

A reflection principle formulated in terms of games *

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Abstract

We introduce a principle formulated in terms of the existence of a winning strategy of a game and prove that this principle is placed between the reflection principle down to internally stationary sets (RP_{IS}) and the reflection principle down to internally club sets (RP_{IC}). In particular, under CH this principle gives a new characterization of Fleissner's Axiom R.

1 Introduction

For a game $\mathcal{G}$ played by Players I and II , let $WS_{II}(\mathcal{G})$ denote the assertion “Player II has a winning strategy in $\mathcal{G}$ ”.

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An extended version of this paper with more details and proofs can be found at:

<http://kurt.scitec.kobe-u.ac.jp/~fuchino/preprints.html>

In [9], we introduced a game $G_\omega^\downarrow(\kappa)$ for uncountable cardinals κ (see Section 3 for the definition of this and other games mentioned here) and proved that the Rado Conjecture (RC, see Section 2 for the definition of this principle) implies the assertion

(G₀) $WS_{II}(G_\omega^\downarrow(\kappa))$ holds for all uncountable κ .

Further, it is proved in [9] that (G₀) implies the Fodor-type Reflection Principle (FRP, see Section 2 for the definition of this principle and, [4] and [5] for basic facts of this principle).

In [1], Philipp Doebler introduced a similar game he called $G_\omega([\kappa]^{\omega_1}, \omega_1)$ and proved that the Rado Conjecture also implies the principle

(G₁) $WS_{II}(G_\omega([\kappa]^{\aleph_1}, \omega_1))$ holds for all $\kappa \geq \aleph_2$.

He then proved that the principle (G₁) implies the Semistationary Reflection (SSR).

In this paper, we introduce a game $G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1})$ which generalizes both $G_\omega^\downarrow(\kappa)$ and $G_\omega([\kappa]^{\omega_1}, \omega_1)$. Unfortunately the principle

(G^{↓↓}) $WS_{II}(G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1}))$ for all $\kappa \geq \aleph_2$

is not a consequence of the Rado Conjecture: In Section 4, we show that the principle (G^{↓↓}) implies the reflection principle RP_{IS} . It is known that RP_{IS} (or even RP) is not a consequence of RC (see Sakai [14]).

2 Reflection Principles

Let us first review the reflection principles we mentioned in the previous section.

We shall call here a partial ordering $T = \langle T, \leq_T \rangle$ a *tree* if the initial segment $\{u \in T : u \leq_T t\}$ in T below each $t \in T$ is well-ordered. In particular, we assume here that a tree may have multiple roots.

A tree T is *special* if there are $T_i \subseteq T$, $i \in \omega$ such that each of T_i 's is pairwise incomparable and $T = \bigcup_{i \in \omega} T_i$.

Rado's Conjecture (RC) is the assertion:

(RC): Any tree T is special if and only if all subtrees of T of cardinality $\aleph_1$ are special.

RC is known to be consistent (modulo a large large cardinal). E.g., Todorćević showed that, if κ is strongly compact and $\mathbb{P} = Col(\omega_1, <\kappa)$, then we have

$\Vdash_{\mathbb{P}}$ "Rado's Conjecture".

For a cardinal κ and a regular cardinal $\delta < \kappa$, we denote

$$E_\delta^\kappa = \{\alpha < \kappa : cf(\alpha) = \delta\}.$$

a mapping $g : E \rightarrow \kappa$ for $E \subseteq E_\delta^\kappa$ is called a ladder system if $\sup g(\alpha) = \alpha$ and $otp(g(\alpha)) = \delta$ hold for all $\alpha \in E$.

For a regular uncountable cardinal κ , we define the Fodor-type Reflection Principle for κ by

FRP(κ): For all stationary $E \subseteq E_\omega^\kappa$ and for all ladder system $g : E \rightarrow [\kappa]^{\aleph_0}$, there exists $\alpha^* \in E_{\omega_1}^\kappa$ such that

$$\{x \in [\alpha^*]^{\aleph_0} : \sup(x) \in E, g(\sup(x)) \subseteq x\}$$

is stationary in $[\alpha^*]^{\aleph_0}$.

The Fodor-type Reflection Principle (FRP) is the assertion:

(FRP): FRP(κ) holds for all regular $\kappa > \aleph_1$.

FRP is known to be equivalent to many mathematical reflection principles over ZFC (see [3], [4], [5], [6], [7], see also [8]).

(2.1) Any locally countably compact topological space X is metrizable if and only if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable

is one of such assertions equivalent to FRP over ZFC (see [4] and [5]).

FRP implies Shelah's Strong Hypothesis and hence, in particular, Singular Cardinal Hypothesis (see [7]). It also implies the total failure of square principles $\square_\kappa$ for all cardinals $\kappa \geq \aleph_1$.

Suppose that $M \prec \mathcal{H}(\lambda)$ for some regular $\lambda \geq \aleph_2$ and $|M| = \aleph_1$.

M is said to be *internally cofinal* (abbreviation: IU)¹⁾ if $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ with respect to $\subseteq$. M is *internally stationary* (abbreviation: IS) if $[M]^{\aleph_0} \cap M$ is stationary in $[M]^{\aleph_0}$. M is *internally club* (abbreviation: IC) if $[M]^{\aleph_0} \cap M$ contains a closed unbounded set in $[M]^{\aleph_0}$. Finally, M is *internally approachable* (abbreviation: IA) if M is the union of a continuously increasing sequence $\langle M_\alpha : \alpha < \omega_1 \rangle$ countable sets such that $\langle M_\alpha : \alpha \leq \delta \rangle \in M_{\delta+1}$ for all $\delta < \omega_1$ ²⁾.

It is clear from the definition that, for any $M \prec \mathcal{H}(\lambda)$, we have the implication: M is IA $\Rightarrow$ M is IC $\Rightarrow$ M is IS $\Rightarrow$ M is IU. It is easy to see that all of these notions can be characterized in terms of filtration (see footnote 2)):

Lemma 2.1 Suppose that $M \prec \mathcal{H}(\lambda)$ for some regular $\lambda \geq \omega_2$ and $|M| = \aleph_1$.

¹⁾ Internally cofinal M is also called *internally unbounded* in the literature (see e.g. Krueger [11]).

²⁾ For a structure M of cardinality $\aleph_1$, we shall call a continuously increasing sequence $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable subsets of M with $\bigcup_{\alpha < \omega_1} M_\alpha = M$ a *filtration* of M . By thinning out the index set ω_1 , we may assume in some cases that the filtration $\langle M_\alpha : \alpha < \omega_1 \rangle$ consists of elementary structures.

(1) M is internally cofinal if and only if there is a filtration $\langle a_\alpha : \alpha < \omega_1 \rangle$ of M such that $a_{\alpha+1} \in M$ for every $\alpha < \omega_1$.

(2) M is internally stationary if and only if $\{\alpha < \omega_1 : M_\alpha \in M\}$ is stationary for a/any filtration $\langle M_\alpha : \alpha < \omega_1 \rangle$ of M .

(3) M is internally club if and only if there is a filtration $\langle M_\alpha : \alpha < \omega_1 \rangle$ of M such that $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \omega_1$. $\square$

These notions can be different: e.g. John Krueger proved under PFA, there are stationarily many internally club but not internally approachable $M \prec \mathcal{H}(\lambda)$ for all regular $\lambda > \aleph_1$ (for this and other results of this line see Krueger [11] and [12]). However this is not the case under CH:

Lemma 2.2 *Under CH, any $M \prec \mathcal{H}(\lambda)$ is IU if and only if it is IS if and only if it is IC if and only if it is IA.* $\square$

In the following, we shall always denote one of the properties IU, IS, IC or IA with $\mathcal{P}$. “ $\sqsubset$ ” in connection with a cardinal, say λ , denotes a(n arbitrary) well-ordering of the set $\mathcal{H}(\lambda)$ of all sets of hereditarily of cardinality $< \lambda$. If we have to emphasize that the well-ordering $\sqsubset$ refers to $\mathcal{H}(\lambda)$, we write $\sqsubset_{\mathcal{H}(\lambda)}$.

For a cardinal $\lambda > \aleph_1$ let

$\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$: For any stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ there is a $\mathcal{P}$ elementary substructure M of the structure $\langle \mathcal{H}(\lambda), \in, \sqsubset \rangle$ (of cardinality $\aleph_1$) such that

$$(2.2) \quad S \cap [M]^{\aleph_0} \text{ is stationary in } [M]^{\aleph_0}.$$

We define the global version of the reflection principle $\text{RP}_{\mathcal{P}}$ down to a structure with the property $\mathcal{P}$ to be $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ for all cardinal $\lambda > \aleph_1$.

$\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ is equivalent with seemingly stronger variants of the assertion:

Lemma 2.3 *the following are equivalent for a regular cardinal $\lambda > \aleph_1$:*

(a) $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$.

(b) *For any stationary $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$ and any expansion $\mathcal{M}$ of the structure $\langle \mathcal{H}(\kappa), \in, \sqsubset \rangle$ in an arbitrary countable language, there is a $\mathcal{P}$ elementary substructure M of $\mathcal{M}$ (of cardinality $\aleph_1$) with (2.2).*

(c) *For any stationary $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$ and any expansion $\mathcal{M}$ of the structure $\langle \mathcal{H}(\kappa), \in, \sqsubset \rangle$ in an arbitrary countable language, there are stationarily many $\mathcal{P}$ elementary substructures M of $\mathcal{M}$ (of cardinality $\aleph_1$) with (2.2).* $\square$

Using Lemma 2.3 we can prove the following downward transfer property of $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$:

³⁾ That is, S intersection with the set of all countable subsets of the underlying set of the structure M .

Lemma 2.4 For regular cardinals $\aleph_1 < \lambda' < \lambda$, if $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ holds then $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda')]^{\aleph_0})$ also holds. $\square$

Lemma 2.5 The following are equivalent: (a) $\text{RP}_{\mathcal{P}}$.

(b) For any uncountable X , stationary $S \subseteq [X]^{\aleph_0}$, regular θ with $X \subseteq \mathcal{H}(\theta)$ and any expansion $\mathcal{M}$ of $\langle \mathcal{H}(\theta), \in, \sqsubset, X \rangle$ in a countable language, there is a $\mathcal{P}$ elementary substructure M of $\mathcal{M}$ of cardinality $\aleph_1$ such that $S \cap [X \cap M]^{\aleph_0}$ is stationary in $[X \cap M]^{\aleph_0}$.

(c) For any uncountable cardinal λ , stationary $S \subseteq [\lambda]^{\aleph_0}$, regular $\theta \geq \lambda$ and any expansion $\mathcal{M}$ of $\langle \mathcal{H}(\theta), \in, \sqsubset, \lambda \rangle$ in a countable language, there is a $\mathcal{P}$ elementary substructure M of $\mathcal{M}$ of cardinality $\aleph_1$ such that $S \cap [\lambda \cap M]^{\aleph_0}$ is stationary in $[\lambda \cap M]^{\aleph_0}$. $\square$

Fleissner's Axiom R ([2]) is equivalent to RP_{IU} in our notation. For a any set X of cardinality $> \aleph_1$, let

$(\text{AR}([X]^{\aleph_0}))$: For any stationary $S \subseteq [X]^{\aleph_0}$ and ω_1 -club⁴⁾ $T \subseteq [X]^{\aleph_1}$, there is $U \in T$ such that $S \cap [U]^{\aleph_0}$ is stationary in $[U]^{\aleph_0}$.

Then we define Axiom R to be the assertion that $\text{AR}([\lambda]^{\aleph_0})$ holds for all cardinal $\alpha > \aleph_1$. Since $\text{AR}([\lambda]^{\aleph_0})$, for cardinals $\lambda > \aleph_1$ also satisfy the downward transfer similar to Lemma 2.4, the following Lemma implies the equivalence of RP_{IU} and Axiom R:

Lemma 2.6 For any $\lambda > \aleph_1$, we have $\text{AR}([2^{<\lambda}]^{\aleph_0})$ if and only if $\text{RP}_{\text{IU}}([\mathcal{H}(\lambda)]^{\aleph_0})$. $\square$

Proof. Note that $|\mathcal{H}(\lambda)| = 2^{<\lambda}$ and hence $\text{AR}([2^{<\lambda}]^{\aleph_0})$ is equivalent to $\text{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$.

First, assume $\text{RP}_{\text{IU}}([\mathcal{H}(\lambda)]^{\aleph_0})$. Suppose that $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ is stationary and $T \subseteq [\mathcal{H}(\lambda)]^{\aleph_1}$ is ω_1 -club.

Let $\mathcal{M} = \langle \mathcal{H}(\lambda), \in, \sqsubset, T \rangle$. By Lemma 2.5, there is $M \prec \mathcal{M}$ such that

$$(2.3) \quad |M| = \aleph_1;$$

$$(2.4) \quad M \models \text{IU} \text{ and}$$

$$(2.5) \quad S \cap [M]^{\aleph_0} \text{ is stationary in } [M]^{\aleph_0}.$$

By (2.3), (2.4) and $M \prec \mathcal{M}$, it is easy to see that M is the union of an ω_1 chain of elements of T . By ω_1 -clubness of T it follows that $M \in T$. This shows that $\text{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$ holds.

Assume now $\text{AR}([2^{<\lambda}]^{\aleph_0})$ and suppose that $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ is stationary. Let

$$T = \{M \in [\mathcal{H}(\lambda)]^{\aleph_1} : M \prec \mathcal{H}(\lambda), M \models \text{IU}\}.$$

⁴⁾ $T \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1 -club (or “tight and unbounded” in Fleissner's terminology in [2]) if T is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and for any increasing chain $\langle U_\alpha : \alpha < \omega_1 \rangle$ in T of length ω_1 , we have $\bigcup_{\alpha < \omega_1} U_\alpha \in T$.

Then T is ω_1 -club. By $\text{AR}([2^{<\lambda}]^{\aleph_0})$ or by its equivalent $\text{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$, there is $M \in T$ such that $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$. This shows that $\text{RP}_{\text{IU}}([\mathcal{H}(\lambda)]^{\aleph_0})$ holds. $\square$ (Lemma 2.6)

3 Definition of the games

For a cardinal κ , let

$$(3.1) \quad \kappa \downarrow \kappa = \{f \in {}^\kappa \kappa : f \text{ is regressive}\}.$$

The game $G_\omega^\downarrow(\kappa)$ for Players I and II is defined as follows: A match in $G_\omega^\downarrow(\kappa)$ is a sequence of the form:

$$\begin{array}{c|ccccccc} I & f_0 \in \kappa \downarrow \kappa & f_1 \in \kappa \downarrow \kappa & \cdots & f_n \in \kappa \downarrow \kappa & \cdots & \\ \hline II & \delta_0 \in \kappa & \delta_1 \in \kappa & \cdots & \delta_n \in \kappa & \cdots & \end{array} \quad (n < \omega)$$

Player II wins in a match of $G_\omega^\downarrow(\kappa)$ as above if

$$(3.2) \quad \{\alpha \in E_{\omega_1}^\kappa : f_n(\alpha) < \sup\{\delta_i : i \in \omega\} \text{ for all } n \in \omega\} \text{ is unbounded.}$$

The game $G_\omega^\downarrow(\kappa)$ was introduced in [9]. It is used there to prove the implication of FRP from RC by showing that the assertion (G_0) as in Section 1 defined in terms of this game interpolates the implication.

The following game $G_\omega([\kappa]^{\aleph_1}, \omega_1)$ for Players I and II for a cardinal κ was introduced by Doebler in [1]: A match in $G_\omega([\kappa]^{\aleph_1}, \omega_1)$ is a sequence of the form:

$$\begin{array}{c|ccccccc} I & f_0 \in [\kappa]^{\aleph_1} \omega_1 & f_1 \in [\kappa]^{\aleph_1} \omega_1 & \cdots & f_n \in [\kappa]^{\aleph_1} \omega_1 & \cdots & \\ \hline II & \delta_0 \in \omega_1 & \delta_1 \in \omega_1 & \cdots & \delta_n \in \omega_1 & \cdots & \end{array} \quad (n < \omega)$$

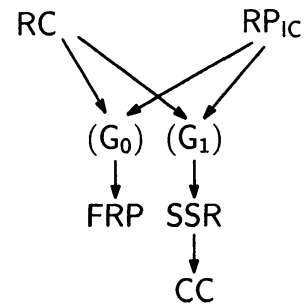
II wins in a match of $G_\omega([\kappa]^{\aleph_1}, \omega_1)$ as above if

$$\{a \in [\kappa]^{\aleph_1} : f_n(a) < \sup\{\delta_i : i \in \omega\} \text{ for all } n \in \omega\}$$

is cofinal in $[\kappa]^{\aleph_1}$.

Doebler proved that the principle (G_1) as defined in Section 1 in terms of this game follows also from RC and it implies SSR.

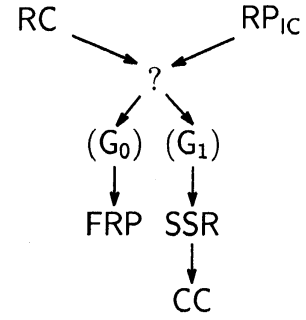
It is easy to see that both of (G_0) and (G_1) are consequences of RP_{IC} (this also follows from Corollary 4.4). Hence we have the diagram on the right:



Since FRP and SSR imply almost all known consequences of $\text{RC}^{(5)}$, it seems to be an interesting question what is the natural principle which is still a consequence of both RC and RP_{IC} while which implies both FRP and SSR.

⁵⁾ Perhaps with the exception of the negation of Martin's Axiom for $\aleph_1$ dense sets which is a consequence of RC while $\text{RC}_{\mathcal{P}}$'s are consistent with Martin's Axiom since they all follow from $\text{MA}^+(\sigma\text{-closed})$.

The assertion of the existence of the winning strategy for player *II* (the principle $(G^{\downarrow\downarrow})$ introduced in Section 1) in the following game $G^{\downarrow\downarrow}_\omega([\kappa]^{\aleph_1})$ for all $\kappa > \aleph_1$ seemed to be a natural candidate for such an interpolant. Unfortunately, this principle turned out to be too strong to be a consequence of RC while it is still a consequence of RP_{IC} as we shall see in Section 4. In [9] we introduce a weakening of $(G^{\downarrow\downarrow})$ which is an interpolant of RC and RP_{IC} on one side and FRP and SSR on the other.



Here is the definition of $G^{\downarrow\downarrow}_\omega([\kappa]^{\aleph_1})$ for a cardinal $\kappa > \aleph_1$. We call a function $f : [\kappa]^{\aleph_1} \rightarrow \kappa$ regressive if $f(a) \in a$ holds for all $a \in [\kappa]^{\aleph_1}$. Similarly to the definition (3.1), let

$$(3.3) \quad [\kappa]^{\aleph_1} \downarrow \kappa = \{f \in [\kappa]^{\aleph_1} : f \text{ is regressive}\}.$$

A match in $G^{\downarrow\downarrow}_\omega([\kappa]^{\aleph_1})$ for Players *I* and *II* is a sequence of the form:

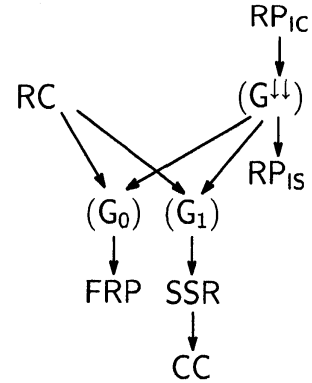
$$\begin{array}{c|ccccccc} I & f_0 \in [\kappa]^{\aleph_1} \downarrow \kappa & f_1 \in [\kappa]^{\aleph_1} \downarrow \kappa & \dots & f_n \in [\kappa]^{\aleph_1} \downarrow \kappa & \dots & \\ \hline II & d_0 \in [\kappa]^{\aleph_0} & d_1 \in [\kappa]^{\aleph_0} & \dots & d_n \in [\kappa]^{\aleph_0} & \dots & \end{array} \quad (n < \omega)$$

II wins in a match in $G^{\downarrow\downarrow}_\omega([\kappa]^{\aleph_1})$ as above if

$$\{a \in [\kappa]^{\aleph_1} : f_n(a) \in \bigcup \{d_i : i \in \omega\} \text{ for all } n \in \omega\}$$

is cofinal in $[\kappa]^{\aleph_1}$.

Note that by the definition of the games, it is clear that $(G^{\downarrow\downarrow})$ implies both of (G_0) and (G_1) .



4 Characterizations of $(G^{\downarrow\downarrow})$

The following characterization of $(G^{\downarrow\downarrow})$ can be obtained easily by regarding the moves of Player *I* in $G^{\downarrow\downarrow}_\omega([\kappa]^{\aleph_1})$ as an enumeration of Skolem functions with parameters in some model M and the moves of Player *II* as the gradual capturing of $\kappa \cap M$:

Lemma 4.1 *For any cardinal $\kappa > \aleph_1$ the following are equivalent:*

(a) $WS_{II}(G^{\downarrow\downarrow}_\omega([\kappa]^{\aleph_1}))$.

(b) *For sufficiently large regular θ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, for any $M \prec \mathcal{M}$ with $|M| = \aleph_0$ and $\kappa \in M$, we have: for any $a \in [\kappa]^{\aleph_1}$, there are $b \in [\kappa]^{\aleph_1}$ and countable $N \prec \mathcal{M}$ such that $a \subseteq b$, $b \in N$, $M \subseteq N$ and $b \cap N = b \cap M$.*

(c) *For sufficiently large regular θ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, for club many⁶⁾ countable $M \prec \mathcal{M}$ with $\kappa \in M$, we have: for any $a \in [\kappa]^{\aleph_1}$, there are $b \in [\kappa]^{\aleph_1}$ and countable $N \prec \mathcal{M}$ such that $a \subseteq b$, $b \in N$, $M \subseteq N$ and $b \cap N = b \cap M$. $\square$*

⁶⁾ We can also express this “club many ...” in terms of expansion of the structure $\mathcal{M}$ similarly to Lemma 2.3 or Lemma 2.4.

By Lemma 4.1, (c), we see immediately that $WS_{II}(G_{\omega}^{\downarrow\downarrow}([\kappa]^{\aleph_1}))$ for cardinals $\kappa > \aleph_1$ also enjoy the downward transfer property:

Corollary 4.2 *Suppose $\aleph_1 < \kappa' < \kappa$ and $WS_{II}(G_{\omega}^{\downarrow\downarrow}([\kappa]^{\aleph_1}))$ holds. Then we also have $WS_{II}(G_{\omega}^{\downarrow\downarrow}([\kappa']^{\aleph_1}))$.*

Theorem 4.3 *The following are equivalent: (a) $(G^{\downarrow\downarrow})$.*

(b) *For all $\kappa > \aleph_1$, for all sufficiently large regular θ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, there are club many countable $M \prec \mathcal{M}$ such that $\kappa \in M$ and for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, there are $Y \in [\mathcal{H}(\kappa)]^{\aleph_1}$ and countable $N \prec \mathcal{M}$ such that $X \subseteq Y$, $Y \in N$, $M \subseteq N$ and $Y \cap N = Y \cap M$.*

(c) *For all $\kappa > \aleph_1$, for all sufficiently large regular θ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, there are club many countable $M \prec \mathcal{M}$ such that $\kappa \in M$ and for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, there are $Z \prec \langle \mathcal{H}(\kappa), \in, \sqsubseteq_{\mathcal{H}(\kappa)} \rangle$ of cardinality $\aleph_1$ and countable $N \prec \mathcal{M}$ such that $X \subseteq Z$, $Z \in N$, $M \subseteq N$ and $Z \cap N = Z \cap M$.*

(d) *For any $\kappa > \aleph_1$ and stationary $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$, for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$ there is a $Z \prec \mathcal{H}(\kappa)$ such that $X \subseteq Z$, $|Z| = \aleph_1$ and $S \cap Z$ is stationary in $[Z]^{\aleph_0}$.*

(e) *For all $\kappa > \aleph_1$, for all sufficiently large regular θ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, there are club many countable $M \prec \mathcal{M}$ such that $\kappa \in M$ and for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, there are $Z \prec \langle \mathcal{H}(\kappa), \in, \sqsubseteq_{\mathcal{H}(\kappa)} \rangle$ of cardinality $\aleph_1$ and countable $N \prec \mathcal{H}(\theta)$ such that $X \subseteq Z$, $Z \in N$, $M \subseteq N$ and $Z \cap N = Z \cap M$.*

Proof. (a) $\Rightarrow$ (b): Let $\lambda = 2^{<\kappa} = |\mathcal{H}(\kappa)|$ and let $\varphi : \lambda \rightarrow \mathcal{H}(\kappa)$ be a bijection. Then all countable $M \prec \mathcal{M}$ with $\varphi \in M$ satisfies the condition in (b): the situation of Lemma 4.1, (b) (for κ there $= \lambda$) is translated to the desired condition in the present (b) by φ .

(b) $\Rightarrow$ (a): The back-translation by the mapping φ as in the proof of (a) $\Rightarrow$ (b) implies $WS_{II}(G_{\omega}^{\downarrow\downarrow}([2^{<\kappa}]^{\aleph_1}))$ for all $\kappa > \aleph_1$. By Corollary 4.2, it follows that $WS_{II}(G_{\omega}^{\downarrow\downarrow}([\kappa]^{\aleph_1}))$ for all $\kappa > \aleph_1$.

(b) $\Rightarrow$ (c): Suppose that $\kappa, \theta, \mathcal{M}, M, X, Y, N$ are as in (b). Then $Z = sk_{\mathcal{M}}(Y)$ witnesses (c).

(c) $\Rightarrow$ (d): Assume that (c) holds and suppose that $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$ is stationary. Let $\theta, \mathcal{M}, M$ be as in (c). Since there are club many M 's as in (c), we may assume that

$$(4.1) \quad S \in M \text{ and } \mathcal{H}(\kappa) \cap M \in S.$$

Let $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$ be defined by

$$(4.2) \quad X = \omega_1 \cup (\mathcal{H}(\kappa) \cap M) \cup \{M \cap \mathcal{H}(\kappa)\}.$$

Let $Z \prec \mathcal{H}(\kappa)$ and $N \prec \mathcal{H}(\theta)$ be as in (c) for this X . Thus we have N is countable, Z is of cardinality $\aleph_1$, $X \subseteq Z$, $Z \in N$, $M \subseteq N$ and

$$(4.3) \quad Z \cap N = Z \cap M.$$

We are done by showing that $S \cap Z$ is stationary in $[Z]^{\aleph_0}$. Since $S, Z \in N$, it is enough to show that any club $C \subseteq [Z]^{\aleph_0}$ with $C \in N$ intersects with S : Note that we have

$$(4.4) \quad Z \cap M = \mathcal{H}(\kappa) \cap M$$

by (4.2). For such a club C we have

$$\begin{aligned} S \cap M \cap \mathcal{H}(\kappa) & \quad \text{by (4.1)} \\ &= Z \cap M \quad \text{by (4.4)} \\ &= Z \cap N \quad \text{by (4.3)} \\ &= \bigcup (C \cap N) \in C. \quad \text{by elementarity, } C \in N \text{ and closedness of } C \end{aligned}$$

Thus $S \cap C \neq \emptyset$ as desired.

(c) $\Rightarrow$ (e): The proof of (c) $\Rightarrow$ (d) above for $S = [\mathcal{H}(\kappa)]^{\aleph_0}$ shows this.

(e) $\Rightarrow$ (c): trivial.

(c) $\Rightarrow$ (b): trivial.

(d) $\Rightarrow$ (e): Assume that (d) holds. For $\kappa > \aleph_1$, let θ a sufficiently large regular cardinal and $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$. Let

$$(4.5) \quad S = \{M \in [\mathcal{M}]^{\aleph_0} : M \prec \mathcal{M}, \kappa \in M, \text{ there is } X_M \in [\mathcal{H}(\kappa)]^{\aleph_1} \text{ such that}$$

$$(4.6) \quad \begin{aligned} &\text{there are no countable } N \prec \mathcal{M} \\ &\text{and } Y \prec \langle \mathcal{H}(\kappa), \in, \sqsubset_{\mathcal{H}(\kappa)} \rangle \text{ such that} \\ &M \prec N, X_M \subseteq Y, Y \text{ is IS and of size } \aleph_1, \\ &Y \in N \text{ and } M \cap Y = N \cap Y \end{aligned} \}.$$

It is enough to show that S is non-stationary. In the following we show this indirectly: We assume that S is stationary and drive a contradiction from this assumption.

For each $M \in S$ we choose $X_M \in [\mathcal{H}]^{\aleph_1}$ such that

$$(4.7) \quad X_M \supseteq M \cup \omega_1, X_M \prec \mathcal{M} \text{ and (4.6) holds for } M \text{ and } X_M.$$

Let $\chi > 2^{<\theta}$ be regular. Note that we have $\mathcal{H}(\theta) \in \mathcal{H}(\chi)$. Let

$$(4.8) \quad \tilde{S} = \{M \in [\mathcal{H}(\chi)]^{\aleph_0} : M \prec \langle \mathcal{H}(\chi), \in, \sqsubset_{\mathcal{H}(\chi)} \rangle, \\ \kappa, \theta, \dots \in M, M \cap \mathcal{H}(\theta) \in S \}.$$

By the assumption of the stationarity of S , $\tilde{S}$ is also stationary. Thus, by (d), there is $Z \prec \langle \mathcal{H}(\chi), \in, \sqsubset \rangle$ such that

$$(4.9) \quad |Z| = \aleph_1, \omega_1 \subseteq Z,$$

$$(4.10) \quad \kappa, \theta, S, \langle X_M : M \in S \rangle, \sqsubset_{\mathcal{H}(\kappa)}, \sqsubset_{\mathcal{H}(\theta)}, \dots \in Z \text{ and}$$

$$(4.11) \quad \tilde{S} \cap Z \text{ is stationary in } [Z]^{\aleph_0}.$$

Let

$$(4.12) \quad Y = Z \cap \mathcal{H}(\kappa).$$

Then we have $\omega_1 \subseteq Y$ and hence $|Y| = \aleph_1$ and $Y \prec \langle \mathcal{H}(\kappa), \in, \sqsubseteq_{\mathcal{H}(\kappa)} \rangle$.

Y is stationary in $[Y]^{\aleph_0}$: Suppose that $C \subseteq [Y]^{\aleph_0}$ is a club. Let $\tilde{C} = \{x \in [Z]^{\aleph_0} : x \cap Y \in C\}$. Then $\tilde{C}$ is a club subset of $[Z]^{\aleph_0}$. By (4.11), $Z \cap [Z]^{\aleph_0}$ is stationary in $[Z]^{\aleph_0}$. Hence there is an $x \in \tilde{C} \cap Z$. By definition of $\tilde{C}$, we have $x \cap \mathcal{H}(\kappa) \in C$.

Since $Z \prec \langle \mathcal{H}(\chi), \in, \sqsubseteq_{\mathcal{H}(\chi)} \rangle$ and $\mathcal{H}(\kappa) \in Z$ by (4.10), we have $x \cap \mathcal{H}(\kappa) \in Z$. By $x \cap \mathcal{H}(\kappa) \in C$ we have $x \cap \mathcal{H}(\kappa) \in \mathcal{H}(\kappa)$. It follows that $x \cap \mathcal{H}(\kappa) \in Y$. Thus $x \cap \mathcal{H}(\kappa) \in C \cap Y$.

For each $M \in \tilde{S} \cap Z$ we have $M \cap \mathcal{H}(\theta) \in S \cap Z$ as we just saw and, by (4.10), $X_{M \cap \mathcal{H}(\theta)} \in Z \cap \mathcal{H}(\kappa) = Y$. Since $\omega_1 \subseteq Y$, it follows that

$$(4.13) \quad X_{M \cap \mathcal{H}(\theta)} \subseteq Y.$$

By (4.11), there is countable $N^* \prec \langle \mathcal{H}(\chi), \in, \sqsubseteq_{\mathcal{H}(\chi)} \rangle$ such that

$$(4.14) \quad N^* \cap Z \in \tilde{S} \cap Z \text{ and}$$

$$(4.15) \quad X, Y, Z, \dots \in N^*.$$

Let $M^* = (N^* \cap Z) \cap \mathcal{H}(\theta)$. Then we have $M^* \in S$ by (4.14). $M^* \subseteq N^* \cap \mathcal{H}(\theta)$ by the definition of M^* and $X_{M^*} \subseteq Y$ by (4.13). $Y \in N^* \cap \mathcal{H}(\theta)$ by (4.12) and (4.15). We also have

$$(4.16) \quad M^* \cap Y = ((N^* \cap Z) \cap \mathcal{H}(\theta)) \cap (Z \cap \mathcal{H}(\kappa)) = M^* \cap \mathcal{H}(\kappa) = (N^* \cap \mathcal{H}(\theta)) \cap Y.$$

Thus $N^* \cap \mathcal{H}(\theta)$ and Y contradict to the choice of X_{M^*} . □ (Theorem 4.3)

Corollary 4.4 *The following implications hold:*

$$\text{RP}_{\text{IC}} \Rightarrow (\text{G}^{\downarrow\downarrow}) \Rightarrow \text{RP}_{\text{IS}}. \quad \square$$

Proof. By Theorem 4.3, (d). The implication “ $\text{RP}_{\text{IC}} \Rightarrow (\text{G}^{\downarrow\downarrow})$ ” follows from the following trivial observation. □ (Corollary 4.4)

Lemma 4.5 *If $M \prec \mathcal{H}(\theta)$ is IC and $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$, then $S \cap (M \cap [M]^{\aleph_0})$ is stationary in $[M]^{\aleph_0}$ as well.* □

Corollary 4.6 *Under the CH, we have:*

$$\text{Axiom R} \Leftrightarrow \text{RP}_{\text{IU}} \Leftrightarrow \text{RP}_{\text{IS}} \Leftrightarrow (\text{G}^{\downarrow\downarrow}) \Leftrightarrow \text{RP}_{\text{IC}} \Leftrightarrow \text{RP}_{\text{IA}}.$$

Proof. By Lemma 2.2, Lemma 2.6 and Corollary 4.4. □ (Corollary 4.6)

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