

Solvability of the heat equation on a half-space with a dynamical boundary condition and unbounded initial data

龍谷大学・先端理工学部 川上 竜樹

Tatsuki Kawakami

Faculty of Advanced Science and Technology,
Ryukoku University

This is a survey article for the paper [5], which is joint work with Marek Fila (Comenius Univ.) and Kazuhiro Ishige (Univ. of Tokyo).

1 Introduction

Let $N \geq 2$ and $\mathbb{R}_+^N := \mathbb{R}^{N-1} \times \mathbb{R}_+$. This paper is concerned with global solvability of the problem

$$\begin{cases} \partial_t u - \Delta u = 0, & x \in \mathbb{R}_+^N, \ t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\mathbb{R}_+^N, \ t > 0, \\ u(x, 0) = \varphi(x), & x \in \mathbb{R}_+^N, \\ u(x, 0) = 0, & x = (x', 0) \in \partial\mathbb{R}_+^N, \ x' := (x_1, x_2, \dots, x_{N-1}), \end{cases} \quad (1)$$

where $\partial_t := \partial/\partial t$, and $\partial_\nu := -\partial/\partial x_N$. The boundary condition from (1) describes diffusion through the boundary in processes such as thermal contact with a perfect conductor or diffusion of solute from a well-stirred fluid or vapour (see e.g. [2]). Various aspects of analysis of parabolic equations with dynamical boundary conditions have been treated by many authors (see e.g. the reference of [5]).

In this paper we focus on the simplest linear problem from a point of view which has not been considered yet (as far as we know). Namely, we are interested in an appropriate choice of the function space of initial functions φ such that problem (1) is solvable.

Throughout this paper we often identify $\mathbb{R}^{N-1}$ with $\partial\mathbb{R}_+^N$. We introduce some notation. Let $\Gamma_D = \Gamma_D(x, y, t)$ be the Dirichlet heat kernel on $\mathbb{R}_+^N$, that is,

$$\Gamma_D(x, y, t) := (4\pi t)^{-\frac{N}{2}} \left[\exp\left(-\frac{|x-y|^2}{4t}\right) - \exp\left(-\frac{|x-y_*|^2}{4t}\right) \right]$$

for $(x, y, t) \in \overline{\mathbb{R}_+^N} \times \mathbb{R}_+^N \times (0, \infty)$, where $y_* = (y', -y_N)$ for $y = (y', y_N) \in \mathbb{R}_+^N$. Set

$$[S_1(t)\phi](x) := \int_{\mathbb{R}_+^N} \Gamma_D(x, y, t)\phi(y) dy, \quad (x, t) \in \mathbb{R}_+^N \times (0, \infty), \quad (2)$$

for any measurable function ϕ in $\mathbb{R}_+^N$ if the right hand side of (2) is well-defined. For $x = (x', x_N) \in \overline{\mathbb{R}_+^N}$ and $t > 0$, set

$$P(x', x_N, t) := C_N(x_N + t) \left(|x'|^2 + (x_N + t)^2 \right)^{-\frac{N}{2}},$$

where C_N is the constant chosen so that

$$\int_{\mathbb{R}^{N-1}} P(x', x_N, t) dx' = 1 \quad \text{for all } x_N \geq 0 \text{ and } t > 0.$$

Then $P = P(x', x_N, t)$ is the fundamental solution of the Laplace equation in $\mathbb{R}_+^N$ with the homogeneous dynamical boundary condition (see e.g. [1]). Set

$$[S_2(t)\psi](x) := \int_{\mathbb{R}^{N-1}} P(x' - y', x_N, t)\psi(y') dy', \quad (x, t) \in \mathbb{R}_+^N \times (0, \infty), \quad (3)$$

for any measurable function ψ in $\mathbb{R}^{N-1}$ if the right hand side of (3) is well-defined. Then the function $\Psi(x, t) := [S_2(t)\psi](x)$ satisfies

$$\begin{cases} -\Delta \Psi = 0, & x \in \mathbb{R}_+^N, \quad t > 0, \\ \partial_t \Psi + \partial_\nu \Psi = 0, & x \in \partial \mathbb{R}_+^N, \quad t > 0, \\ \Psi(x, 0) = \psi(x'), & x = (x', 0) \in \partial \mathbb{R}_+^N. \end{cases}$$

Consider

$$\begin{cases} \partial_t v = \Delta v - F[v], \quad \Delta w = 0, & x \in \mathbb{R}_+^N, \quad t > 0, \\ v = 0, \quad \partial_t w - \partial_{x_N} w = \partial_{x_N} v, & x \in \partial \mathbb{R}_+^N, \quad t > 0, \\ v(x, 0) = \varphi(x), & x \in \mathbb{R}_+^N, \\ w(x, 0) = 0, & x = (x', 0) \in \partial \mathbb{R}_+^N, \end{cases} \quad (4)$$

where

$$\begin{aligned} F[v](x, t) &:= \int_{\mathbb{R}^{N-1}} P(x' - y', x_N, 0) \partial_{x_N} v(y', 0, t) dy' \\ &\quad + \int_0^t \int_{\mathbb{R}^{N-1}} \partial_t P(x' - y', x_N, t - s) \partial_{x_N} v(y', 0, s) dy' ds. \end{aligned} \quad (5)$$

Following [4], we formulate the definition of a solution of (1).

Definition 1.1 Let φ be measurable function in $\mathbb{R}_+^N$. Let $0 < T \leq \infty$ and

$$v, \partial_{x_N} v, w \in C(\overline{\mathbb{R}_+^N} \times (0, T)).$$

We call (v, w) a solution of (4) in $\mathbb{R}_+^N \times (0, T)$ if

$$[S_1(t)\varphi](x), \quad \int_0^t [S_1(t-s)F[v](s)](x) ds, \quad \int_0^t [S_2(t-s)\partial_{x_N} v(\cdot, 0, s)](x) ds$$

are well-defined and functions v and w satisfy

$$\begin{aligned} v(x, t) &= [S_1(t)\varphi](x) - \int_0^t [S_1(t-s)F[v](s)](x) ds, \\ w(x, t) &= \int_0^t [S_2(t-s)\partial_{x_N} v(\cdot, 0, s)](x) ds, \end{aligned}$$

for $x \in \overline{\mathbb{R}_+^N}$ and $t \in (0, T)$, respectively. Then we say that $u := v + w$ is a solution of (1) in $\mathbb{R}_+^N \times (0, T)$. In the case of $T = \infty$, we call (v, w) a global-in-time solution of (4) and u a global-in-time solution of (1).

For $1 \leq r \leq \infty$, we write $\|\cdot\|_{L^r} := \|\cdot\|_{L^r(\partial\mathbb{R}_+^N)}$ and $\|\cdot\|_{L^r} := \|\cdot\|_{L^r(\mathbb{R}_+^N)}$ for simplicity. Furthermore, for $1 \leq r \leq \infty$ and $\alpha \geq 0$, we define

$$L_\alpha^r := \{f \in L^r(\mathbb{R}_+^N) : \|f\|_{L_\alpha^r} < \infty\},$$

where

$$\|f\|_{L_\alpha^r} := \begin{cases} \left(\int_{\mathbb{R}_+^N} |f(x)|^r h(x_N)^{-\alpha r} dx \right)^{\frac{1}{r}} & \text{if } 1 \leq r < \infty, \\ \|f\|_{L^\infty} & \text{if } r = \infty, \end{cases} \quad \text{with } h(x_N) := \frac{x_N}{x_N + 1}.$$

Then we can easily show that $\|f\|_{L_\alpha^r} \leq \|f\|_{L_\beta^r}$ for $r \in [1, \infty]$ and $0 \leq \alpha \leq \beta$.

Now we are ready to state the main results of this paper.

Theorem 1.1 Let $N \geq 2$ and $1 \leq q \leq \infty$. Furthermore, let

$$p \in (Nq/(N-1), \infty] \quad \text{if } q < \infty \quad \text{and} \quad p = \infty \quad \text{if } q = \infty.$$

For $r \in [q, \infty]$, put

$$\alpha(r) = (N-1) \left(\frac{1}{q} - \frac{1}{r} \right) + \frac{1}{q}. \quad (6)$$

Assume $\varphi \in L_{\alpha(p)}^q$. Then problem (4) possesses a unique global-in-time solution (v, w) with the following property: For any $T > 0$ there exists $C_T > 0$ such that

$$\begin{aligned} \sup_{0 < t < T} \left[t^{\frac{N}{2}(\frac{1}{q} - \frac{1}{p})} \left(\|v(t)\|_{L^p} + t^{\frac{1}{2}} \|\partial_{x_N} v(t)\|_{L^p} \right) + t^{\frac{1}{2}} |\partial_{x_N} v(t)|_{L^r} \right] &\leq C_T \|\varphi\|_{L_{\alpha(p)}^q}, \\ \sup_{0 < t < T} \left[\|w(t)\|_{L^p} + |w(t)|_{L^r} \right] &\leq C_T \|\varphi\|_{L_{\alpha(p)}^q}, \end{aligned} \quad (7)$$

for $r \in [q, p]$. Furthermore, v and w are bounded and smooth in $\overline{\mathbb{R}_+^N} \times I$ for any bounded interval $I \subset (0, \infty)$.

Remark 1.1 We explain the role of the space $L_{\alpha(p)}^q$ in our study. Let $1 \leq q \leq \infty$ and take arbitrary functions $\Phi \in L^q(\mathbb{R}^{N-1})$, $\vartheta \in L^q(1, \infty)$. Now set $\varphi(x) := \Phi(x')\Psi(x_N)$ for $x = (x', x_N) \in \mathbb{R}_+^N$, where

$$\Psi(x_N) := \begin{cases} x_N^\lambda & \text{if } 0 < x_N \leq 1, \quad \lambda \in \mathbb{R}, \\ \vartheta(x_N) & \text{if } x_N > 1. \end{cases}$$

Choose p as in Theorem 1.1. Then it is easy to check that $\varphi \in L_{\alpha(p)}^q$ if and only if

$$\lambda > (N-1) \left(\frac{1}{q} - \frac{1}{p} \right) (> 0 \quad \text{if } q < \infty).$$

If $\lambda > 0$, then $\lim_{x_N \rightarrow 0} \varphi(x) = 0$ which means that the condition $u(x', 0, 0) = 0$ in (1) is satisfied. This indicates that the choice of the space of initial functions is natural and also optimal in some sense since λ can be arbitrarily close to 0 if q is large enough.

We have not observed the importance of the behavior of φ near $\partial\mathbb{R}_+^N$ in the L^∞ -setting in [4]. The main novelty of this paper consists in working in an appropriate weighted L^q -space by which we extend a result from [4] significantly, as we explain below.

In [4] we studied the problem

$$\begin{cases} \partial_t u - \Delta u = 0, & x \in \mathbb{R}_+^N, \quad t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\mathbb{R}_+^N, \quad t > 0, \\ u(x, 0) = \varphi(x), & x \in \mathbb{R}_+^N, \\ u(x, 0) = \varphi_b(x'), & x = (x', 0) \in \partial\mathbb{R}_+^N, \end{cases} \quad (8)$$

where φ and φ_b are bounded functions. A part of Theorem 1.1 in [4] reads as follows:

Theorem 1.2 Let $N \geq 2$, $\varphi \in L^\infty(\mathbb{R}_+^N)$ and $\varphi_b \in L^\infty(\mathbb{R}^{N-1})$. Then problem (8) possesses a unique global-in-time solution u which is bounded and smooth in $\overline{\mathbb{R}_+^N} \times I$ for any bounded interval $I \subset (0, \infty)$.

Hence, if $\varphi_b \equiv 0$ then Theorem 1.2 is a very special case of Theorem 1.1. If $\varphi_b \in L^\infty(\mathbb{R}^{N-1})$ and $\varphi \in L_{\alpha(p)}^q$ with p, q as in Theorem 1.1, then we can combine Theorem 1.1 with Theorem 1.2 to obtain the existence of a solution of (8) easily, since the problem is linear.

2 Preliminaries

In this section we prove several lemmata on $S_1(t)\phi$ and $F[v]$, and recall some properties of $S_2(t)\psi$. In what follows, by the letter C we denote generic positive constants (independent of x and t) and they may have different values also within the same line.

We first recall some properties of $S_1(t)\phi$ (see e.g., [6] and [4, Lemma 2.1]).

(G₁) For any $1 \leq q \leq r \leq \infty$,

$$\|S_1(t)\phi\|_{L^r} \leq c_1 t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})} \|\phi\|_{L^q}, \quad t > 0,$$

for all $\phi \in L^q(\mathbb{R}_+^N)$, where c_1 is a positive constant, independent of q and r . In particular, if $q = r$, then

$$\sup_{t>0} \|S_1(t)\phi\|_{L^r} \leq \|\phi\|_{L^r}.$$

Furthermore, for any $1 \leq q \leq r \leq \infty$,

$$\|\partial_{x_N} S_1(t)\phi\|_{L^r} \leq c_2 t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}} \|\phi\|_{L^q}, \quad t > 0,$$

for all $\phi \in L^q(\mathbb{R}_+^N)$, where c_2 is a positive constant, independent of q and r .

(G₂) Let $\phi \in L^q(\mathbb{R}_+^N)$ with $1 \leq q \leq \infty$ and $T > 0$. Then $S_1(t)\phi$ is bounded and smooth with respect to x and t in $\mathbb{R}_+^N \times (T, \infty)$.

Applying an argument similar to the proof of Young's inequality, we have the following.

Lemma 2.1 *Let $1 \leq q \leq r \leq \infty$. Assume $\phi \in L_{\alpha(r)}^q$ with $\alpha(r)$ as in (6). Then there exists $c_3 = c_3(N) > 0$ such that*

$$|\partial_{x_N} [S_1(t)\phi]|_{L^r} \leq c_3 t^{-\frac{1}{2}} \|\phi\|_{L_{\alpha(r)}^q}, \quad t > 0. \quad (9)$$

Next we recall some properties of $S_2(t)\psi$.

(P₁) Let $\psi \in L^r(\mathbb{R}^{N-1})$ for some $r \in [1, \infty]$ and $t, t' > 0$. Then

$$\begin{aligned} [S_2(t)\psi](x', x_N) &= [S_2(t + x_N)\psi](x', 0), \\ [S_2(t + t')\psi](x) &= [S_2(t)(S_2(t')\psi)](x), \end{aligned}$$

for $x = (x', x_N) \in \overline{\mathbb{R}_+^N}$. Furthermore,

$$\lim_{t \rightarrow 0} |S_2(t)\psi - \psi|_r = 0 \quad \text{if } 1 \leq r < \infty.$$

(P₂) For any $1 \leq r \leq q \leq \infty$,

$$|S_2(t)\psi|_{L^q} \leq C t^{-(N-1)(\frac{1}{r}-\frac{1}{q})} |\psi|_{L^r}, \quad t > 0,$$

for all $\psi \in L^r(\mathbb{R}^{N-1})$. In particular, if $q = r$, then

$$\sup_{t>0} |S_2(t)\psi|_{L^q} \leq |\psi|_{L^q}.$$

(P_3) Let $1 \leq r < \infty$ and $Nr/(N-1) < q \leq \infty$. Then

$$\|S_2(t)\psi\|_{L^q} \leq Ct^{-(N-1)(\frac{1}{r}-\frac{1}{q})+\frac{1}{q}}|\psi|_{L^r}, \quad t > 0,$$

for all $\psi \in L^r(\mathbb{R}^{N-1})$. Furthermore,

$$\sup_{t>0} \|S_2(t)\psi\|_{L^q} \leq C(|\psi|_{L^q} + |\psi|_{L^r}) \quad (10)$$

for all $\psi \in L^q(\mathbb{R}^{N-1}) \cap L^r(\mathbb{R}^{N-1})$.

Properties (P_1), (P_2), and (P_3) easily follow from (3) (see e.g. [3]) and imply that

$$\sup_{t>0} \|S_2(t)\psi\|_{L^\infty} \leq |\psi|_{L^\infty}$$

for all $\psi \in L^\infty(\mathbb{R}^{N-1})$. Furthermore, by an argument similar to that in the proof of property (G_2) we have:

(P_4) Let $\psi \in L^r(\mathbb{R}^{N-1})$ with $1 \leq r \leq \infty$. Then, for any $T > 0$, $S_2(t)\psi$ is bounded and smooth in $\mathbb{R}_+^N \times (T, \infty)$.

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. By [4, Theorem 1.1] with $\varepsilon = 1$ and $\varphi_b \equiv 0$ we have Theorem 1.1 for the case $p = q = \infty$. So we focus on the case $q < \infty$.

Let $T > 0$, $M \geq 1$, $1 \leq q < \infty$, and $p \in (Nq/(N-1), \infty]$. Set

$$X_{T,M} := \left\{ v : v, \partial_{x_N} v \in C(\overline{\mathbb{R}_+^N} \times (0, T)), \|v\|_{X_{T,M}} < \infty \right\}, \quad \|v\|_{X_{T,M}} := \sup_{0 < t < T} e^{-Mt} E[v](t),$$

where

$$E[v](t) := t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \left[\|v(t)\|_{L^p} + t^{\frac{1}{2}} \|\partial_{x_N} v(t)\|_{L^p} \right] + \sup_{q \leq r \leq p} t^{\frac{1}{2}} |\partial_{x_N} v(t)|_{L^r}.$$

Then $X_{T,M}$ is a Banach space equipped with the norm $\|\cdot\|_{X_{T,M}}$. We apply the Banach contraction mapping principle in $X_{T,M}$ to find a fixed point of the functional

$$Q[v](t) := S_1(t)\varphi - D[v](t) \quad (11)$$

on $X_{T,M}$, where $D[v]$ is the function defined by

$$D[v](t) := \int_0^t S_1(t-s)F[v](s)ds \quad (12)$$

and $F[v]$ is the function defined by (5). For the function $F[v]$, we have the following.

Lemma 3.1 *Let $T > 0$, $M \geq 1$, $1 \leq q < \infty$, and $p \in (Nq/(N-1), \infty]$. Assume that $v \in X_{T,M}$. Then there exists $C > 0$, independent of T and M , such that, for $p \in (Nq/(N-1), \infty)$, it holds that*

$$\|F[v](t)\|_{L^p} \leq C(1+t^{\frac{1}{p}})t^{-\frac{1}{2}}e^{Mt}\|v\|_{X_{T,M}}$$

for $0 < t < T$. Furthermore, for any $r \in [q, p]$,

$$\|F[v](\cdot, x_N, t)\|_{L^r(\mathbb{R}^{N-1})} \leq C\left(1+(x_N^{-1}t)^{\frac{1}{2}}\right)t^{-\frac{1}{2}}e^{Mt}\|v\|_{X_{T,M}}$$

for $x_N \in (0, \infty)$ and $0 < t < T$.

Applying Lemma 3.1, we obtain the following estimate for the function $D[v]$.

Lemma 3.2 *Assume the same conditions as in Lemma 3.1. Let $D[v]$ be the function defined by (12). Then there exists $M_* \geq 1$ such that*

$$\|D[v]\|_{X_{T,M}} \leq \frac{1}{2}\|v\|_{X_{T,M}} \quad (13)$$

for $v \in X_{T,M}$ and $M \geq M_*$. Furthermore, $D[v]$ is bounded and smooth in $\overline{\mathbb{R}_+^N} \times (\tau, T)$ for any $0 < \tau < T$.

Now we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $T > 0$, $M \geq 1$, $1 \leq q < \infty$, and $p \in (Nq/(N-1), \infty]$. Then, since $\|\varphi\|_{L_\alpha^\alpha} \leq \|\varphi\|_{L_\beta^r}$ for $r \in [1, \infty]$ and $0 \leq \alpha \leq \beta$, by (G_1) and (9) we have

$$e^{-Mt}E[S_1(t)\varphi](t) \leq (c_1 + c_2 + c_3)\|\varphi\|_{L_{\alpha(p)}^q} \quad (14)$$

for $t > 0$, where c_1 , c_2 , and c_3 are positive constants given in (G_1) and Lemma 2.1, respectively, and $\alpha(p)$ is given in (6). Furthermore, by Lemma 3.2, taking a sufficiently large $M \geq 1$ if necessary, we see that

$$\|D[v]\|_{X_{T,M}} \leq \frac{1}{2}\|v\|_{X_{T,M}}, \quad v \in X_{T,M}, \quad (15)$$

for $0 < t < T$. Set

$$m := 2(c_1 + c_2 + c_3)\|\varphi\|_{L_{\alpha(p)}^q}. \quad (16)$$

We deduce from (11), (14), (15), and (16) that

$$\begin{aligned} \|Q[v]\|_{X_{T,M}} &\leq \sup_{0 < t < T} e^{-Mt}E[S_1(t)\varphi](t) + \|D[v]\|_{X_{T,M}} \\ &\leq (c_1 + c_2 + c_3)\|\varphi\|_{L_{\alpha(p)}^q} + \frac{1}{2}\|v\|_{X_{T,M}} \leq m \end{aligned} \quad (17)$$

for $v \in X_{T,M}$ with $\|v\|_{X_{T,M}} \leq m$. Similarly, it follows from (15) that

$$\|Q[v_1] - Q[v_2]\|_{X_{T,M}} = \|D[v_1 - v_2]\|_{X_{T,M}} \leq \frac{1}{2}\|v_1 - v_2\|_{X_{T,M}} \quad (18)$$

for $v_i \in X_{T,M}$ ($i = 1, 2$). Then, by (17) and (18), applying the contraction mapping theorem, we find a unique solution $v \in X_{T,M}$ with $\|v\|_{X_{T,M}} \leq m$ such that

$$v = Q[v] = S_1(t)\varphi - D[v](t) \quad \text{in } X_{T,M}.$$

In particular, we see that

$$\|v\|_{X_{T,M}} \leq C\|\varphi\|_{L_{\alpha(p)}^q}.$$

Moreover, by (G_2) and Lemma 3.2, we see that v is bounded and smooth in $\overline{\mathbb{R}_+^N} \times (T_1, T)$ for any $0 < T_1 < T$.

Set

$$w(x, t) = \int_0^t [S_2(t-s)\partial_{x_N}v(\cdot, 0, s)](x) ds$$

for $x \in \overline{\mathbb{R}_+^N}$ and $t \in (0, T)$. By (10) and (16) we obtain

$$\begin{aligned} \|w(t)\|_{L^p} &\leq \int_0^t \|S_2(t-s)\partial_{x_N}v(\cdot, 0, s)\|_{L^p} ds \\ &\leq C \int_0^t \left(|\partial_{x_N}v(s)|_{L^q} + |\partial_{x_N}v(s)|_{L^p} \right) ds \\ &\leq C \int_0^t e^{Ms} s^{-\frac{1}{2}} \|v\|_{X_{T,M}} ds \leq Ce^{MT} T^{\frac{1}{2}} \|\varphi\|_{L_{\alpha(p)}^q} < \infty, \end{aligned}$$

and

$$\begin{aligned} |w(t)|_{L^r} &\leq \int_0^t |S_2(t-s)\partial_{x_N}v(\cdot, 0, s)|_{L^r} ds \\ &\leq C \int_0^t |\partial_{x_N}v(s)|_{L^r} ds \\ &\leq C \int_0^t e^{Ms} s^{-\frac{1}{2}} \|v\|_{X_{T,M}} ds \leq Ce^{MT} T^{\frac{1}{2}} \|\varphi\|_{L_{\alpha(p)}^q} < \infty, \end{aligned}$$

for $0 < t < T$. Furthermore, by (P_3) we apply an argument similar to that in the proof of Lemma 3.2 and see that w is bounded and smooth in $\overline{\mathbb{R}_+^N} \times (T_1, T)$ for any $0 < T_1 < T$. Therefore we deduce that (v, w) is a solution of (4) in $\mathbb{R}_+^N \times (0, T)$ satisfying (7).

Let $(\tilde{v}, \tilde{w})$ be a solution of (4) in $\mathbb{R}_+^N \times (0, T_*)$ for any $T_* > T$ and such that $\tilde{v} \in X_{T_*, M_*}$ with some $M_* > 0$. Then $\tilde{v} \in X_{T,M}$ and since

$$v - \tilde{v} = Q[v] - Q[\tilde{v}] = D[v - \tilde{v}] \quad \text{in } X_{T,M},$$

by (13) we have

$$\|v - \tilde{v}\|_{X_{T,M}} \leq \frac{1}{2} \|v - \tilde{v}\|_{X_{T,M}}.$$

This implies that $v = \tilde{v}$ in $X_{T,M}$. Therefore we deduce that (v, w) is a unique global-in-time solution of (4) satisfying (7). Thus Theorem 1.1 holds for the case $q < \infty$. Furthermore, by [4, Theorem 1.1] with $\varepsilon = 1$ and $\varphi_b \equiv 0$ we have Theorem 1.1 for the case $p = q = \infty$, and the proof of Theorem 1.1 is complete. $\square$

References

- [1] H. Amann and M. Fila, A Fujita-type theorem for the Laplace equation with a dynamical boundary condition, *Acta Math. Univ. Comenianae* 66 (1997), 321–328.
- [2] J. Crank, *The Mathematics of Diffusion*, Clarendon Press, Oxford, second edition (1975).
- [3] M. Fila, K. Ishige, and T. Kawakami, Large-time behavior of solutions of a semilinear elliptic equation with a dynamical boundary condition, *Adv. Differential Equations* 18 (2013), 69–100.
- [4] M. Fila, K. Ishige, and T. Kawakami, The large diffusion limit for the heat equation with a dynamical boundary condition, *Commun. Contemp. Math.* 23 (2021), Paper No. 2050003, 20 pp.
- [5] M. Fila, K. Ishige, and T. Kawakami, Solvability of the heat equation on a half-space with a dynamical boundary condition and unbounded initial data, *Z. Angew. Math. Phys.* 74 (2023), Paper No. 143, 17 pp.
- [6] M. Giga, Y. Giga and J. Saal, *Nonlinear Partial Differential Equations, Asymptotic Behavior of Solutions and Self-Similar Solutions*, Birkhäuser, Boston, 2010.