evaluates the estimation error in the L_∞ norm, which one might consider associating with the L_1 norm for the input w considered in the preceding section through duality, because 1 and ∞ are conjugate indices to each other. However, it is quite important to note the well-known fact that the dual space of L_∞ is *not* L_1 [8]. This observation hence suggests a serious risk of relying on a merely superficial and ungrounded ‘duality concept’ one might imagine, and does surely indicate the importance of careful attention and arguments on what ‘duality’ should precisely mean or how it should be interpreted in this present context. This section is devoted to providing solid arguments through which the ‘duality’ in this particular context can be better interpreted and understood. Simply put, we must proceed deeper to the recent advances on the L_2/L_1 and L_∞/L_2 analysis of LPTV systems [9], [12], at the level of which we can associate the L_2/L_1 control and L_∞/L_2 estimation problems by their key monotonicity properties of Gramians; one the observability and the other the controllability Gramians, which are undoubtedly mutually dual in the standard sense.

In the following, we consider the generalized plant $\mathcal{P}$ with the disturbance input w , control input u , performance output z , and measurement output y described by

$$\mathcal{P} : \begin{cases} \dot{x}(t) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t), \\ z(t) = C_1(t)x(t) + D_{12}(t)u(t), \\ y(t) = C_2(t)x(t) + D_{21}(t)w(t), \end{cases} \quad (30)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $w(t) \in \mathbb{R}^{n_w}$, $z(t) \in \mathbb{R}^{n_z}$, and $y(t) \in \mathbb{R}^{n_y}$. We assume that each of the coefficient matrices is piecewise continuous and h -periodic, and the following assumptions are satisfied.

Assumption 3.1: The pair $(A(t), C_2(t))$ is detectable.

Assumption 3.2: $\forall t \in \mathbb{R}$, $R(t) := D_{21}(t)D_{21}(t)^T > 0$.

Assumption 3.3: For $\hat{A}(t)$ and $\hat{B}(t)$ defined as

$$\hat{A}(t) := A(t) - B_1(t)D_{21}(t)^T R(t)^{-1} C_2(t), \quad (31)$$

$$\hat{B}(t) := B_1(t) - B_1(t)D_{21}(t)^T R(t)^{-1} D_{21}(t), \quad (32)$$

each of the characteristic multipliers of $\hat{A}(t)$ on the unit circle, if one exists, is $(\hat{A}(t), \hat{B}(t))$ -reachable.

A similar comment for Assumption 2.3 applies to Assumption 3.3. With the above settings, we study the problem of estimating the performance output z from the information on the measurement output y and the control input u . The goal is to optimally reject the effect of the disturbance input w in the estimation in the sense of the L_∞/L_2 induced norm precisely described later. Specifically, let us consider constructing the estimated performance output $\hat{z}$ with the estimator

$$\begin{cases} \dot{\hat{x}}(t) = A(t)\hat{x}(t) + B_2(t)u(t) - L(t)\eta(t), \\ \hat{z}(t) = C_1(t)\hat{x}(t) + D_{12}(t)u(t), \\ \hat{y}(t) = C_2(t)\hat{x}(t), \end{cases} \quad (33)$$

where $L(t)$ is the h -periodic estimator gain, and $\eta := y - \hat{y}$. By further defining $\xi := x - \hat{x}$ and $\zeta := z - \hat{z}$, we can describe

the error system $\mathcal{E}$ by

$$\mathcal{E} : \begin{cases} \dot{\xi}(t) = \tilde{A}(t)\xi(t) + \tilde{B}(t)w(t), \\ \zeta(t) = C_1(t)\xi(t), \end{cases} \quad (34)$$

where

$$\tilde{A}(t) := A(t) + L(t)C_2(t), \quad (35)$$

$$\tilde{B}(t) := B_1(t) + L(t)D_{21}(t). \quad (36)$$

By Assumption 3.1, there exists an estimator gain $L(t)$ such that the above periodic function $\tilde{A}(t)$ is stable, i.e., the error system $\mathcal{E}$ is stable. Under such $L(t)$, the $L_{\infty, p'}/L_{2,2}$ induced norm of $\mathcal{E}$ is characterized in [9] as

$$\|\mathcal{E}\|_{L_{\infty, p'}/L_{2,2}} = \sup_{\tau \in [0, h]} \mu_{q'}^{1/2}(G(\tau)), \quad (37)$$

where q' is the conjugate index of $p' \in [1, \infty]$. Here, $G(\tau) \in \mathbb{R}^{n_z \times n_z}$ is defined as

$$G(\tau) := C_1(\tau)W_c(\tau)C_1(\tau)^T \quad (38)$$

with the controllability Gramian $W_c(\tau)$ of the system $\mathcal{E}$, which admits the representation

$$W_c(\tau) = \int_{-\infty}^{\tau} \text{sq} \left(\tilde{B}(\theta)^T \tilde{\Phi}(\tau; \theta)^T \right) d\theta \quad (39)$$

with the state transition matrix $\tilde{\Phi}(\tau; \theta) \in \mathbb{R}^{n \times n}$ associated with $\tilde{A}(t)$. The controllability Gramian can also be characterized as the unique symmetric and h -periodic solution of the PLE [13]

$$\dot{P}(t) = P(t)\tilde{A}(t)^T + \tilde{A}(t)P(t) + \tilde{B}(t)\tilde{B}(t)^T. \quad (40)$$

On the other hand, the time-domain definition of the H_2 norm [13], [14] of the system $\mathcal{E}$ can be described as

$$\|\mathcal{E}\|_{H_2} = \left(\frac{1}{h} \int_0^h \text{tr}(G(\tau)) d\tau \right)^{1/2}. \quad (41)$$

Now we are in a position to formulate the L_{∞}/L_2 and H_2 optimal output estimation problems.

Problem 3.4 (The L_{∞}/L_2 optimal estimation problem): Given $p' \in [1, \infty]$, find an h -periodic estimator gain $L(t)$ stabilizing the error system $\mathcal{E}$ and minimizing the $L_{\infty, p'}/L_{2,2}$ induced norm (37).

Problem 3.5 (The H_2 optimal estimation problem): Find an h -periodic estimator gain $L(t)$ stabilizing the error system $\mathcal{E}$ and minimizing the H_2 norm (41).

Regarding the above two estimation problems, the following result holds.

Theorem 3.6: The h -periodic estimator gain

$$L_+(t) := -(P_+(t)C_2(t)^T + B_1(t)D_{21}(t)^T)R(t)^{-1} \quad (42)$$

stabilizes the error system $\mathcal{E}$ and minimizes the $L_{\infty, p'}/L_{2,2}$ induced norm regardless of $p' \in [1, \infty]$, where $P_+(t)$ is given as the unique stabilizing solution (i.e., the symmetric and h -periodic solution $P(t)$ such that $\hat{A}(t) - P(t)\hat{C}(t)^T\hat{C}(t)$ is stable) of the PRE

$$\begin{aligned} \dot{P}(t) = & P(t)\hat{A}(t)^T + \hat{A}(t)P(t) \\ & - P(t)\hat{C}(t)^T\hat{C}(t)P(t) + \hat{B}(t)\hat{B}(t)^T, \end{aligned} \quad (43)$$

with $\hat{A}(t)$ and $\hat{B}(t)$ defined in (31) and (32), respectively, and $\hat{C}(t)$ defined as $R(t)^{-1/2}C_2(t)$. Furthermore, the estimator gain $L_+(t)$ also minimizes the H_2 norm of the error system $\mathcal{E}$; in other words, Problems 3.4 and 3.5 share the same optimal estimator gain (42).

Proof: A brief sketch of the proof is as follows. The recent analysis [9] shows that L_{∞}/L_2 induced norm (37) can be expressed with the controllability Gramian $W_c(\tau)$; recall (38). Since the controllability Gramian $W_c(\tau)$ allows the dual treatment to the observability Gramian $W_o(\tau)$ in the usual sense, the arguments based on the monotonicity property of the PLE (40) corresponding to the controllability Gramian $W_c(\tau)$ can be developed in much the same way as those in the preceding section through the monotonicity property of the PLE (14) associated with the observability Gramian. Thus, the above theorem is eventually established. ■

IV. NUMERICAL EXAMPLE

This section deals with the following system described by the lossy (damped) Mathieu equation [5] (with the period $h = 2\pi$), which appears in many engineering fields.

$$\ddot{x}_1(t) + \gamma \dot{x}_1(t) + (\alpha - \beta \cos t)x_1(t) = b_1 w_1(t) + v(t) \quad (44)$$

Here, w_1 is the disturbance and v is the manipulated variable assumed to be generated by an actuator with the first order dynamics

$$v(t) = x_3(t)/a, \quad \dot{x}_3(t) = -ax_3(t) + b_2 w_2(t) + u(t) \quad (45)$$

subject to another disturbance w_2 . The actuator input $u(t)$ is assumed to be generated by the h -periodic state feedback gain $K(t) = [k_1(t) \ k_2(t) \ k_3(t)]$ for $x_2(t) := \dot{x}_1(t)$. Let us consider $z := [c_1 x_1 \ c_2 x_2 \ c_3 u]^T$, together with the closed-loop LPTV system $\mathcal{G} : w := [w_1 \ w_2]^T \mapsto z$, where the above performance output signal z is assumed to be taken as a signal such that the ‘smaller’ it is, the ‘better performance’ we may regard the state feedback gain $K(t)$ has achieved; however, the precise meaning of ‘smaller’ and ‘better performance’ may not be clear enough at this stage. We assume the parameters $\alpha = 2$, $\beta = 2$, $\gamma = 1$, $a = 3$, $b_1 = 1$, $b_2 = 10$, $c_1 = 10$, $c_2 = 10$, and $c_3 = 1$. In this case, Assumptions 2.1, 2.2, and 2.3 for the plant are satisfied.

Let us consider the system $\mathcal{G}_+$ with $K(t) = K_+(t)$ defined by (16) and calculated numerically through the unique stabilizing solution of the PRE (17) obtained by the algorithm called the multi-shot method [17]. For the sake of comparison, let us further consider a fictitious situation where we solidly had an alternative objective in which what should be small were rather a different performance output taken as $z_{\text{stab}} := [30x_1 \ 5x_2 \ u]^T$. This example treats the corresponding $K_+(t)$ obtained for z_{stab} as an example of stabilizing gains for $\mathcal{G}$, which was defined to have the original performance output z rather than z_{stab} , leading to the closed-loop system denoted by $\mathcal{G}_{\text{stab}}$. The matrix-valued function $F(\tau) = B_1(\tau)^T P(\tau) B_1(\tau)$ for the two systems are denoted by $F_+(\tau)$ and $F_{\text{stab}}(\tau)$, respectively, where $P(t)$ is the observability Gramian for each closed-loop system.

This section deals with three types of norms: the $L_{2,2}/L_{1,p'}$ induced norms $\|\mathcal{G}\|_{L_{2,2}/L_{1,p'}}$ for $p' = 1, 2$ and the H_2 norm $\|\mathcal{G}\|_{H_2}$. Fig. 1 shows $\mu_1^{1/2}(F(\tau))$ (blue), $\mu_2^{1/2}(F(\tau))$ (orange), and $\text{tr}^{1/2}(F(\tau))$ (green), contained in the expressions (11) with $p' = 1$, that with $p' = 2$, and (15), for these norms, respectively, using the fact that $\mu_{p'}^{1/2}(M)$ for $p' = 1$ and $p' = 2$ are the maximum diagonal entry and maximum eigenvalue of the positive semidefinite matrix M , respectively. The solid and dashed lines correspond to $\mathcal{G}_+$ and $\mathcal{G}_{\text{stab}}$, respectively, where the observations given below correspond to confirming the property $F_{\text{stab}}(\tau) \geq F_+(\tau)$ that follows from the arguments in Section II-B as a core behind Theorem 2.6 (recall the last part of the proof in Section II-B, immediately followed by Corollary 2.8). For each of the three pairs under the same performance index (i.e., the same color) with the original z , the curve for the optimal closed-loop system (solid) can indeed be confirmed to stay, at every $\tau \in [0, h]$, no greater than that for the other *non-optimal* closed-loop system (dashed). This figure thus enables us to observe the performance improvement by $K_+(t)$ over another stabilizing gain *regardless of the three different performance measures*. Furthermore, a similar observation applies to the fourth performance measure of the worst-timing-type H_2 norm $\|\mathcal{G}\|_{H_2^*}$ shown in (29) through the comparison of the peak values in the green solid and dashed curves.

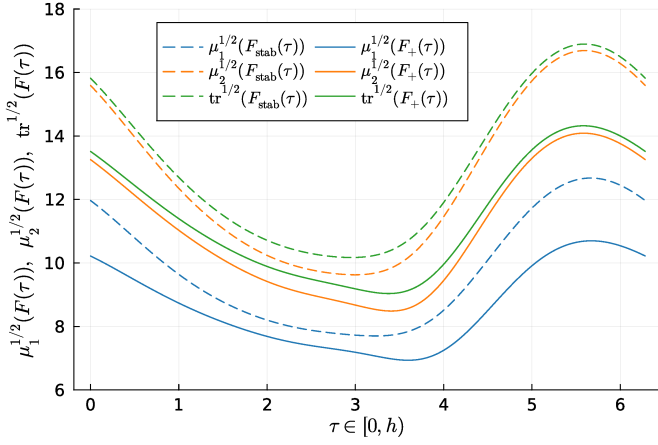


Fig. 1. Variations of $\mu_1^{1/2}(F(\tau))$, $\mu_2^{1/2}(F(\tau))$, and $\text{tr}^{1/2}(F(\tau))$.

Although numerically confirming the same for all the stabilizing state feedback gains $K(t)$ is impossible, it is important to note, as a summary of our theoretical arguments and their relevance to the present numerical example, that the same has actually been established for any other stabilizing gain $K(t)$. Therefore, the resulting gain $K_+(t)$ is optimal regardless of the four aforementioned performance measures with respect to the original performance output z , and thus it can be used as the ‘definitely optimal one’ *without being puzzled about which performance measure to minimize among these candidates, when the ‘most effective closed-loop performance’ is aimed at in controller synthesis but is often prescribed only vaguely, in practice*.

V. CONCLUSION

This paper developed arguments on the connection between the induced norms and the H_2 norm from the perspective of the norm minimization problems. Section II studied the state feedback control problems and established the crucial fact that the $L_{2,2}/L_{1,p'}$ induced norm of the closed-loop system $\mathcal{G}$ is minimized by a common optimal state feedback gain regardless of $p' \in [1, \infty]$, and the common gain further minimizes the H_2 norm of the system. Section III studied the output estimation problems, and showed that the $L_{\infty,p'}/L_{2,2}$ induced norm for $p' \in [1, \infty]$ and H_2 norm of the associated error system $\mathcal{E}$ are all minimized by a common optimal estimator gain.

More generally, the present paper showed that the optimal control gain $K_+(t)$ in the L_2/L_1 and H_2 sense given by (16) further minimizes all the cost functions described by $\nu(W_o)$ with the *observability* Gramian W_o of the closed-loop system $\mathcal{G}$ and a monotonic map ν in the sense of (28), and similar claims on the estimation problems are also established. A practical advantage of establishing such theoretical results is also discussed.

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