

Locally finite higher
derivations and its applications

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§1. Definition.

Let A be a reduced ring with identity element. A higher derivation $\underline{D}$ is a system of additive endomorphism D_i of A ($i = 0, 1, 2, \dots$, $D_0 = \text{id}$) such that they satisfy

$$(1) \quad D_n(ab) = \sum_{i=0}^n D_i(a)D_{n-i}(b) \quad (n = 1, 2, \dots)$$

for any element a, b of A . This is equivalent to saying that the mapping ϕ of A into a formal power series ring $A[[T]]$ defined by

$$(2) \quad \phi(a) = a + \sum_{i \geq 1} D_i(a)T^i$$

is a ring homomorphism of A into $A[[T]]$ such that $\epsilon\phi = \text{id}$, where ϵ is an augmentation $A[[T]] \rightarrow A$ defined by $\epsilon(a) = a$ and $\epsilon(T) = 0$. A higher derivation D of A is called iterative if they satisfy moreover the identity

$$(3) \quad D_i D_j = \binom{i+j}{i} D_{i+j}.$$

A higher derivation D is called locally finite if $I_m(\phi)$ is contained in a polynomial ring $A[T]$.

Lemma 1. Let A be a reduced ring and let D be a locally finite higher derivation (ℓ fhd). Then any unit in A is killed by D , i. e., if a is a unit of A then $D_i(a) = 0$ for $i \geq 1$.

Corollary. The following rings have no non-trivial ℓ fhd.

(1) A field: (2) A local ring (3) An integral domain with non-zero Jacobson radical.

§2. Application to invariance of rings.

A ring A is called a strongly invariant ring whenever the relation $A[X_1, \dots, X_n] = B[Y_1, \dots, Y_n]$ implies $A = B$ where $X_1, \dots, X_n$ are indeterminates and $Y_1, \dots, Y_n$ are independent variables over B .

A ring A is called an invariant ring whenever the relation $A[X_1, \dots, X_n] = B[Y_1, \dots, Y_n]$ implies A and B are isomorphic.

Theorem 2. Let A be a reduced ring, then A is strongly invariant if A has no non-trivial ℓ fhd. A is not strongly invariant if A has a non-trivial ℓ fihd.

Theorem 3. Let A be an affine domain over a field k such that $\text{Spec}(A)(k)$ is dense in $\text{Spec}(A)$. If A has a non-trivial ℓ fhd, then $Q(A)$ is rational over k . If, moreover, A is normal A is isomorphic to a polynomial ring $k[t]$.

§3. An integral domain with non-trivial ℓ fhds.

Let A be an integral domain and let D be a ℓ fhd. Let A_i be defined by

$$A_i = \{a \in A \mid D_n(a) = 0 \text{ for all } n > i\}.$$

An integer n such that $A_{n-1} \subsetneq A_n$ is called a jump index.

Proposition 4. Let A, D, A_i be as above. If the characteristic of an integral domain A is zero. Then the first jump index is 1. If the characteristic of A is a positive prime integer p , then we have the following:

- (i) The first jump index is a power of p , say, $q = p^s$.
- (ii) The m -th jump index is mp^s ($m = 1, 2, \dots$).
- (iii) Let a be an element of $A_q \setminus A_{q-1}$. Then $\text{Supp}(a)$ consists of powers of p where $\text{Supp}(a) = \{k \in \mathbb{N} \mid D_k(a) \neq 0\}$. Moreover if $k \in \text{Supp}(a)$, then $D_k(a)$'s are D -constant, i. e., $D_k(a) \in A_0$.

Let $a (\in A)$ be an element of A . Then $\ell(a)$ is, by definition, the largest integer m such $a \in A_{mq} \setminus A_{(m-1)q}$. In this case m is called the index of a and will be denoted by $i(a)$.

Let A be an integral domain. Let $\underline{D}^{(1)}, \dots, \underline{D}^{(n)}$ be n - ℓ fhds of A . We say that $\underline{D}^{(1)}, \dots, \underline{D}^{(n)}$ are independent if

$$B_1 \cap \dots \cap B_i \cap \dots \cap B_n \not\subset B_i \quad (i = 1, \dots, n)$$

where $B_i = (\underline{D}^{(i)})^{-1}(0) = \{a \in A \mid D_j^{(i)}(a) = 0 \text{ for } j = 1, 2, \dots\}$.

They are said to be commutative if we have

$$D_j^{(i)} D_\ell^{(k)} = D_\ell^{(k)} D_j^{(i)} \quad (\text{for all } j, \ell = 1, 2, \dots).$$

Theorem 5. Let A be an integral domain and let $\underline{D}^{(i)}$ $i = 1, 2, \dots, n$ be n -independent, mutually commutative, non-trivial ℓ fihd, and let $B^i = (\underline{D}^{(i)})^{-1}(0)$. Then there exists an element ω in $A_0 = \bigcap_{i=1}^n B_i$, such that

$$A[\omega^{-1}] = A_0[\omega^{-1}][x_1, \dots, x_n]$$

where $x_1, \dots, x_n$ are independent variables over $Q(A_0)$.

Corollary 6. Let A be an affine domain of transcendence degree n over a field k such that $\bar{k} \cap A = k$. Assume that A has n -independent, mutually commutative, non-trivial ℓ fihd $\underline{D}^i$ ($i = 1, \dots, n$). Then A is a polynomial ring in n -variables over k . Moreover if $x_1, \dots, x_n$ are elements of A such that

- (1) x_i is an element of index 1 with respect to $\underline{D}^i$.
- (2) $x_i \in \bigcap_{j \neq i} (\underline{D}^j)^{-1}(0)$.

Then $A = k[x_1, x_2, \dots, x_n]$.

Corollary 7. Let $F_1, \dots, F_n$ be polynomials in n -variables $x_1, \dots, x_n$ over a field k of characteristic zero. Assume

$$\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \in k^*.$$

If the derivation D_i defined by

$$D_i(g) = \frac{\partial(F_1, \dots, \overset{i}{g}, \dots, F_n)}{\partial(x_1, \dots, x_i, \dots, x_n)}, \quad (i = 1, 2, \dots, n)$$

is locally nilpotent for any i , i. e., any element is killed by some power of D_i , then we have $k[F_1, \dots, F_n] = k[x_1, \dots, x_n]$.

Corollary 8. Separable forms of affine n -spaces
($n = 1, 2$) are trivial.

§4. Characterization of a polynomial ring $k[x, y]$.

Theorem 9. Let k be an algebraically closed field of arbitrary characteristic and let A be an integral domain satisfying conditions:

- (1) There exists a non-trivial lfihd $\underline{A}$ over k .
- (2) The constant ring A_0 of $\underline{A}$ is either (2,1) a principal ideal domain finitely generated over k or (2,2) a DVR whose residue field is k .
- (3) Any prime element of A_0 remains prime in A .

Then A is a polynomial ring in one variable over A_0 .

As application of Theorem 9 we have the following two theorems.

Theorem 10 (T. Kambayashi). Let (D, M) be a DVR with algebraically closed residue field $k = D/M$. Let K be the field of fractions of D . Let A be a flat D -algebra of finite type. Assume that $A \otimes_D K = K[t]$ and $A \otimes_D K$ is an integral domain. Then A is isomorphic to a polynomial ring $D[T]$.

Theorem 11. Let k be an algebraically closed field and let A be a normal affine domain over k such that

- (i) $\dim A = 2$.
- (ii) $A^* = k^*$ where $*$ denote the set of units.

(iii) Either A is UFD or $Q(A)$ is unirational.

Let Δ be a non-trivial lfihd of A over k . Then the constant ring A_0 of Δ is a polynomial ring over k .

Theorem 12 (M. Miyanishi). Let k be an algebraically closed field and let A be a finitely generated integral domain over k . Assume the following:

- (i) $\dim A = 2$.
- (ii) $A^* = k^*$.
- (iii) A is UFD.
- (iv) A has a non-trivial lfihd.

Then $A \cong k[x, y]$ where x, y are independent variables.

§5. Lines in an affine 2-space.

An affine plane curve C defined over k by the equation: $f(x, y) = 0$ is called a quasi-line if the coordinate ring $A = k[x, y]/(f)$ is isomorphic to a polynomial ring $k[t]$. C is called a line if there exists a curve $\Gamma: g(x, y) = 0$ such that $k[x, y] = k[f, g]$.

Theorem 13. Let k be an algebraically closed field and let $C: f(x, y) = 0$ be a curve defined over k . Then the following conditions are equivalent to each other

- (1) C is a line.
- (2) There exists a lfihd Δ such that $\Delta(f) = 0$.
- (3) $C_u: f(x, y) - u = 0$ is a quasi-line over $k(u)$ where u is an indeterminate.

Theorem 14. Let k be an algebraically closed field of characteristic zero and let $C : f(x,y) = 0$ be an irreducible curve over k . Then C is a line if and only if the derivation

$$D_f = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}$$

is locally nilpotent.

References

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