

A generalization of Hardy spaces on spaces of homogeneous type

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1. INTRODUCTION

This is an announcement of my recent work [10].

Let $X = (X, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss [1, 2] (see the next section for the definition). Using atoms, Coifman and Weiss [2] introduced the Hardy space $H^p(X)$. The purpose of this report is to generalize the definition of Hardy space $H^p(X)$ and prove that the generalized Hardy spaces have the same property as $H^p(X)$. Our definition includes a kind of Hardy spaces with variable exponent. The results are new even for the $\mathbb{R}^n$ case.

First we state definitions of Campanato and Hölder spaces. Let $1 \leq p < \infty$ and $\phi : X \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$. For a ball $B = B(x, r)$, we shall write $\phi(B)$ in place of $\phi(x, r)$. For a function $f \in L^1_{\text{loc}}(X)$ and for a ball B , let $f_B = \mu(B)^{-1} \int_B f(x) d\mu(x)$. Then the Campanato spaces $\mathcal{L}_{p,\phi}(X)$ and the Hölder spaces $\Lambda_\phi(X)$ are defined to be the sets of all f such that $\|f\|_{\mathcal{L}_{p,\phi}} < \infty$ and $\|f\|_{\Lambda_\phi} < \infty$, respectively, where

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p d\mu(x) \right)^{1/p},$$

$$\|f\|_{\Lambda_\phi} = \sup_{x,y \in X, x \neq y} \frac{2|f(x) - f(y)|}{\phi(x, d(x, y)) + \phi(y, d(y, x))}.$$

Let $\mathcal{C}$ be the space of all constant functions. Then $\mathcal{L}_{p,\phi}(X)/\mathcal{C}$ and $\Lambda_\phi(X)/\mathcal{C}$ are Banach spaces with the norm $\|f\|_{\mathcal{L}_{p,\phi}}$ and $\|f\|_{\Lambda_\phi}$, respectively. Campanato spaces of these type were studied in [11, 7, 8, 12, 9]. See [9] for relations among these spaces. When $p = 1$, we denote $\mathcal{L}_{1,\phi}(X)$ by $\text{BMO}_\phi(X)$. If $\phi \equiv 1$, then $\mathcal{L}_{1,\phi}(X) = \text{BMO}(X)$.

2000 *Mathematics Subject Classification*. 42B30, 46E30, 42B35, 46E15.

Key words and phrases. atom, Hardy space, Lipschitz space, variable exponent, Campanato space, Hölder space, space of homogeneous type.

For $\phi(x, r) = r^{\alpha(x)}$, $\alpha(x) > 0$, we denote $\Lambda_\phi(X)$ by $\text{Lip}_{\alpha(\cdot)}(X)$. Then

$$\|f\|_{\text{Lip}_{\alpha(\cdot)}} = \sup_{x, y \in X, x \neq y} \frac{2|f(x) - f(y)|}{d(x, y)^{\alpha(x)} + d(y, x)^{\alpha(y)}}.$$

If $\alpha(\cdot)$ satisfies a certain condition, then $\text{Lip}_{\alpha(\cdot)}(X) = \mathcal{L}_{p, \phi}(X)$ for all $p \in [1, \infty)$.

Using atoms, Coifman and Weiss [2] defined the Hardy space $H^p(X)$ as a subspace of the dual of $\text{Lip}_\alpha(X)$ and they proved that $\text{Lip}_\alpha(X)$ is the dual of $H^p(X)$. Their results are generalization of the case $X = \mathbb{R}^n$. In [2] $\text{Lip}_\alpha(X)$ was regarded as the space of functions modulo constants. Therefore, we denote by $(H^p(X))^* = \text{Lip}_\alpha(X)/\mathcal{C}$ the fact above.

In this report, using $[\phi, q]$ -atoms, we define a generalized Hardy space $H_U^{[\phi, q]}(X)$ as a subspace of the dual of $\mathcal{L}_{q', \phi}(X)/\mathcal{C}$ and prove that $\mathcal{L}_{q', \phi}(X)/\mathcal{C}$ is the dual of $H_U^{[\phi, q]}(X)$, i.e. $(H_U^{[\phi, q]}(X))^* = \mathcal{L}_{q', \phi}(X)/\mathcal{C}$, where $1 < q \leq \infty$, $1/q + 1/q' = 1$, U is a concave strictly increasing function from $[0, \infty)$ to itself and $U(0) = 0$ (see the third section for the precise definition of $H_U^{[\phi, q]}(X)$). The definition of $H^p(X)$ in [2], $0 < p \leq 1$, is a special case of ours, since $\text{Lip}_\alpha(X)$ is a special case of $\mathcal{L}_{q', \phi}(X)$.

Coifman and Weiss [2] first defined $H^{p, q}(X)$, and then proved $H^{p, q}(X) = H^{p, \infty}(X)$, which was denoted by $H^p(X)$. We will prove that $H_U^{[\phi, q]}(X) = H_U^{[\phi, \infty]}(X)$ under a certain condition. In particular, for Hardy spaces with variable exponent $p(x)$, we use the condition that $p(x)$ is log-Hölder continuous (see Corollary 4.2).

The log-Hölder continuity was used to prove boundedness of the Hardy-Littlewood maximal operator on $L^{p(x)}$, Lebesgue spaces with variable exponent, as follows.

Let $G \subset \mathbb{R}^n$ be bounded. For a function $p : G \rightarrow [1, \infty)$, let

$$L^{p(x)}(G) = \left\{ f \in L^1(G) : \int_G (c|f(x)|)^{p(x)} dx < \infty \text{ for some } c > 0 \right\}.$$

For $f \in L^{p(x)}(G)$, let

$$\|f\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_G \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

Then $\|\cdot\|_{p(x)}$ is a norm and thereby $L^{p(x)}(G)$ is a Banach space. For a function f on G , the Hardy-Littlewood maximal function of f is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B \cap G} |f(y)| dy,$$

where the supremum is taken over all balls B containing x . By the definition we have

$$\|Mf\|_\infty \leq \|f\|_\infty.$$

We say that $p(x)$ is log-Hölder continuous if

$$|p(x) - p(y)| \leq \frac{c}{|\log|x - y||} \quad \text{for } |x - y| \leq \frac{1}{2}.$$

Theorem 1.1 (Diening [3]). *If $p(x)$ is log-Hölder continuous, then the operator M is bounded on $L^{p(x)}(G)$.*

Remark 1.1. Let

$$p(x) = \begin{cases} 4 & (-1 < x \leq 0) \\ 2 & (0 < x < 1). \end{cases}$$

If $f(x) = \begin{cases} 0 & (-1 < x \leq 0) \\ x^{-1/3} & (0 < x < 1), \end{cases}$ then $Mf(x) \geq c|x|^{-1/3}$. In this case $f \in L^{p(x)}(-1, 1)$ and $Mf \notin L^{p(x)}(-1, 1)$.

2. SPACE OF HOMOGENEOUS TYPE

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e. X is a topological space endowed with a quasi-distance d and a nonnegative measure μ such that

$$d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \text{ if and only if } x = y,$$

$$d(x, y) = d(y, x),$$

$$(2.1) \quad d(x, y) \leq K_1 (d(x, z) + d(z, y)),$$

the balls (d -balls) $B(x, r) = B^d(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x , μ is defined on a σ -algebra of subsets of X which contains the balls, and

$$(2.2) \quad 0 < \mu(B(x, 2r)) \leq K_2 \mu(B(x, r)) < \infty,$$

If there are constants θ ($0 < \theta \leq 1$) and $K_3 \geq 1$ such that

$$(2.3) \quad |d(x, z) - d(y, z)| \leq K_3 (d(x, z) + d(y, z))^{1-\theta} d(x, y)^\theta, \quad x, y, z \in X,$$

then the balls are open sets. Note that (2.1) for some $K_1 \geq 1$ follows from (2.3) (Lemarié [4]). Conversely, from (2.1) it follows that there exist $\theta > 0$, $K_3 \geq 1$ and a quasi-distance which is equivalent to the original d such that (2.3) holds (Macías and Segovia [5]). Therefore We always assume (2.3) in this report.

It is known that, if $\mu(X) < +\infty$, then there is a constant $R_0 > 0$ such that

$$(2.4) \quad X = B(x, R_0) \quad \text{for all } x \in X$$

(see [12, Lemma 5.1]).

3. DEFINITIONS

Definition 3.1 ($[\phi, q]$ -atom (resp. $(p(\cdot), q)$ -atom)). Let $\phi : X \times (0, \infty) \rightarrow (0, \infty)$ and $1 < q \leq \infty$. A function a on X is called a $[\phi, q]$ -atom (resp. $(p(\cdot), q)$ -atom) if there exists a ball B such that

$$\begin{aligned} & \text{(i) } \text{supp } a \subset B, \\ & \text{(ii) } \|a\|_q \leq \frac{1}{\mu(B)^{1/q'} \phi(B)} \end{aligned}$$

(resp. $\|a\|_q \leq \mu(B)^{1/q-1/p(x)}$, where x is the center of B),

$$\text{(iii) } \int_X a(x) d\mu(x) = 0,$$

where $\|a\|_q$ is the L^q norm of a and $1/q + 1/q' = 1$. We denote by $A[\phi, q]$ the set of all $[\phi, q]$ -atoms. (We denote by $A(p(\cdot), q)$ the set of all $(p(\cdot), q)$ -atoms.)

We note that $(p(\cdot), q)$ -atoms are special cases of $[\phi, q]$ -atoms. If $p(x) \equiv p$, then the $(p(\cdot), q)$ -atom is the usual (p, q) -atom. Let $p_- = \inf p(x)$ and $p_+ = \sup p(x)$.

Remark 3.1. Assume that $\mu(B(x, r)) \sim r^Q$ ($Q > 0$) for $x \in X$ and $0 < r < \infty$ ($0 < r < R_0$ if $\mu(X) < \infty$). Let $\alpha(x) = Q(1/p(x) - 1)$. If $Q/(\theta + Q) \leq p_- \leq p_+ < 1$, then $0 < \alpha_- \leq \alpha_+ \leq \theta$ and $\text{Lip}_{\alpha(\cdot)}(X) = \mathcal{L}_{q', \phi}(X)$ for all $q' \in [1, \infty)$.

If a is a $[\phi, q]$ -atom and a ball B satisfies (i)–(iii), then

$$\begin{aligned} (3.1) \quad \left| \int_X a(x) g(x) d\mu(x) \right| &= \left| \int_B a(x) (g(x) - g_B) d\mu(x) \right| \\ &\leq \|a\|_q \left(\int_B |g(x) - g_B|^{q'} d\mu(x) \right)^{1/q'} \\ &\leq \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |g(x) - g_B|^{q'} d\mu(x) \right)^{1/q'} \\ &\leq \|g\|_{\mathcal{L}_{q', \phi}}. \end{aligned}$$

That is, the mapping $g \mapsto \int_X a g d\mu$ is a bounded linear functional on $\mathcal{L}_{q', \phi}(X)/\mathcal{C}$ with norm not exceeding 1.

Definition 3.2 ($H_U^{[\phi, q]}(X)$). Let $\phi : X \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $1 < q \leq \infty$ and $1/q + 1/q' = 1$. Let U be a continuous, concave, increasing and bijective function from $[0, +\infty)$ to itself. Assume that $\mathcal{L}_{q', \phi}(X)/\mathcal{C} \neq \{0\}$. We define the space $H_U^{[\phi, q]}(X) \subset (\mathcal{L}_{q', \phi}(X)/\mathcal{C})^*$ as follows:

$f \in H_U^{[\phi, q]}(X)$ if and only if there exist sequences $\{a_j\} \subset A[\phi, q]$ and positive numbers $\{\lambda_j\}$ such that

$$(3.2) \quad f = \sum_j \lambda_j a_j \text{ in } (\mathcal{L}_{q', \phi}(X)/\mathcal{C})^* \quad \text{and} \quad \sum_j U(\lambda_j) < \infty.$$

From $U(0) = 0$ and the concavity of U it follows that

$$(3.3) \quad U(Cr) \leq CU(r), \quad 1 \leq C < \infty, \quad 0 \leq r < \infty,$$

$$(3.4) \quad U(r+s) \leq U(r) + U(s), \quad 0 \leq r, s < \infty.$$

Then $H_U^{[\phi, q]}(X)$ is a linear space. (3.4) implies

$$(3.5) \quad \sum_j \lambda_j \leq U^{-1} \left(\sum_j U(\lambda_j) \right).$$

Therefore, if $\sum_j U(\lambda_j) < \infty$, then $\sum_j \lambda_j < \infty$ and $\sum_j \lambda_j a_j$ converges in $(\mathcal{L}_{q', \phi}(X)/\mathcal{C})^*$.

In general, the expression (3.2) is not unique. We define

$$\|f\|_{H_U^{[\phi, q]}} = \inf \left\{ U^{-1} \left(\sum_j U(\lambda_j) \right) \right\},$$

where the infimum is taken over all expressions as in (3.2). We note that $\|f\|_{H_U^{[\phi, q]}}$ is not a norm in general. Let $d(f, g) = U(\|f - g\|_{H_U^{[\phi, q]}})$ for $f, g \in H_U^{[\phi, q]}(X)$. Then $d(f, g)$ is a metric and $H_U^{[\phi, q]}(X)$ is complete with respect to this metric. If $I(r) = r$, then $\|f\|_{H_U^{[\phi, q]}}$ is a norm and $H_U^{[\phi, q]}(X)$ is a Banach space.

In the case of $(p(\cdot), q)$ -atoms instead of $[\phi, q]$ -atoms, we denote $H_U^{[\phi, q]}(X)$ by $H_U^{p(\cdot), q}(X)$.

4. RESULTS

Theorem 4.1. *If there exists a constant $C_* > 0$ such that*

$$(4.1) \quad U(rs) \leq C_* U(r)U(s) \quad \text{for } 0 < r, s \leq 1,$$

$$(4.2) \quad U \left(\frac{\mu(B_1)\phi(B_1)}{\mu(B_2)\phi(B_2)} \right) \leq C_* \frac{\mu(B_1)}{\mu(B_2)} \quad \text{for all balls } B_1 \text{ and } B_2 \text{ with } B_1 \subset B_2,$$

then

$$H_U^{[\phi, q]}(X) = H_U^{[\phi, \infty]}(X),$$

with equivalent topologies.

Corollary 4.2. Let $Q > 0$. Assume that $\mu(X) < \infty$ and that $\mu(B(x, r)) \sim r^Q$ for all $x \in X$ and $0 < r < R_0$, where R_0 is the constant in (2.4). Let $U(r) = r^{p_+}$ with $0 < p_- \leq p_+ \leq 1$, where $p_- = \inf p(x)$ and $p_+ = \sup p(x)$. If there exists a constant $C_0 > 0$ such that

$$(4.3) \quad |p(x) - p(y)| \leq \frac{C_0}{\log(1/d(x, y))} \quad \text{for } d(x, y) < 1/2,$$

then

$$H_U^{p(\cdot), q}(X) = H_U^{p(\cdot), \infty}(X),$$

with equivalent topologies.

In this case we denote $H_U^{p(\cdot), q}(X)$ by $H^{p(\cdot)}(X)$ simply, which is a kind of Hardy spaces with variable exponent.

Proof of Corollary 4.2. The inequality (4.1) holds clearly. We show (4.2).

For $B(x, r) \subset B(y, s)$,

$$\frac{U\left(\frac{\phi(x, r)\mu(B(x, r))}{\phi(y, s)\mu(B(y, s))}\right)}{\frac{\mu(B(x, r))}{\mu(B(y, s))}} \sim \left(\frac{r}{s}\right)^{Qp_+(1/p(x)-1/p_+)} s^{Qp_+(1/p(x)-1/p(y))} \leq s^{Qp_+(1/p(x)-1/p(y))},$$

since $r/s \leq 1$. If $1/2 < s < R_0$, then

$$s^{Qp_+(1/p(x)-1/p(y))} \leq R_0^{Qp_+/p_-}.$$

If $s \leq 1/2$, then $d(x, y) < s$ and

$$\begin{aligned} \log s^{Qp_+(1/p(x)-1/p(y))} &\leq Qp_+ \left| \frac{1}{p(y)} - \frac{1}{p(x)} \right| \log(1/s) \\ &\leq Qp_+ \left| \frac{p(x) - p(y)}{p(x)p(y)} \right| \log(1/d(x, y)) \leq \frac{C_0 Qp_+}{p_-^2}. \quad \square \end{aligned}$$

Lemma 4.3. Let $E = H_U^{[\phi, q]}(X)$. If

$$(4.4) \quad \sup_{0 < s \leq 1} \frac{U(rs)}{U(s)} \rightarrow 0 \quad (r \rightarrow 0),$$

then

$$\|\ell\|_{E^*} = \sup \{|\ell(f)| : \|f\|_E \leq 1\}$$

is finite for all $\ell \in E^*$, and $\|\ell\|_{E^*}$ is a norm.

Remark 4.1. If (4.1) holds, then (4.4) holds. If (4.4) holds, then there exist constants $C > 0$ and $p > 0$ such that $U(r) \leq Cr^p$ for $r \in (0, 1]$. If $\alpha > 0$ and $U(r) = (\log(1/r))^{-\alpha}$ for small $r > 0$, then U does not satisfy (4.4).

Let $L_c^q(X)$ be the set of all L^q -functions with bounded support, and let

$$L_c^{q,0}(X) = \left\{ f \in L_c^q(X) : \int_X f \, d\mu = 0 \right\}.$$

Then, for $1 < q \leq \infty$, $L_c^{q,0}(X)$ is dense in $H_U^{[\phi,q]}(X)$.

Theorem 4.4. *If U satisfies (4.4), then*

$$\left(H_U^{[\phi,q]}(X) \right)^* = \mathcal{L}_{q',\phi}(X)/\mathcal{C}.$$

More precisely, if $g \in \mathcal{L}_{q',\phi}(X)/\mathcal{C}$, then the mapping $\ell : f \mapsto \int_X f(g+c) \, d\mu$, for $f \in L_c^{q,0}(X)$, can be extended to a continuous linear functional on $H_U^{[\phi,q]}(X)$. Conversely, if ℓ is a continuous linear functional on $H_U^{[\phi,q]}(X)$, then there exists $g \in \mathcal{L}_{q',\phi}(X)/\mathcal{C}$ such that $\ell(f) = \int_X f(g+c) \, d\mu$ for $f \in L_c^{q,0}(X)$. The norm $\|\ell\|$ is equivalent to $\|g\|_{\mathcal{L}_{q',\phi}}$.

Corollary 4.5. *Assume the conditions in Remark 3.1 and Corollary 4.2. Then*

$$\left(H^{p(\cdot)}(X) \right)^* = \text{Lip}_{\alpha(\cdot)}(X)/\mathcal{C}.$$

REFERENCES

- [1] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math., vol.242, Springer-Verlag, Berlin and New York, 1971.
- [2] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.
- [3] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$* , Math. Inequal. Appl. **7** (2004), no. 2, 245–253.
- [4] P. G. Lemarié, *Algèbres d'opérateurs et semi-groupes de Poisson sur un espace de nature homogène*, Publ. Math. Orsay **84-3** (1984).
- [5] R. A. Macías and C. Segovia *Lipschitz functions on spaces of homogeneous type*, Adv. Math. **33** (1979), 257–270.
- [6] R. A. Macías and C. Segovia *A decomposition into atoms of distributions on spaces of homogeneous type*, Adv. Math. **33** (1979), 270–309.
- [7] E. Nakai, *Pointwise multipliers for functions of weighted bounded mean oscillation*, Studia Math., **105** (1993), 105–119.
- [8] E. Nakai, *Pointwise multipliers on weighted BMO spaces*, Studia Math., **125** (1997), 35–56.
- [9] E. Nakai, *The Campanato, Morrey and Hölder spaces on spaces of homogeneous type*, Studia Math., **176** (2006), 1–19.
- [10] E. Nakai, *A generalization of Hardy spaces H^p by using atoms*, to appear in Acta Math. Sinica.
- [11] E. Nakai and K. Yabuta, *Pointwise multipliers for functions of bounded mean oscillation*, J. Math. Soc. Japan, **37** (1985), 207–218.

- [12] E. Nakai and K. Yabuta, *Pointwise multipliers for functions of weighted bounded mean oscillation on spaces of homogeneous type*, Math. Japon. **46** (1997), 15–28.

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