

On diffeomorphisms over non-orientable surfaces embedded in the 4-sphere

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1. INTRODUCTION

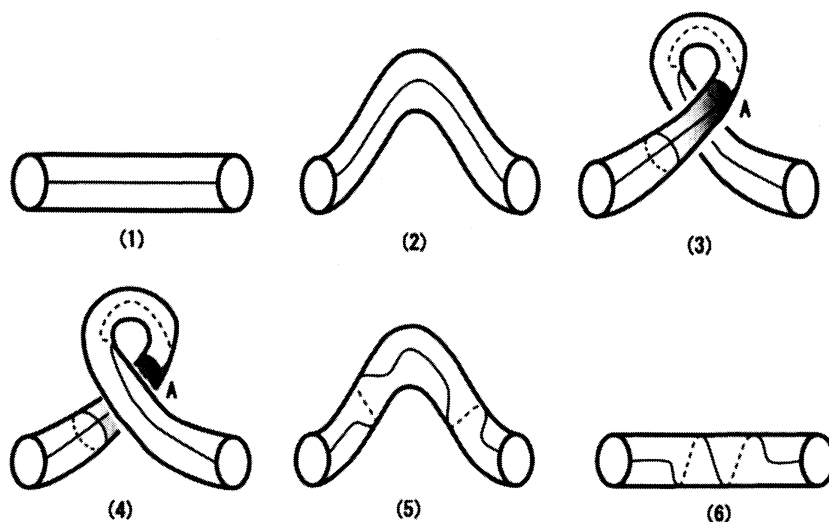


FIGURE 1

We put an annulus in $\mathbb{R}^4$, and deform this in $\mathbb{R}^4$ with fixing its boundary as shown in Figure 1. We can change crossing from (3) to (4) because this annulus is in $\mathbb{R}^4$. After this deformation, this annulus is twisted two times along the core. This means that this double twist can be extended to the ambient $\mathbb{R}^4$. In this note, we will discuss how many diffeomorphisms over the embedded surface are extendable to the ambient 4-space.

For some special embeddings of closed surfaces in 4-manifolds, we have answers to the above problem (for example, [9], [3], [4]). An embedding e of the orientable surface Σ_g into S^4 is called standard if there is an embedding of 3-dimensional handlebody into S^4 such that whose boundary is the image of e . In [9] and [3], we showed:

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Theorem 1.1 ([9] ($g = 1$), [3] ($g \geq 2$)). *Let Σ_g be standardly embedded in S^4 . A diffeomorphism ϕ over the Σ_g is extendable to S^4 if and only if ϕ preserves the Rokhlin quadratic form of the Σ_g .*

In this note, we will introduce some approach to the same kind of problem for non-orientable surfaces embedded in S^4 .

2. SETTING

Let N_g be a connected non-orientable surface constructed from g projective planes by connected sum. We call N_g the *closed non-orientable surface of genus g* . For a smooth embedding e of N_g into S^4 , Guillou and Marin ([2] see also [8]) defined a quadratic form $q_e : H_1(N_g; \mathbb{Z}_2) \rightarrow \mathbb{Z}_4$ as follows: Let C be an immersed circle on N_g , and D be a connected orientable surface immersed in S^4 such that $\partial D = C$, and D is not tangent to N_g . Let ν_D be the normal bundle of D , then $\nu_D|_C$ is a solid torus with the unique trivialization induced from any trivialization of ν_D . Let $N_{N_g}(C)$ be the tubular neighborhood of C in N_g , then $N_{N_g}(C)$ is a twisted annulus or Möbius band in $\nu_D|_C$. We denote by $n(D)$ the number of right hand half-twist of $N_{N_g}(C)$ with respect to the trivialization of $\nu_D|_C$. Let $D \cdot F$ be mod-2 intersection number of D and F , $Self(C)$ be mod-2 double points number of C , and 2 be an injection $\mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ defined by $2[n]_2 = [2n]_4$. Then the number $n(D) + 2D \cdot F + 2Self(C)$ (mod 4) depend only on the mod-2 homology class $[C]$ of C . Hence, we define

$$q_e([C]) := n(D) + 2D \cdot F + 2Self(C) \pmod{4}.$$

This map q_e is called Guillou and Marin *quadratic form*, since q_e satisfies

$$q_e(x + y) = q_e(x) + q_e(y) + 2 \langle x, y \rangle_2,$$

where $\langle x, y \rangle_2$ means mod-2 intersection number between x and y . This quadratic form q_e is a non-orientable analogy of Rokhlin quadratic form.

A diffeomorphism ϕ over N_g is *e-extendable* if there is an orientation preserving diffeomorphism Φ of S^4 such that the following diagram is commutative,

$$\begin{array}{ccc} N_g & \xrightarrow{e} & S^4 \\ \phi \downarrow & & \downarrow \Phi \\ N_g & \xrightarrow{e} & S^4. \end{array}$$

If the diffeomorphisms ϕ_1 over N_g is *e-extendable*, and ϕ_1 and ϕ_2 are isotopic, then ϕ_2 is *e-extendable*. Therefore, *e-extendability* is a property about isotopy classes of diffeomorphisms over N_g . The group $\mathcal{M}(N_g)$ of isotopy classes of diffeomorphisms over N_g is called *the mapping class group of N_g* . An element ϕ of $\mathcal{M}(N_g)$ is *e-extendable* if there is an *e-extendable* representative of ϕ . By the definition of q_e , we can see that if $\phi \in \mathcal{M}(N_g)$ is *e-extendable* then ϕ preserves q_e , i.e. $q_e(\phi_*(x)) = q_e(x)$ for any $x \in H_1(N_g; \mathbb{Z}_2)$. What we would like to know is whether $\phi \in \mathcal{M}(N_g)$ is *e-extendable* when ϕ preserves q_e . But the answer for this problem would be depend on the embedding e . So, we will introduce an embedding which seems to be *simplest*.

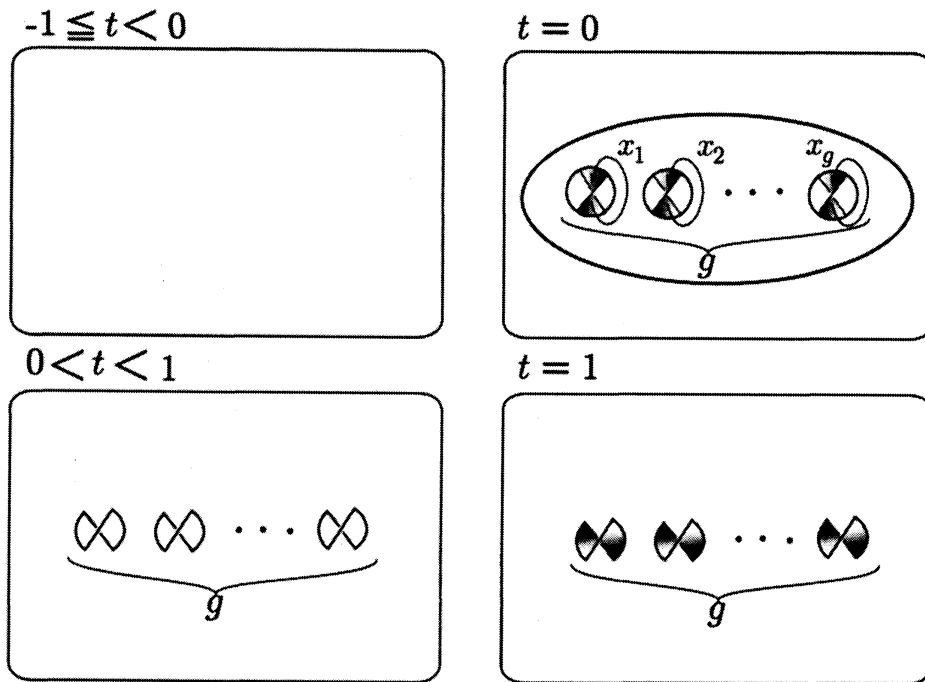


FIGURE 2

Let $S^3 \times [-1, 1]$ be a closed tubular neighborhood of the equator S^3 in S^4 . Then $S^4 - S^3 \times (-1, 1)$ consists of two 4-balls. Let D_+^4 be the northern component of

them, and D_-^4 be the southern component of them. An embedding $ps : N_g \hookrightarrow S^4$ is *p-standard* if $ps(N_g) \subset S^3 \times [-1, 1]$ and as shown in Figure 2. For the basis $\{e_1, \dots, e_g\}$ of $H_1(N_g; \mathbb{Z}_2)$ shown in Figure 2, $q_{ps}(e_i) = 1$. Since $\langle e_i, e_j \rangle_2 = \delta_{ij}$, $q_{ps}(e_{i_1} + e_{i_2} + \dots + e_{i_t}) = t$. The problem which we consider is the following:

Problem 2.1. If ϕ preserves q_{ps} , is $\phi \in \mathcal{M}(N_g)$ *ps-extendable*?

In order to approach this problem, we review the generators for $\mathcal{M}(N_g)$.

3. GENERATORS FOR $\mathcal{M}(N_g)$

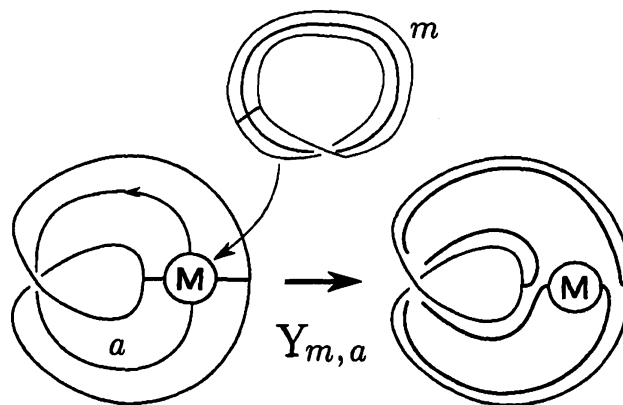


FIGURE 3. M with circle indicates a place where to attach a Möbius band

A simple closed curve c on N_g is A-circle (resp. M-circle), if the tubular neighborhood of c is an annulus (resp. a Möbius band). We denote by t_c the Dehn twist about an A-circle c on N_g . Lickorish [6] showed that $\mathcal{M}(N_g)$ is not generated by Dehn twists, and that Dehn twists and *Y-homeomorphisms* generate $\mathcal{M}(N_g)$. We review the definition of Y-homeomorphism. Let m be an M-circle and a be an oriented A-circle in N_g such that m and a transversely intersect in one point. Let $K \subset N_g$ be a regular neighborhood of $m \cup a$, which is homeomorphic to the Klein bottle with a hole, and let M be a regular neighborhood of m , which is a Möbius band. We denote by $Y_{m,a}$ a homeomorphism over N_g which may be described as the result of pushing M once along a keeping the boundary of K fixed (see Figure 3). We call

$Y_{m,a}$ a Y -homeomorphism. Since Y -homeomorphisms act on $H_1(N_g; \mathbb{Z}_2)$ trivially, Y -homeomorphisms do not generate $\mathcal{M}(N_g)$. Szepietowski [11] showed an interesting results on the proper subgroup of $\mathcal{M}(N_g)$ generated by all Y -homeomorphisms.

Theorem 3.1 ([11]). $\Gamma_2(N_g) = \{\phi \in \mathcal{M}(N_g) \mid \phi_* : H_1(N_g; \mathbb{Z}_2) \rightarrow H_1(N_g; \mathbb{Z}_2) = id\}$ is generated by Y -homeomorphisms.

In Appendix, we give a quick proof for this Theorem.

Chillingwirth showed that $\mathcal{M}(N_g)$ is finitely generated.

Theorem 3.2 ([1]). $t_{a_1}, \dots, t_{a_{g-1}}, t_{b_2}, \dots, t_{b_{\lfloor \frac{g}{2} \rfloor}}, Y_{m_{g-1}, a_{g-1}}$ generate $\mathcal{M}(N_g)$.

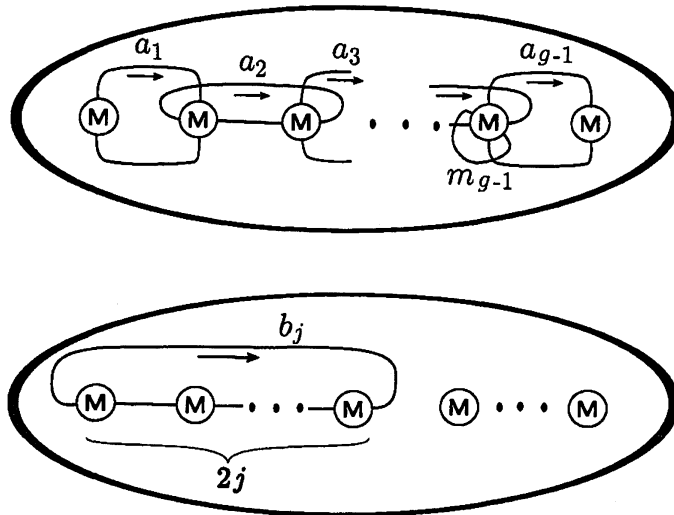


FIGURE 4

4. LOWER GENUS CASES

When genus g is at most 3, Problem 2.1 has a trivial answer.

The case where genus $g = 1$: $\mathcal{M}(N_1)$ is trivial.

The case where genus $g = 2$: $\mathcal{M}(N_2)$ is generated by two elements t_{a_1} and Y_{m_1, a_1} . Since the tubular neighborhood of a_1 in N_2 is a Hopf-band in $S^3 \times \{0\}$, t_{a_1} is ps -extendable by [4, §2]. Since a sliding of a Möbius band along the tube illustrated in Figure 5 is an extension of Y_{m_1, a_1} , Y_{m_1, a_1} is ps -extendable. Therefore, any element of $\mathcal{M}(N_2)$ is ps -extendable.

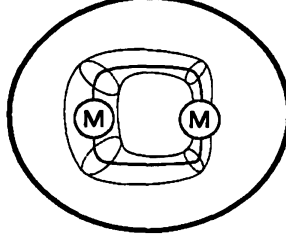


FIGURE 5

The case where genus $g = 3$: $\mathcal{M}(N_3)$ is generated by three elements t_{a_1} , t_{a_2} and Y_{m_2, a_2} . By the same argument as in the above case, it is shown that any element of $\mathcal{M}(N_3)$ is ps -extendable.

5. HIGHER GENUS CASES

In the case where genus $g = 4$, t_{b_4} does not preserve q_{ps} because $q_{ps}((t_{b_4})_*(x_1)) = q_{ps}(x_2 + x_3 + x_4) = 3 \neq 1 = q_{ps}(x_1)$. Therefore, t_{b_4} is not ps -extendable. We should consider the following subgroup of $\mathcal{M}(N_g)$,

$$\mathcal{N}_g = \{\phi \in \mathcal{M}(N_g) \mid q_{ps}(\phi_*(x)) = q_{ps}(x) \text{ for any } x \in H_1(N_g; \mathbb{Z}_2)\}.$$

In order to find a finite system of generators of $\mathcal{N}_g$, we introduce a group

$$\mathcal{O}_g = \{\phi_* \in \text{Aut}(H_1(N_g; \mathbb{Z}_2), \langle, \rangle_2) \mid \phi \in \mathcal{N}_g\}.$$

Then we have a natural short exact sequence

$$0 \rightarrow \Gamma_2(N_g) \rightarrow \mathcal{N}_g \rightarrow \mathcal{O}_g \rightarrow 0.$$

Since $\Gamma_2(N_g)$ is a finite index subgroup of $\mathcal{M}(N_g)$ and $\mathcal{O}_g$ is a finite group, theoretically, there is a finite system of generators for $\mathcal{N}_g$. But we would like to find an *explicit* system of generators. Nowik found a system of generators for $\mathcal{O}_g$ explicitly. For $a \in H_1(N_g; \mathbb{Z}_2)$, define $T_a : H_1(N_g; \mathbb{Z}_2) \rightarrow H_1(N_g; \mathbb{Z}_2)$ (transvection) by $T_a(x) = x + \langle x, a \rangle_2 a$, where $\langle, \rangle_2$ means mod-2 intersection form. We remark that if l is a simple closed curve on N_g such that $[l] = a \in H_1(N_g; \mathbb{Z}_2)$, then $(t_l)_* = T_a$. Nowik proved:

Theorem 5.1. $\mathcal{O}_g$ is generated by T_a about a with $q_{ps}(a) = 2$.

If we can find a finite system of generators for $\Gamma_2(N_g)$ explicitly, we can get a finite system of generators for $\mathcal{N}_g$. In the case where genus $g = 4$, we find that $\Gamma_2(N_4)$ is generated by the elements shown in Figure 6. Considering the action of Dehn twists corresponding to Nowik's generators of $\mathcal{O}_4$ on our system of generators for $\Gamma_2(N_4)$ by the conjugation, we see that $\mathcal{N}_4$ is generated by the 7 elements shown in Figure 7.

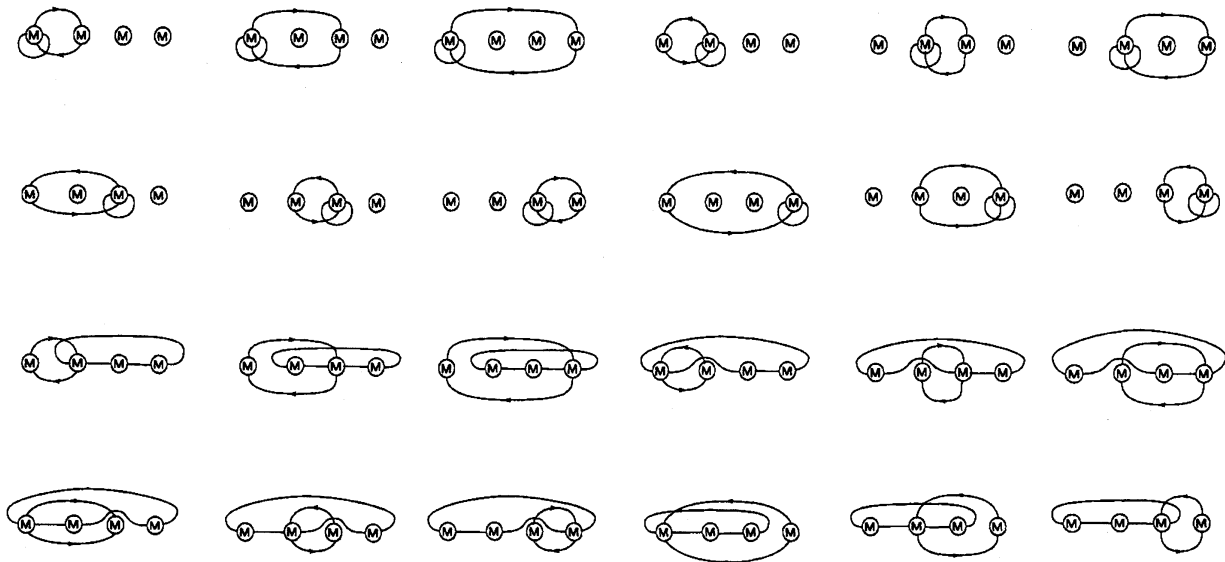


FIGURE 6

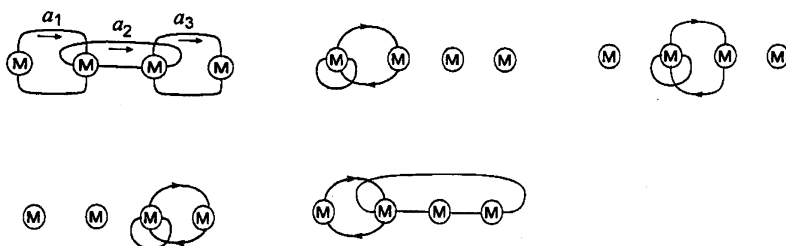
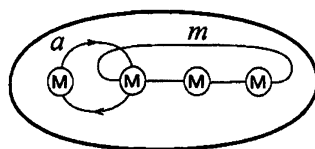


FIGURE 7

We can get an affirmative answer to Problem 2.1 when genus $g = 4$, if we answer the following Problem positively.



Problem 5.2. Is Y -homeomorphism $Y_{m,a}$ indicated in the above figure ps -extendable?

Appendix. A quick proof of Theorem 3.1

Nowik showed,

Theorem 5.3 ([10]). $\Gamma_2(N_g)$ is generated by the following three types of elements:

- (1) $(t_c)^2$ about non-separating A -circles c (i.e., $N_g - c$ is connected) in N_g ,
- (2) t_c about separating A -circles c (i.e., $N_g - c$ is not connected) in N_g ,
- (3) Y -homeomorphisms.

This Theorem is proved by the same type of argument as in [5]. If we see that any elements of type (1) and (2) are products of Y -homeomorphisms, then we see Theorem 3.1. For (1), Szepietowski showed,

Lemma 5.4 (Lemma 3.1 of [11]). *For any non-separating A -circle c in N_g , $(t_c)^2$ is a product of two Y -homeomorphisms.*

Proof. There exists an M -circle m which intersects c in one point. Since $Y_{m,c}$ exchanges the two sides of c , we see $Y_{m,c} t_c Y_{m,c}^{-1} = t_c^{-1}$. Therefore, $(t_c)^2 = t_c (Y_{m,c} t_c Y_{m,c}^{-1})^{-1} = t_c Y_{m,c} t_c^{-1} \cdot Y_{m,c}^{-1} = Y_{t_c(m),c} Y_{m,c}^{-1}$. $\square$

Let c be a separating A -circle, then at least one component F of $N_g - c$ is non-orientable. Let k be the genus of F .

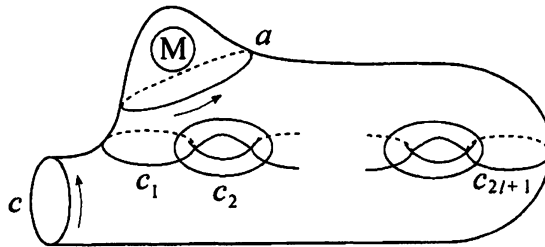


FIGURE 8

If k is an odd integer, we set $l = (k - 1)/2$. Then F is as shown in Figure 8. By the chain relation, $t_a t_c = (t_{c_1} t_{c_2} \cdots t_{c_{2l+1}})^{2l+2}$. Let G_F be a subgroup of $\mathcal{M}(N_g)$ generated by $t_{c_1}, t_{c_2}, \dots, t_{c_{2l+1}}$, and B_{2l+2} be the group of $2l + 2$ string braid group generated by $\sigma_1, \sigma_2, \dots, \sigma_{2l+1}$ where σ_i is a braid exchanging the i -th string and the $i + 1$ -st string. Then there is a surjection $\pi : B_{2l+2} \rightarrow G_F$ defined by $\pi(\sigma_i) = t_{c_i}$. By

this surjection, $\pi(\text{a full twist}) = (t_{c_1} t_{c_2} \cdots t_{c_{2l+1}})^{2l+2}$. Since a full twist is a pure braid and the subgroup of pure braids in B_{2l+2} is generated by $(\sigma_1 \cdots \sigma_i) \sigma_{i+1}^2 (\sigma_1 \cdots \sigma_i)^{-1}$, $(t_{c_1} t_{c_2} \cdots t_{c_{2l+1}})^{2l+2}$ is a product of $(t_{c_1} \cdots t_{c_i}) t_{c_{i+1}}^2 (t_{c_1} \cdots t_{c_i})^{-1} = (t_{c_1 \cdots c_i(c_{i+1})})^2$. By the above Lemma, $(t_{c_1 \cdots c_i(c_{i+1})})^2$ is a product of Y -homeomorphisms. Since t_a is isotopic to the identity, t_c is a product of Y -homeomorphisms.

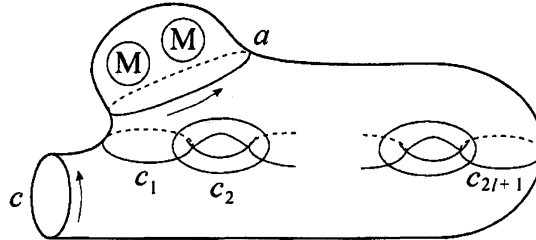


FIGURE 9

If k is an even integer, we set $l = (k - 2)/2$. Then F is as shown in Figure 9. By the chain relation, $t_a t_c = (t_{c_1} t_{c_2} \cdots t_{c_{2l+1}})^{2l+2}$. By the same argument as above, we see that $t_a t_c$ is a product of Y -homeomorphisms. Let y be a Y -homeomorphism whose support is a Klein bottle with one boundary a , then $t_a = y^2$. Therefore, t_c is a product of Y -homeomorphisms.

Remark 5.5. Szepietowski showed that (2) is a product of Y -homeomorphisms in Lemma 3.2 of [11] by using the lantern relation.

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