

完全 2 組グラフの S_5 因子分解
(S_5 -FACTORIZATION OF COMPLETE BIPARTITE GRAPHS)

近畿大 理工 潮 和彦 (Kazuhiko Ushio)

Department of Industrial Engineering, Faculty of Science and Technology, Kinki University, 577 Osaka, Japan

In this paper, trivial necessary conditions for the existence of an S_5 -factorization of $K_{m,n}$ are given. Several types of construction algorithms of S_5 -factorization of $K_{m,n}$ are also given.

1. Introduction

Let S_5 be a *star* on 5 vertices and $K_{m,n}$ be a *complete bipartite graph* with partite sets V_1 and V_2 of m and n vertices each. A spanning subgraph F of $K_{m,n}$ is called an S_5 -*factor* if each component of F is isomorphic to S_5 . If $K_{m,n}$ is expressed as an edge-disjoint sum of S_5 -factors, then this sum is called an S_5 -*factorization* of $K_{m,n}$.

In this paper, trivial necessary conditions for the existence of an S_5 -factorization of $K_{m,n}$ are given. Several types of construction algorithms of S_5 -factorization of $K_{m,n}$ are also given.

2. S_5 -factor of $K_{m,n}$

The following theorem is on the existence of S_5 -factors of $K_{m,n}$.

Theorem 1. $K_{m,n}$ has an S_5 -factor if and only if (i) $m+n \equiv 0 \pmod{5}$, (ii) $4n-m \equiv 0 \pmod{15}$, (iii) $4m-n \equiv 0 \pmod{15}$, (iv) $m \leq 4n$ and (v) $n \leq 4m$.

Proof. Suppose that $K_{m,n}$ has an S_5 -factor F . Let t be the number of components of F . Then $t=(m+n)/5$. Hence, Condition (i) is necessary. Among these t components, let x and y be the number of components whose endvertices are in V_2 and V_1 , respectively. Then, since F is a spanning subgraph of $K_{m,n}$, we have $x+4y=m$ and $4x+y=n$. Hence $x=(4n-m)/15$ and $y=(4m-n)/15$. From $0 \leq x \leq m$ and $0 \leq y \leq n$, we must have $m \leq 4n$ and $n \leq 4m$. Conditions (ii)-(v) are, therefore, necessary.

For those parameters m and n satisfying (i)-(v), let $x=(4n-m)/15$ and $y=(4m-n)/15$. Then x and y are integers such that $0 \leq x \leq m$ and $0 \leq y \leq n$.

Hence, $x+4y=m$ and $4x+y=n$. Using x vertices in V_1 and $4x$ vertices in V_2 , consider x S_5 's whose endvertices are in V_2 . Using the remaining $4y$ vertices in V_1 and the remaining y vertices in V_2 , consider y S_5 's whose endvertices are in V_1 . Then these $x+y$ S_5 's are edge-disjoint and they form an S_5 -factor of $K_{m,n}$.

□

Corollary 1. $K_{m,n}$ has an S_5 -factor if and only if $n \equiv 0 \pmod{5}$.

3. S_5 -factorization of $K_{m,n}$

We use the following notations.

Notation 1. r, t, b : number of S_5 -factors, number of S_5 -components of each S_5 -factor, and total number of S_5 -components, respectively, in an S_5 -factorization of $K_{m,n}$.

t_1 (t_2) : number of components whose centers are in V_1 (V_2), respectively, among t S_5 -components of each S_5 -factor.

$r_1(u)$ ($r_2(v)$) : number of components whose centers are all u (v) for any u (v) in V_1 (V_2), respectively, among b S_5 -components.

3.1. Trivial necessary conditions of S_5 -factorization of $K_{m,n}$

We give the following trivial necessary conditions for the existence of S_5 -factorization of $K_{m,n}$.

Theorem 2. If $K_{m,n}$ has an S_5 -factorization then (i) $b=mn/4$, (ii) $t=(m+n)/5$, (iii) $r=5mn/4(m+n)$, (iv) $t_1=(4n-m)/15$, (v) $t_2=(4m-n)/15$, (vi) $r_1=(4n-m)n/12(m+n)$, (vii) $r_2=(4m-n)m/12(m+n)$, (viii) $m \leq 4n$ and (ix) $n \leq 4m$.

Proof. Suppose that $K_{m,n}$ has an S_5 -factorization. Then it holds that $b=mn/4$, $t=(m+n)/5$, $r=b/t=5mn/4(m+n)$, $t_1=(4n-m)/15$, $t_2=(4m-n)/15$, $m \leq 4n$ and $n \leq 4m$. Let $s_1(u)$ ($s_2(v)$) be the number of components which have endvertex u (v) for any u (v) in V_1 (V_2), respectively, among b S_5 -components. Then it holds that $r_1(u)+s_1(u)=r$, $4r_1(u)+s_1(u)=n$, $r_2(v)+s_2(v)=r$ and $4r_2(v)+s_2(v)=m$. Hence we have $r_1(u)=(4n-m)n/12(m+n)$ and $r_2(v)=(4m-n)m/12(m+n)$. $r_1(u)$ ($r_2(v)$) doesn't depend on u (v), respectively. Therefore, Conditions (i)-(ix) are necessary. □

Corollary 2. If $K_{m,n}$ has an S_5 -factorization then $n \equiv 0 \pmod{40}$.

3.2. Extension theorem of S_5 -factorization of $K_{m,n}$

We prove the following extension theorem, which we use later in this paper.

Theorem 3. If $K_{m,n}$ has an S_5 -factorization, then $K_{sm,sn}$ has an S_5 -factorization for every positive integer s .

Proof. Let V_1, V_2 be the independent sets of $K_{sm,sn}$, where $|V_1| = sm$ and $|V_2| = sn$. Divide V_1 and V_2 into s subsets of m and n vertices each, respectively. Construct a new graph G with a vertex set consisting of the subsets which were just constructed. In this graph, two vertices are adjacent if and only if the subsets come from disjoint independent sets of $K_{sm,sn}$. G is a complete bipartite graph $K_{s,s}$. Noting that the cardinality of each subset identified with a vertex set of G is m or n and that $K_{s,s}$ has a 1-factorization, we see that the desired result is obtained. 1-factorization of $K_{s,s}$ is discussed in [1,3]. $\square$

3.3. Sufficient conditions of S_5 -factorization of $K_{m,n}$

We consider the following three cases.

Case (1) $m=4n$: In this case, from Theorem 3, $K_{4n,n}$ has an S_5 -factorization since $K_{4,1}$ is just S_5 .

Case (2) $n=4m$: Obviously, $K_{m,4m}$ has an S_5 -factorization.

Case (3) $m < 4n$ and $n < 4m$: In this case, let $x = (4n-m)/15$ and $y = (4m-n)/15$. Then from Conditions (iv)-(v), x and y are integers such that $0 < x < m$ and $0 < y < n$. We have $x+4y=m$ and $4x+y=n$. Hence it holds that $b = (x^2+4xy+y^2)+xy/4$, $t=x+y$, $r=(x+y)+9xy/4(x+y)$, $t_1=x$, $t_2=y$, $r_1=x-3xy/4(x+y)$ and $r_2=y-3xy/4(x+y)$. Let $z=3xy/4(x+y)$, which is a positive integer. And let $(x,4y)=d$, $x=dp$, $4y=dq$, where $(p,q)=1$. Then $dq/4$ is an integer and $z=3dpq/4(4p+q)$. The following lemmas can be verified.

Lemma 1. $(p,q)=1 \implies (pq, p+q)=1$.

Lemma 2. $(p,q)=1 \implies (pq, 4p+q)=1$ (q is an odd integer), 2 ($q/2$ is an odd

integer) and $4 \mid (q/4)$ is an integer).

Using these p, q, d , the parameters m and n satisfying Conditions (i)–(ix) are expressed as follows:

Lemma 3. $(p, q) = 1$ and $3dpq/4(4p+q)$ is an integer

====> (I) $m=4(p+q)(4p+q)s$, $n=(16p+q)(4p+q)s$ $((4p+q)/3:\text{not integer})$
or $m=4(p+q)(4p+q)s/3$, $n=(16p+q)(4p+q)s/3$ $((4p+q)/3:\text{integer})$

when q is an odd integer,

(II) $m=4(p+2q')(2p+q')s$, $n=2(8p+q')(2p+q')s$ $((2p+q')/3:\text{not integer})$
or $m=4(p+2q')(2p+q')s/3$, $n=2(8p+q')(2p+q')s/3$ $((2p+q')/3:\text{integer})$

when $q=2q'$ and q' is an odd integer,

(III) $m=4(p+4q'')(p+q'')s$, $n=4(4p+q'')(p+q'')s$ $((p+q'')/3:\text{not integer})$
or $m=4(p+4q'')(p+q'')s/3$, $n=4(4p+q'')(p+q'')s/3$ $((p+q'')/3:\text{integer})$

when $q=4q''$,

where s is a positive integer.

We use the following notations for sequences.

Notation 2. Let A and B be two sequences of the same size such as

$A: a_1, a_2, \dots, a_u$

$B: b_1, b_2, \dots, b_u$.

If $b_i = a_i + c$ ($i=1, 2, \dots, u$), then we write $B = A + c$. If $b_i = ((a_i + c) \bmod w)$ ($i=1, 2, \dots, u$), then we write $B = A + c \bmod w$, where the residuals $a_i + c \bmod w$ are integers in the set $\{1, 2, \dots, w\}$.

Lemma 4. $(p, q) = 1$ and q is an odd integer

$m=4(p+q)(4p+q)s$, $n=(16p+q)(4p+q)s$, where s is a positive integer

====> $K_{m, n}$ has an S_5 -factorization.

Proof. When $s=1$, the proof is by construction (Algorithm I). Let $x=(4n-m)/15$, $y=(4m-n)/15$, $t=(m+n)/5$, $r=5mn/4(m+n)$. Then we have $x=4p(4p+q)$, $y=q(4p+q)$, $t=(4p+q)^2$, $r=(p+q)(16p+q)$. Let $r_m=p+q$, $r_n=16p+q$, $m_0=m/r_m=4(4p+q)$, $n_0=n/r_n=4p+q$. Consider two sequences R and C of the same size $16(4p+q)$.

$R: 1, 1, 1, 1, 2, 2, 2, 2, \dots, 4(4p+q), 4(4p+q), 4(4p+q), 4(4p+q)$

$C: 1, 2, \dots, 16(4p+q)-1, 16(4p+q)$.

Construct p sequences R_i such that $R_i = R + 4(i-1)(4p+q)$ ($i=1, 2, \dots, p$).

Construct p sequences C_i such that $C_i = (C + 4(i-1) \bmod 16(4p+q)) + 16(i-1)(4p+q)$

($i=1,2,\dots,p$). Consider two sequences R' and C' of the same size $4(4p+q)$.

R' : $r_1, r_2, \dots, r_{4(4p+q)}$, where $r_i = (i-1)p+1 \bmod 4(4p+q)$ ($i=1,2,\dots,4(4p+q)$)

C' : $c_1, c_2, \dots, c_{4(4p+q)}$, where $c_i = n - (i-1)q \bmod q(4p+q)$ ($i=1,2,\dots,4(4p+q)$).

Construct q sequences R_i' such that $R_i' = R' + 4(i-1)(4p+q) + 4p(4p+q)$ ($i=1,2,\dots,q$).

Construct q sequences C_i' such that $C_i' = (C' - (i-1) \bmod q(4p+q)) + 16p(4p+q)$ ($i=1,2,\dots,q$). Consider two sequences I and J of the same size.

I : $R_1, R_2, \dots, R_p, R_1', R_2', \dots, R_q'$

J : $C_1, C_2, \dots, C_p, C_1', C_2', \dots, C_q'$.

Then the size of I or J is $4t$. Let i_k and j_k be the k -th element of I and J , respectively ($k=1,2,\dots,4t$). Join two vertices i_k in V_1 and j_k in V_2 with an edge (i_k, j_k) ($k=1,2,\dots,4t$). Construct a graph F with two vertex sets $\{i_k\}$ and $\{j_k\}$ and an edge set $\{(i_k, j_k)\}$. Then F is an S_5 -factor of $K_{m,n}$. This graph is called an *S_5 -factor constructed with two sequences I and J* .

Construct r_m sequences I_i such that $I_i = I + (i-1)m_0 \bmod m$ ($i=1,2,\dots,r_m$).

Construct r_n sequences J_j such that $J_j = J + (j-1)n_0 \bmod n$ ($j=1,2,\dots,r_n$).

Construct $r_m r_n$ S_5 -factors F_{ij} with I_i and J_j ($i=1,2,\dots,r_m; j=1,2,\dots,r_n$). Then it is easy to show that F_{ij} are edge-disjoint and that their sum is an S_5 -factorization of $K_{m,n}$. By Theorem 3, $K_{m,n}$ has an S_5 -factorization for every positive integer s . $\square$

Lemma 5. $(p,q)=1$ and $q=2q'$ (q' is an odd integer)

$m=4(p+2q')(2p+q')s$, $n=2(8p+q')(2p+q')s$, where s is a positive integer

$\implies K_{m,n}$ has an S_5 -factorization.

Proof. When $s=1$, the proof is by construction (Algorithm II). Let $x=(4n-m)/15$, $y=(4m-n)/15$, $t=(m+n)/5$, $r=5mn/4(m+n)$. Then we have $x=4p(2p+q')$, $y=2q'(2p+q')$, $t=2(2p+q')^2$, $r=(p+2q')(8p+q')$. Let $r_m=p+2q'$, $r_n=8p+q'$, $m_0=m/r_m=4(2p+q')$, $n_0=n/r_n=2(2p+q')$. Consider two sequences R and C of the same size $16(2p+q')$.

R : $1,1,1,1,2,2,2,2,\dots,4(2p+q'),4(2p+q'),4(2p+q'),4(2p+q')$

C : $1,2,\dots,16(2p+q')-1,16(2p+q')$.

Construct p sequences R_i such that $R_i = R + 4(i-1)(2p+q')$ ($i=1,2,\dots,p$).

Construct p sequences C_i such that $C_i = (C + 4(i-1) \bmod 16(2p+q')) + 16(i-1)(2p+q')$ ($i=1,2,\dots,p$). Consider two sequences R' and C' of the same size $4(2p+q')$.

R' : $r_1, r_2, \dots, r_{4(2p+q')}$, where $r_i = (i-1)p+1 \bmod 4(2p+q')$ ($i=1,2,\dots,4(2p+q')$)

C' : $c_1, c_2, \dots, c_{4(2p+q')}$, where $c_i = n - 2(i-1)q' \bmod 2q'(2p+q')$ ($i=1,2,\dots,4(2p+q')$).

Construct $2q'$ sequences R_i' such that $R_i' = R' + 4(i-1)(2p+q') + 4p(2p+q')$ ($i=1,2,\dots,2q'$). Construct $2q'$ sequences C_i' such that $C_i' = (C' - (i-1) \bmod 2q'(2p+q')) + 16p(2p+q')$ ($i=1,2,\dots,2q'$). Consider two sequences I and J of the

same size $4t$.

I: $R_1, R_2, \dots, R_p, R_1', R_2', \dots, R_{2q}'$

J: $C_1, C_2, \dots, C_p, C_1', C_2', \dots, C_{2q}'$.

Construct r_m sequences I_i such that $I_i = I + (i-1)m_0 \bmod m$ ($i=1, 2, \dots, r_m$).

Construct r_n sequences J_j such that $J_j = J + (j-1)n_0 \bmod n$ ($j=1, 2, \dots, r_n$).

Construct $r_m r_n$ S_5 -factors F_{ij} with I_i and J_j ($i=1, 2, \dots, r_m; j=1, 2, \dots, r_n$). Then it is easy to show that F_{ij} are edge-disjoint and that their sum is an S_5 -factorization of $K_{m,n}$. By Theorem 3, $K_{m,n}$ has an S_5 -factorization for every positive integer s . $\square$

Lemma 6. $(p, q)=1$ and $q=4q''$

$m=4(p+4q'')(p+q'')s$, $n=4(4p+q'')(p+q'')s$, where s is a positive integer

$\implies K_{m,n}$ has an S_5 -factorization.

Proof. When $s=1$, the proof is by construction (Algorithm III). Let $x=(4n-m)/15$, $y=(4m-n)/15$, $t=(m+n)/5$, $r=5mn/4(m+n)$. Then we have $x=4p(p+q'')$, $y=4q''(p+q'')$, $t=4(p+q'')^2$, $r=(p+4q'')(4p+q'')$. Let $r_m=p+4q''$, $r_n=4p+q''$, $m_0=m/r_m=4(p+q'')$, $n_0=n/r_n=4(p+q'')$. Consider two sequences R and C of the same size $16(p+q'')$.

$R: 1, 1, 1, 1, 2, 2, 2, 2, \dots, 4(p+q''), 4(p+q''), 4(p+q''), 4(p+q'')$

$C: 1, 2, \dots, 16(p+q'')-1, 16(p+q'')$.

Construct p sequences R_i such that $R_i = R + 4(i-1)(p+q'')$ ($i=1, 2, \dots, p$).

Construct p sequences C_i such that $C_i = (C + 4(i-1) \bmod 16(p+q'')) + 16(i-1)(p+q'')$ ($i=1, 2, \dots, p$). Consider two sequences R' and C' of the same size $4(p+q'')$.

$R': r_1, r_2, \dots, r_{4(p+q'')}$, where $r_i = (i-1)p+1 \bmod 4(p+q'')$ ($i=1, 2, \dots, 4(p+q'')$)

$C': c_1, c_2, \dots, c_{4(p+q'')}$, where $c_i = n - 4q''(i-1) \bmod 4q''(p+q'')$ ($i=1, 2, \dots, 4(p+q'')$).

Construct $4q''$ sequences R_i' such that $R_i' = R' + 4(i-1)(p+q'') + 4p(p+q'')$ ($i=1, 2, \dots, 4q''$). Construct $4q''$ sequences C_i' such that $C_i' = (C' - (i-1) \bmod 4q''(p+q'')) + 16p(p+q'')$ ($i=1, 2, \dots, 4q''$). Consider two sequences I and J of the same size $4t$.

I: $R_1, R_2, \dots, R_p, R_1', R_2', \dots, R_{4q''}'$

J: $C_1, C_2, \dots, C_p, C_1', C_2', \dots, C_{4q''}'$.

Construct r_m sequences I_i such that $I_i = I + (i-1)m_0 \bmod m$ ($i=1, 2, \dots, r_m$).

Construct r_n sequences J_j such that $J_j = J + (j-1)n_0 \bmod n$ ($j=1, 2, \dots, r_n$).

Construct $r_m r_n$ S_5 -factors F_{ij} with I_i and J_j ($i=1, 2, \dots, r_m; j=1, 2, \dots, r_n$). Then it is easy to show that F_{ij} are edge-disjoint and that their sum is an S_5 -factorization of $K_{m,n}$. By Theorem 3, $K_{m,n}$ has an S_5 -factorization for every positive integer s . $\square$

In Lemma 6, put $p=1$, $q=4q''=4$. Then we have the following example.

Example 1. $K_{40s, 40s}$ has an S_5 -factorization.

By Corollary 2 and Example 1, we have the following theorem.

Theorem 4. $K_{n, n}$ has an S_5 -factorization if and only if $n \equiv 0 \pmod{40}$.

Conjecture 1. $(p, q)=1$, q is an odd integer and $(4p+q)/3$ is an integer
 $m=4(p+q)(4p+q)s/3$, $n=(16p+q)(4p+q)s/3$,
 where s is a positive integer and $s/3$ is not an integer
 $\implies K_{m, n}$ has an S_5 -factorization.

Conjecture 2. $(p, q)=1$, $q=2q'$ (q' is an odd integer) and $(2p+q)/3$ is an integer
 $m=4(p+2q')(2p+q')s/3$, $n=2(8p+q')(2p+q')s/3$,
 where s is a positive integer and $s/3$ is not an integer
 $\implies K_{m, n}$ has an S_5 -factorization.

Conjecture 3. $(p, q)=1$, $q=4q''$ and $(p+q'')/3$ is an integer
 $m=4(p+4q'')(p+q'')s/3$, $n=4(4p+q'')(p+q'')s/3$,
 where s is a positive integer and $s/3$ is not an integer
 $\implies K_{m, n}$ has an S_5 -factorization.

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