

Convergence analysis of elliptic regularization for advection equations in a bounded domain

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Abstract

We discuss the elliptic regularization applied to linear advection equations in a bounded domain, where the homogeneous Dirichlet condition is imposed on the inflow boundary. The adopted regularization is to add a diffusion term with a small positive coefficient ε . Our regularization, also known as singular perturbation, requires additional boundary conditions on the remaining part of the boundary. We focus on L^2 asymptotic behavior of regularized solutions u^ε as $\varepsilon \rightarrow +0$, particularly on their convergence rates under different types of additional boundary conditions, and we propose a new technique to derive these estimates. Furthermore, we also give some numerical experiments based on the finite element methods in order to check optimality of our convergence rates.

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1 Introduction

The study of first-order partial differential equations (PDEs) is one of the fundamentals of the theories of PDEs and has been pursued from diverse perspectives. Our research aims to contribute to the establishment of stable numerical methods for systems of first-order quasi-linear equations. As a fundamental step to the goal, we focus on the elliptic regularization method for a single first-order advection equation in a bounded domain, especially on the estimation of the asymptotic behavior of regularized solutions primarily from the standpoint of mathematical analysis.

First-order PDEs are discussed in relations with surface theory, as seen in the historical works such as Darboux [5] and Goursat [13], and they are also be discussed from the standpoint of mathematical physics, such as flow problems. In this study, we adopt the latter standpoint, treating the equations as a mathematical model of advection phenomena.

A mathematical model of advection phenomena is given as

$$u_t + \nabla \cdot F(x, u) + G(x, u) = 0, \quad (1.1)$$

where $u = u(x, t)$ with spatial variables $x \in \mathbb{R}^d$ and time t , F is a $\mathbb{R}^d$ -valued flux function, and G is a source term. It is well-known that solutions to (1.1) may have discontinuities (e.g., shock waves) even with smooth initial data. In discussions within mathematical physics, one of the common approaches to handle such discontinuities is to regularize the problem by introducing a small diffusion term. That is to say, the regularized equation

$$u_t^\varepsilon + \nabla \cdot F(x, u^\varepsilon) + G(x, u^\varepsilon) = \varepsilon \Delta u^\varepsilon, \quad (1.2)$$

is considered in parallel with (1.1). Here, Δ denotes the Laplacian corresponding to diffusion, and ε is a small positive parameter representing small diffusion effect. This approach, also known as the vanishing viscosity method, analyzes the solution u to (1.1) by studying the asymptotic behavior of the solution u^ε to (1.2) as $\varepsilon \rightarrow +0$ [3,21].

When considering (1.1) in a bounded domain $\Omega \subset \mathbb{R}^d$, we need to impose boundary conditions. For the first-order equation (1.1), they are typically imposed on the inflow part of the boundary $\partial\Omega$. However, difficulty arises in the general nonlinear case, because the inflow boundary itself may depend on the unknown solution u . In contrast with (1.1), since the regularized equation (1.2) (or its stationary one) is second-order with respect to the spatial variables, it requires boundary conditions on the whole boundary $\partial\Omega$. A significant advantage of this regularization is that an extensive body of mathematical results is available for such second-order (e.g., parabolic or elliptic) boundary value problems, enabling more precise analyses of u^ε and its asymptotic behavior. On the other hand, this approach should require new challenges. For instance, if the homogeneous Dirichlet condition

is imposed on the whole boundary $\partial\Omega$ for u^ε , it forces the regularized solution to be zero on the parts of the boundary where the solution u to the equation (1.1) would not necessarily be zero. This mismatch gives rise to a new issue concerning the behavior of u and u^ε near such boundaries, so called the boundary layer problem.

Furthermore, in case the setting for the equation (1.1) allows discontinuities of a solution u within the interior of the domain (corresponding to, e.g., shock waves), internal layers may appear for the smooth function u^ε . These internal layers and the boundary layers are mathematically distinct phenomena, and we should remark, specially, internal layers arise from the quasi-linear nature of equation (1.1). Focusing on boundary layers in this research, we restrict ourselves to the cases where the term $\nabla \cdot F(x, u)$ in (1.1) is linear. This leads us to concentrate on the detailed analysis of the error estimate $u^\varepsilon - u$, particularly near the boundary. We aim to discuss the refinement of estimates for $u^\varepsilon - u$.

Thus, let us consider the stationary advection-reaction equation in a bounded domain $\Omega \subset \mathbb{R}^d$:

$$\beta \cdot \nabla u + \mu u = f \quad \text{in } \Omega, \quad (1.3)$$

with the boundary condition

$$u = 0 \quad \text{on } \Gamma_- = \{x \in \partial\Omega \mid \beta(x) \cdot n(x) < 0\},$$

where β is a given vector field on $\overline{\Omega}$, μ and f are given scalar functions on Ω , and $n(x)$ is the outward unit normal vector at $x \in \partial\Omega$. The boundary Γ_- corresponds to the inflow boundary, which is determined a priori for the linear equation (1.3). The corresponding regularized problem, which is sometimes called an elliptic singularly perturbed problem, is

$$-\varepsilon \Delta u^\varepsilon + \beta \cdot \nabla u^\varepsilon + \mu u^\varepsilon = f \quad \text{in } \Omega, \quad (1.4)$$

where $\varepsilon > 0$ is a small parameter. For u^ε , a boundary condition on Γ_- is naturally inherited as

$$u^\varepsilon = 0 \quad \text{on } \Gamma_-.$$

However, since (1.4) is a second-order elliptic equation, boundary conditions for u^ε should be imposed on the whole boundary $\partial\Omega$. We may impose various types of boundary conditions on $\partial\Omega \setminus \Gamma_-$. various conditions can be considered. In this study, we will focus on two typical conditions; the homogeneous Dirichlet condition and the homogeneous Neumann condition. A precise formulation of these problems will be given in the subsequent sections.

There have been numerous studies about the case where the Dirichlet boundary condition is imposed on the whole boundary $\partial\Omega$ for u^ε , as

this setting is related to the vanishing viscosity problem for the Navier–Stokes equations [4]. The boundary layer analysis is often employed to investigate the behavior of u^ε near the boundary. However, obtaining a global estimate such as $\|u^\varepsilon - u\|_{L^2(\Omega)}$ is not easy and the shape of domains have offered some restriction [7,20]. This difficulty arises from serious difference of the boundary layers appearing at the characteristic boundary $\Gamma_0 = \{x \in \partial\Omega \mid \beta(x) \cdot n(x) = 0\}$ and at the outflow boundary $\Gamma_+ = \{x \in \partial\Omega \mid \beta(x) \cdot n(x) > 0\}$. The situation should be more difficult near the points where these boundary segments meet; it is complicated to construct corrector terms for the estimation. To deal with these challenges, we will derive convergence estimates in Section 3.1 by using a priori estimates for advection equations, achieving this for certain domains like an interval in $\mathbb{R}$ or an annulus in $\mathbb{R}^2$ without resorting to explicit boundary layer correctors.

It is known that the boundary layer arisen from the homogeneous Neumann condition on $\partial\Omega \setminus \Gamma_-$ tends to be weaker than from the Dirichlet case [28], and an estimate for the convergence rate has been provided for the case of $u \in H^1(\Omega)$ [26]. In Section 3.2, we will derive convergence estimates under the assumption $u \in H^2(\Omega)$. It will be clarified that the convergence rate depends on both the presence or absence of the characteristic boundary Γ_0 and the degeneracy of $\beta \cdot n$ on the outflow boundary Γ_+ .

The convergence results obtained in Section 3 provide upper bounds for $\|u^\varepsilon - u\|_{L^2(\Omega)}$, which in turn give lower bounds for the convergence rate of $\|u^\varepsilon - u\|_{L^2(\Omega)}$ as $\varepsilon \rightarrow +0$. Some numerical examples will be given in Section 4 to verify the optimality of our convergence rates for cases where the Neumann boundary condition is imposed for u^ε on $\partial\Omega \setminus \Gamma_-$.

2 Preliminaries

We formulate our boundary value problem of the steady advection equation together with its regularized problem. We state the well-posedness of the problems, following [6] with slight modifications.

Let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$. We consider the homogeneous Dirichlet problem of the linear steady advection equation:

$$\begin{cases} \beta \cdot \nabla u + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_-, \end{cases} \quad (2.1)$$

$$(2.2)$$

where β is a given vector field on $\overline{\Omega}$, and μ and f are given as scalar functions on Ω . To formulate the boundary condition, we decompose the boundary

$\partial\Omega$ into $\partial\Omega = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$, where each component is defined as follows:

$$\begin{aligned}\Gamma_- &= \{x \in \partial\Omega \mid \beta(x) \cdot n(x) < 0\}, \\ \Gamma_+ &= \{x \in \partial\Omega \mid \beta(x) \cdot n(x) > 0\}. \\ \Gamma_0 &= \partial\Omega \setminus (\Gamma_- \cup \Gamma_+).\end{aligned}$$

Since Ω has a Lipschitz boundary, the outward unit normal vector $n(x)$ is defined almost everywhere on $\partial\Omega$. These subsets are called the inflow boundary, the outflow boundary and the characteristic boundary, respectively. We impose the Dirichlet boundary condition only on the inflow boundary Γ_- .

We assume that $\beta \in [W^{1,\infty}(\Omega)]^d$ and $\mu \in L^\infty(\Omega)$. Note that the Sobolev imbedding theorem implies $W^{1,\infty}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ (see, e.g., [1]). We also assume that a positivity condition

$$\mu_0 := \operatorname{ess\,inf}_{\Omega} \left(\mu - \frac{1}{2} \nabla \cdot \beta \right) > 0. \quad (2.3)$$

The condition (2.3) is a technical requirement necessary for our proof given in Section 3. We remark that it is satisfied for sufficiently large μ , that is, the reaction term μu dominates the advection term $\beta \cdot \nabla u$. The condition (2.3) is a sufficient condition for our results and almost the same results can be shown by the existence of a smooth auxiliary function ζ satisfying $|\beta \cdot \nabla \zeta| > 0$ on Ω (see Remark 2.2).

In this study, we focus on an L^2 -theory for the equation (2.1), which is suitable naturally with its numerical computing by the finite element method (FEM) with non-conforming elements.

We also describe an alternative approach, which is called the method of characteristics. When a curve $c(t)$ in $\mathbb{R}^d$ satisfies a system of ordinary differential equations(ODEs)

$$\frac{dc}{dt} = \beta(c(t)), \quad (2.4)$$

it is called a characteristic curve corresponding to the equation (2.2). For a smooth solution u to (2.1), set $U(t) = u(c(t))$, and we have

$$\frac{dU}{dt} = \nabla u \cdot \frac{dc}{dt} = (\beta \cdot \nabla u)(c(t)) = -\mu(c(t))U(t) + f(c(t)). \quad (2.5)$$

Thus, one might expect to obtain a solution to (2.1) by solving ODEs (2.5) along each characteristic curve. However, certain assumptions are needed to justify this method. One of the most strict restriction is the regularity requirement for the coefficients in (2.1) such as $\beta \in C^1(\overline{\Omega})$, $\mu, f \in C^0(\overline{\Omega})$, which ensures unique solvability of ODEs (2.4) and (2.5). Furthermore, from the viewpoint of boundary value problems, only the regularity requirement is not enough. This method may fail to provide a solution to (2.1)-(2.2)

if, for example, the system (2.4) have a limit cycle within Ω . Additional conditions are needed to avoid such issues. The functional analytic method employed in this study, supported by condition (2.3), is applicable even in some situations where the method of characteristics faces difficulties.

We state the boundary condition (2.2). Recalling the method of characteristics, we may impose the boundary condition either on the inflow boundary Γ_- or on the outflow boundary Γ_+ : The ODE (2.4) is solved both forward and backward with respect to t . However, in our method based on the L^2 theory, the boundary condition must be given on the inflow boundary in terms of the well-posedness of the problem. It may be understood after introduction of regularized problem later.

We will state the well-posedness of the boundary value problem (2.1)-(2.2) based on Chapter 2 of [6]. Let us introduce a function space $H_\beta(\Omega)$ defined by

$$H_\beta(\Omega) = \{v \in L^2(\Omega) \mid \beta \cdot \nabla v \in L^2(\Omega)\}$$

with the inner product

$$(v, w)_{H_\beta(\Omega)} = \int_{\Omega} v w dx + \int_{\Omega} (\beta \cdot \nabla v)(\beta \cdot \nabla w) dx, \quad (2.6)$$

where $\beta \cdot \nabla v$ means the weak derivative. Precisely, for $\beta \in [W^{1,\infty}(\Omega)]^d$ and $v \in L^2(\Omega)$, we consider the linear functional $F_v : C_0^\infty(\Omega) \rightarrow \mathbb{R}$ defined by

$$F_v(\varphi) = - \int_{\Omega} v \nabla \cdot (\beta \varphi) dx.$$

We say that v belongs to $H_\beta(\Omega)$ if there exists a positive number C_v satisfying

$$|F_v(\varphi)| \leq C_v \|\varphi\|_{L^2(\Omega)}$$

for all $\varphi \in C_0^\infty(\Omega)$. In that case, F_v extends uniquely to a bounded linear functional on $L^2(\Omega)$. By the Riesz representation theorem, there exists a unique element $w \in L^2(\Omega)$ satisfying

$$F_v(\varphi) = \int_{\Omega} w \varphi dx$$

for all $\varphi \in L^2(\Omega)$, and the function $\beta \cdot \nabla v$ is defined as this (unique) representation function w .

Lemma 2.1 (Proposition 2.2 of [6]). *The function space $H_\beta(\Omega)$ with its inner product (2.6) is a Hilbert space.*

The inflow and outflow boundaries are called well-separated when the separation condition $\overline{\Gamma_-} \cap \overline{\Gamma_+} = \emptyset$ is satisfied. Under this condition, we can define the trace operator on $H_\beta(\Omega)$.

Proposition 2.2. *Under the separation condition, the trace operators $\gamma_{\pm} : H_{\beta}(\Omega) \rightarrow L^2(\Gamma_{\pm}; |\beta \cdot n|)$ are defined and are bounded, where $L^2(\Gamma_{\pm}; |\beta \cdot n|)$ are Lebesgue spaces on $\Gamma_{\pm}$ with respect to the weighted surface measure $|\beta \cdot n|d\sigma$. Moreover, for all $v, w \in H_{\beta}(\Omega)$, the following integration by parts formula holds:*

$$\int_{\Omega} ((\beta \cdot \nabla v)w + v(\beta \cdot \nabla w) + (\nabla \cdot \beta)vw)dx = \int_{\partial\Omega} vw(\beta \cdot n)d\sigma.$$

This proposition is almost the same with Lemma 2.5 in [6], and we follow its proof.

Proof. Owing to the separation condition, there exist two function $\psi^{\pm} \in C^1(\overline{\Omega})$ such that,

$$\psi^- + \psi^+ \equiv 1 \quad \text{on } \overline{\Omega}, \quad \psi^-|_{\Gamma_+} = 0, \quad \text{and} \quad \psi^+|_{\Gamma_-} = 0.$$

For $\varphi \in C^{\infty}(\overline{\Omega})$, we have

$$\begin{aligned} \int_{\Omega} v^2 |\beta \cdot n| &= \int_{\partial\Omega} (-\psi^- + \psi^+) v^2 (\beta \cdot n) dx \\ &= \int_{\Omega} \nabla \cdot ((-\psi^- + \psi^+) v^2 \beta) dx \\ &= \int_{\Omega} v^2 \nabla \cdot ((-\psi^- + \psi^+) \beta) dx + \int_{\Omega} 2(-\psi^- + \psi^+) (\beta \cdot \nabla v) v dx \\ &\leq C_{\gamma}^2 \|v\|_{H_{\beta}(\Omega)}^2, \end{aligned}$$

where $C_{\gamma}^2 = \|\nabla \cdot ((-\psi^- + \psi^+) \beta)\|_{L^{\infty}(\Omega)} + \|-\psi^- + \psi^+\|_{L^{\infty}(\Omega)}$. Thus, we have

$$\|v\|_{L^2(\Gamma_-; |\beta \cdot n|)} + \|v\|_{L^2(\Gamma_+; |\beta \cdot n|)} \leq C_{\gamma} \|v\|_{H_{\beta}(\Omega)}. \quad (2.7)$$

Since $C^{\infty}(\overline{\Omega})$ is dense in $H_{\beta}(\Omega)$ (see, e.g., [18]), the inequality (2.7) enables us to define the trace operators $\gamma_{\pm} : H_{\beta}(\Omega) \rightarrow L^2(\Gamma_{\pm}; |\beta \cdot n|)$ as bounded operators, and we have

$$\|\gamma_{\pm}(v)\|_{L^2(\Gamma_{\pm}; |\beta \cdot n|)} \leq C_{\gamma} \|v\|_{H_{\beta}(\Omega)}.$$

This proof also means the integration by parts formula. □

Remark 2.1. It is worthy of remarking that the trace of a function in $H_{\beta}(\Omega)$ is defined and belongs to $L_{\text{loc}}^2(\Gamma_{\pm}; \beta \cdot n)$, but does not to $L^2(\Gamma_{\pm}; |\beta \cdot n|)$ in general without the separation condition of Γ_- and Γ_+ (see [2]). In order to check the fact, we take arbitrary compact sets $\Gamma'_- \subset \Gamma_-$ and $\Gamma'_+ \subset \Gamma_+$. Since

$\text{dist}(\Gamma'_-, \Gamma_+) > 0$ and $\text{dist}(\Gamma'_+, \Gamma_-) > 0$, there exist functions $\chi_-, \chi_+ \in C(\overline{\Omega})$ with

$$\begin{aligned} 0 \leq \chi_- \leq 1, \quad \chi_- = 1 \quad \text{on} \quad \Gamma'_-, \quad \chi_- = 0 \quad \text{on} \quad \Gamma_+, \\ 0 \leq \chi_+ \leq 1, \quad \chi_+ = 1 \quad \text{on} \quad \Gamma'_+, \quad \chi_+ = 0 \quad \text{on} \quad \Gamma_-. \end{aligned}$$

Then, for $v \in C^\infty(\overline{\Omega})$ we have

$$\begin{aligned} \int_{\Gamma'_-} v^2 |\beta \cdot n| d\sigma + \int_{\Gamma'_+} v^2 |\beta \cdot n| d\sigma &\leq \int_{\Gamma_-} v^2 \chi_- |\beta \cdot n| d\sigma + \int_{\Gamma_+} v^2 \chi_+ |\beta \cdot n| d\sigma \\ &= \int_{\partial\Omega} (-\chi_- + \chi_+) v^2 (\beta \cdot n) d\sigma. \end{aligned}$$

The rest of the argument is same as the proof of Proposition 2.2. We also note that we can take the trace of $v \in H_\beta(\Omega)$ vanishing on Γ_- (in the sense of the local trace) mapped to $L^2(\Gamma_+; \beta \cdot n)$ even if the separation condition does not hold. To see this, we take a sequence $\{v_m\}_m$ in $C^\infty(\overline{\Omega})$ vanishing on Γ_- and converging to v [18], and by the integration by parts we have

$$\int_{\Gamma_+} v_m^2 |\beta \cdot n| d\sigma = \int_{\Omega} \nabla \cdot (v_m^2 \beta) dx \leq \|v_m\|_{L^2(\Gamma_-; |\beta \cdot n|)}^2 + C \|v_m\|_{H_\beta(\Omega)}^2,$$

where C is a constant independent of v_m . By using the argument about the local trace with $v = 0$ on Γ_- , as $m \rightarrow \infty$, we obtain $\|v\|_{L^2(\Gamma_+; |\beta \cdot n|)} \leq C \|v\|_{H_\beta(\Omega)}$. This leads the well-posedness of the boundary value problem (2.1)-(2.2). For more details, see [17].

Let us define subspaces of $H_\beta(\Omega)$ by

$$H_{\beta, \pm}(\Omega) = \left\{ v \in H_\beta(\Omega) \mid v|_{\Gamma_\pm} = 0 \right\}.$$

In case that the separation condition does not hold, we regard $v|_{\Gamma_\pm}$ as the local trace. In any case, $H_{\beta, -}(\Omega)$ is a closed subspace of $H_\beta(\Omega)$, that is, $H_{\beta, -}(\Omega)$ is a Banach space with the norm $\|\cdot\|_{H_\beta(\Omega)}$.

To show the well-posedness of the boundary value problem (2.1)-(2.2), we prepare the Banach-Nečas-Babuška theorem. A proof is seen in e.g. [6].

Proposition 2.3 (Banach-Nečas-Babuška theorem). *Let V and W be Banach spaces over $\mathbb{R}$, where W is reflexive, and let a be a bounded bilinear functional on $V \times W$. Then, the following three conditions (a)–(c) are equivalent to one another:*

(a) *For any $F \in W'$, there exists a unique $u \in V$ such that, for all $w \in W$,*

$$a(u, w) = F(w).$$

(b) The following two conditions hold:

$$\alpha_1 := \inf_{v \in V \setminus \{0\}} \sup_{w \in W \setminus \{0\}} \frac{a(v, w)}{\|v\|_V \|w\|_W} > 0,$$

$$\{w \in W \mid \text{For all } v \in V, a(v, w) = 0\} = \{0\}.$$

(c) The following two conditions hold:

$$\alpha_1 := \inf_{v \in V \setminus \{0\}} \sup_{w \in W \setminus \{0\}} \frac{a(v, w)}{\|v\|_V \|w\|_W} > 0,$$

$$\alpha_2 := \inf_{w \in W \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{a(v, w)}{\|v\|_V \|w\|_W} > 0.$$

Furthermore, it holds that $\|u\|_V \leq \alpha_1^{-1} \|F\|_{W'}$ for u in (a), and that $\alpha_1 = \alpha_2$ in (c).

The well-posedness of the boundary value problem of (2.1)-(2.2) is deduced from the above proposition. We give such a precise statement as in Chapter 2 of [6], but we do not assume the separation condition.

Proposition 2.4. *Under the positivity condition (2.3) of $\mu_0 > 0$, we conclude the following results:*

1. *For any $f \in L^2(\Omega)$, there uniquely exists a solution $u \in H_{\beta,-}(\Omega)$ to (2.1)-(2.2). Moreover, there exists a positive number C independent of f such that*

$$\|u\|_{H_{\beta}(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

2. *For any $f \in (H_{\beta,+}(\Omega))'$, the dual space of $H_{\beta,+}(\Omega)$, there uniquely exists a function $u \in L^2(\Omega)$ satisfying*

$$\int_{\Omega} u(-\nabla \cdot (\beta v) + \mu v) dx = \langle f, v \rangle$$

for all $v \in H_{\beta,+}(\Omega)$, whose bracket $\langle \cdot, \cdot \rangle$ denotes the duality pairing. Moreover, there exists a positive number C independent of f such that

$$\|u\|_{L^2(\Omega)} \leq C \|f\|_{H'_{\beta,+}(\Omega)}.$$

Proof. Let B be a bilinear form defined on $H_{\beta,-}(\Omega) \times L^2(\Omega)$ by

$$B(v, w) = \int_{\Omega} (\beta \cdot \nabla v + \mu v) w dx, \quad \text{for } (v, w) \in H_{\beta,-}(\Omega) \times L^2(\Omega).$$

We have the following two results

$$\alpha := \inf_{v \in H_{\beta,-}(\Omega) \setminus \{0\}} \sup_{w \in L^2(\Omega) \setminus \{0\}} \frac{B(v, w)}{\|v\|_{H_{\beta}(\Omega)} \|w\|_{L^2(\Omega)}} > 0, \quad (2.8)$$

and

$$\{w \in L^2(\Omega) \mid B(v, w) = 0 \text{ for all } v \in H_{\beta,-}(\Omega)\}. \quad (2.9)$$

The conclusion of the first half of the proposition follows from (2.8) and (2.9) owing to the Proposition 2.3. Their proof are as follows: For (2.8), for any $v \in H_{\beta,-}(\Omega)$, there exists a sequence $\{v_m\}_m \subset C^\infty(\Omega)$ approximating v in $H_\beta(\Omega)$. Thanks to the positivity condition (2.3), we have

$$\begin{aligned} B(v_m, v_m) &= \int_{\Omega} (\beta \cdot \nabla v_m + \mu v_m) v_m dx \\ &= \int_{\Omega} \left(\mu - \frac{1}{2} \nabla \cdot \beta \right) v_m^2 dx - \frac{1}{2} \int_{\Gamma_-} v_m^2 |\beta \cdot n| d\sigma + \frac{1}{2} \int_{\Gamma_+} v_m^2 |\beta \cdot n| d\sigma \\ &\geq \mu_0 \|v_m\|_{L^2(\Omega)}^2 - \frac{1}{2} \|v_m\|_{L^2(\Gamma_-; |\beta \cdot n|)}^2 + \frac{1}{2} \|v_m\|_{L^2(\Gamma_+; |\beta \cdot n|)}^2, \end{aligned}$$

hence we have

$$B(v, v) \geq \mu_0 \|v\|_{L^2(\Omega)}^2 + \frac{1}{2} \|v\|_{L^2(\Gamma_+; |\beta \cdot n|)}^2. \quad (2.10)$$

Therefore,

$$\|v\|_{L^2(\Omega)}^2 \leq \mu_0^{-1} B(v, v) \leq \mu_0^{-1} \left(\sup_{w \in L^2(\Omega) \setminus \{0\}} \frac{B(v, w)}{\|w\|_{L^2(\Omega)}} \right) \|v\|_{L^2(\Omega)},$$

and it leads us to

$$\|v\|_{L^2(\Omega)} \leq \mu_0^{-1} \sup_{w \in L^2(\Omega) \setminus \{0\}} \frac{B(v, w)}{\|w\|_{L^2(\Omega)}}.$$

On the other hand, it holds that

$$\begin{aligned} \|\beta \cdot \nabla v\|_{L^2(\Omega)} &= \sup_{w \in L^2(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (\beta \cdot \nabla v) w dx}{\|w\|_{L^2(\Omega)}} \\ &= \sup_{w \in L^2(\Omega) \setminus \{0\}} \frac{B(v, w) - \int_{\Omega} \mu v w dx}{\|w\|_{L^2(\Omega)}} \\ &\leq \left(1 + \|\mu\|_{L^\infty(\Omega)} \mu_0^{-1} \right) \sup_{w \in L^2(\Omega) \setminus \{0\}} \frac{B(v, w)}{\|w\|_{L^2(\Omega)}}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \left\{ \mu_0^{-2} + (1 + \|\mu\|_{L^\infty(\Omega)} \mu_0^{-1})^2 \right\}^{-1/2} &\leq \\ &\inf_{v \in H_{\beta,-}(\Omega) \setminus \{0\}} \sup_{w \in L^2(\Omega) \setminus \{0\}} \frac{B(v, w)}{\|v\|_{H_\beta(\Omega)} \|w\|_{L^2(\Omega)}}, \end{aligned}$$

which implies to the inf-sup condition (2.8).

In order to show (2.9), take $w \in L^2(\Omega)$ satisfying $B(v, w) = 0$ for all $v \in H_{\beta,-}(\Omega)$, and we have, for $v = \varphi \in C_0^\infty(\Omega)$,

$$\int_{\Omega} (\beta \cdot \nabla \varphi + \mu \varphi) v dx = 0.$$

Therefore, since $-(\nabla \cdot \beta)v + \mu v \in L^2(\Omega)$, integration by parts shows that $v \in H_{\beta}(\Omega)$ and $\beta \cdot \nabla v = -(\nabla \cdot \beta)v + \mu v$.

Let Γ'_+ be a subset of Γ_+ with $\text{dist}(\Gamma_-, \Gamma'_+) > 0$. We can take, following the notice in Remark 2.1, a function $\chi \in C^\infty(\overline{\Omega})$ satisfying

$$0 \leq \chi \leq 1 \quad \text{on } \overline{\Omega}, \quad \chi = 1 \quad \text{on } \Gamma'_+, \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma_-.$$

Note that we can take $\Gamma'_+ = \Gamma_+$ if we assume the separation condition. For $v \in H_{\beta}(\Omega)$, it holds that $\chi v \in H_{\beta,-}(\Omega)$, and we have

$$\begin{aligned} \int_{\Gamma'_+} v^2 (\beta \cdot n) d\sigma &\leq \int_{\partial\Omega} \chi v^2 (\beta \cdot n) d\sigma \\ &= \int_{\Omega} (\beta \cdot \nabla (\chi v)) v + (\chi v) \nabla \cdot (\beta v) dx \\ &= \int_{\Omega} (\beta \cdot \nabla (\chi v) + \mu (\chi v)) v dx \\ &= B(\chi v, v) = 0, \end{aligned}$$

and hence $v = 0$ almost everywhere on Γ'_+ , and $v = 0 \in L^2_{\text{loc}}(\Omega)$, which implies $v \in H_{\beta,+}(\Omega)$.

Therefore, we have

$$\begin{aligned} 0 &= \int_{\Omega} (-\nabla \cdot (\beta v) + \mu v) v dx \\ &= \int_{\Omega} \left(\mu - \frac{1}{2} (\nabla \cdot \beta) \right) v^2 dx + \frac{1}{2} \int_{\Gamma_-} v^2 |\beta \cdot n| d\sigma \\ &\geq \mu_0 \|v\|_{L^2(\Omega)}^2. \end{aligned}$$

For the latter part of the proposition, we define a bilinear form $B' : L^2(\Omega) \times H_{\beta,+}(\Omega)$ by

$$B'(v, w) = \int_{\Omega} v (-\nabla \cdot (\beta w) + \mu w) dx \quad \text{for all } (v, w) \in L^2(\Omega) \times H_{\beta,+}(\Omega).$$

Similarly to the former discussion, we show the well-posedness of the adjoint problem to (2.1)-(2.2);

$$\begin{cases} -\nabla \cdot (\beta u) + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_+. \end{cases}$$

Define a bilinear form $\overline{B'} : H_{\beta,+}(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}$ by

$$\overline{B'}(v, w) = \int_{\Omega} (-\nabla \cdot (\beta v) + \mu v) w dx \quad \text{for all } (v, w) \in H_{\beta,+}(\Omega) \times L^2(\Omega),$$

and it suffices the statement (c) in Proposition 2.3. Finally we conclude $B'(w, v) = \overline{B'}(v, w)$, which implies the latter part of the proposition. $\square$

Remark 2.2. The positivity condition (2.3) is used to derive the inequality (2.10), but this condition is replaced by the existence of the following auxiliary function $\zeta \in L^\infty(\Omega)$. Let a function $\zeta \in L^\infty(\Omega)$ satisfy both $\beta \cdot \nabla v \in L^\infty(\Omega)$ and $\text{essinf}_{\Omega}(\beta \cdot \nabla \zeta) > 0$. For a solution $u \in H_{\beta,-}(\Omega)$ to (2.1) and for any positive number λ , set $z = e^{-\lambda \zeta} u \in H_{\beta,-}(\Omega)$. Then

$$\beta \cdot \nabla z = e^{-\lambda \zeta}(\beta \cdot \nabla u) - \lambda e^{-\lambda \zeta}(\beta \cdot \nabla \zeta)u = e^{-\lambda \zeta}f - e^{-\lambda \zeta}\{\mu + \lambda(\beta \cdot \nabla \zeta)\}u,$$

and z is a solution to $\beta \cdot \nabla z + \tilde{\mu}z = e^{-\lambda \zeta}f$ for $\tilde{\mu} = \mu + \lambda(\beta \cdot \nabla \zeta)$. Since $\text{essinf}_{\Omega}(\tilde{\mu} - \frac{1}{2}\nabla \cdot \beta) > 0$ for sufficiently large λ , we return to the discussion above to estimate $\|e^{-\lambda \zeta}u\|_{L^2(\Omega)}$ to obtain (2.10). One of the sufficient conditions for the existence of such ζ is that the vector fields β never vanishes on Ω and has a scalar potential ϕ such that $\beta = \nabla \phi$. In this case we can take $\zeta = \phi$. Another sufficient condition is that β is a filling field; for all $x \in \Omega$ there is a characteristic curve defined by (2.4) which starts from Γ_- and that reaches x in finite time. In this case we can solve $\beta \cdot \nabla \zeta = 1$ by using the method of characteristics (see [8]).

We should also show the well-posedness for the regularized equation

$$-\varepsilon \Delta u^\varepsilon + \beta \cdot \nabla u^\varepsilon + \mu u^\varepsilon = f \quad \text{in } \Omega. \quad (2.11)$$

Because (2.11) are second-order equations, they requires additional boundary condition on $\partial\Omega \setminus \Gamma_-$, and we consider either of the boundary conditions such that

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega, \quad (2.12)$$

$$u^\varepsilon = 0 \quad \text{on } \Gamma_-, \quad \frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_-. \quad (2.13)$$

Proposition 2.5. *1. For all $\varepsilon > 0$, the boundary value problem (2.11)-(2.12) has a unique solution $u^\varepsilon \in H_0^1(\Omega)$ in the sense that*

$$\varepsilon \int_{\Omega} \nabla u^\varepsilon \cdot \nabla v dx + \int_{\Omega} (\beta \cdot \nabla u^\varepsilon + \mu u^\varepsilon) v dx = \int_{\Omega} f v dx,$$

for all $v \in H_0^1(\Omega)$.

2. For all $\varepsilon > 0$, the boundary value problem (2.11)-(2.13) has a unique solution $u^\varepsilon \in H_{\Gamma_-}^1(\Omega) = \left\{ v \in H^1(\Omega) \mid v|_{\Gamma_-} = 0 \right\}$ in the sense that

$$\varepsilon \int_{\Omega} \nabla u^\varepsilon \cdot \nabla v dx + \int_{\Omega} (\beta \cdot \nabla u^\varepsilon + \mu u^\varepsilon) v dx = \int_{\Omega} f v dx,$$

for all $v \in H_{\Gamma_-}^1(\Omega)$.

Proof. These well-posedness results are well known in the theory of elliptic equations (e.g. [9]), but we dare to derive them here because we will use the inequalities below in subsequent sections.

By virtue of the positivity condition (2.3), the bilinear form

$$B^\varepsilon(u, v) = \varepsilon \int_{\Omega} (\nabla u \cdot \nabla v) dx + \int_{\Omega} (\beta \cdot \nabla u + \mu u) v dx \quad (2.14)$$

is coercive, i.e., for all $v \in H_{\Gamma_-}^1(\Omega)$ we have

$$\begin{aligned} B^\varepsilon(v, v) &= \varepsilon \|v\|_{L^2(\Omega)}^2 + \int_{\Omega} \sum_{i=1}^d \frac{1}{2} \beta_i (v^2)_{x_i} dx + \int_{\Omega} \mu v^2 dx \\ &= \varepsilon \|v\|_{L^2(\Omega)}^2 + \int_{\Omega} \left(\mu - \frac{1}{2} \nabla \cdot \beta \right) v^2 dx + \frac{1}{2} \int_{\Gamma_+} v^2 (\beta \cdot n) d\alpha \\ &\geq \min \{ \varepsilon, \mu_0 \} \|v\|_{H^1(\Omega)}^2. \end{aligned} \quad (2.15)$$

The conclusion follows from the Lax–Milgram theorem. $\square$

Remark 2.3. For the both boundary value problems above, we have

$$\varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|u^\varepsilon\|_{L^2(\Omega)}^2 \leq \|f\|_{L^2(\Omega)} \|u^\varepsilon\|_{L^2(\Omega)},$$

and

$$\varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|u^\varepsilon\|_{L^2(\Omega)}^2 \leq \mu_0^{-1} \|f\|_{L^2(\Omega)}^2. \quad (2.16)$$

Remark 2.4. As is mentioned in Remark 2.2, the positivity condition is replaced by existence of the auxiliary function ζ satisfying $|\beta \cdot \nabla \zeta| > 0$: Under this assumption, the bilinear form B^ε is still coercive for sufficiently small ε .

At the end of this section, we introduce two inequalities, which are used later, for the trace of a function in H^1 .

Proposition 2.6. Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$. Then, there exists a constant C such that

$$\|u\|_{L^2(\partial\Omega)} \leq C (\eta \|\nabla u\|_{L^2(\Omega)} + \eta^{-1} \|u\|_{L^2(\Omega)})$$

for all $u \in H^1(\Omega)$ and $0 < \eta < 1$.

The following is called the Sobolev trace embedding theorem, which plays an important role in Section 3.2. A proof is found in [11]

Proposition 2.7. *Let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary, and suppose that $d > 2$ and $2 \leq q \leq 2^* := 2(d-1)/(d-2)$. Then, the trace operator $\gamma_{0,q} : H^1(\Omega) \rightarrow L^q(\partial\Omega)$ is bounded. If $d = 2$, then the trace operator $\gamma_{0,q} : H^1(\Omega) \rightarrow L^q(\partial\Omega)$ is bounded for $2 \leq q < \infty$.*

3 Convergence estimates

This section is dedicated to our main results: deriving convergence estimates for the error $u^\varepsilon - u$ in the $L^2(\Omega)$ as the perturbation parameter $\varepsilon \rightarrow +0$. The solution u is from the advection problem (2.1)-(2.2), and u^ε is from its elliptic regularization (2.11) with either Dirichlet or Neumann boundary conditions on $\partial\Omega \setminus \Gamma_-$. The behavior of u^ε near the boundary, particularly the formation of boundary layers, significantly influences these estimates. Understanding these rates is crucial, for instance, when using u^ε in numerical computations, as it informs how discretization parameters might need to relate to ε .

3.1 The case of the homogeneous Dirichlet boundary condition

We first consider the case where u^ε satisfies the homogeneous Dirichlet condition on the entire boundary $\partial\Omega$, i.e., problem (2.11)-(2.12).

The behavior of u^ε for this problem has been extensively studied, often using asymptotic expansion techniques (see, e.g., [7,25,27] and the review [28]). These studies reveal that u^ε can typically be decomposed into the solution u of the limit problem, boundary layer correctors, and a remainder term.

For instance, let Ω be a bounded domain in $\mathbb{R}^2$, and assume that the boundary $\partial\Omega$ and the coefficients in (2.1) are sufficiently smooth. Let S_- and S_+ be connected subsets of Γ_- and Γ_+ , respectively, satisfying that the characteristics of (2.1) in Ω emanating from S_- exit Ω through S_+ , and conversely. We call a regular quadrilateral for the closed and simply connected subset of $\bar{\Omega}$ bounded by S_- , S_+ and by the two characteristics of (2.1) joining the end points of S_- and S_+ . It is stated in [25] that the solution u^ε to the boundary value problem (2.11)-(2.12) can be decomposed in a regular quadrilateral Ω_0 in $\bar{\Omega}$ as

$$u^\varepsilon(x) = u(x) + z^\varepsilon(x) + w^\varepsilon(x)$$

where u is the solution to the boundary value problem to (2.1)-(2.2), and $w^\varepsilon = O(\varepsilon^{1/2})$ uniformly in Ω_0 , namely, there exists a number C independent of ε such that

$$\max_{x \in \Omega_0} |w^\varepsilon(x)| \leq C\varepsilon^{1/2} \quad \text{as } \varepsilon \rightarrow +0.$$

Moreover on some neighborhood of S_+ in Ω_0 we have

$$z^\varepsilon(x) = e^{-g(x)/\varepsilon} h(x) \quad (3.1)$$

where the functions g and h are of class C^2 , $g = 0$ on S_+ and $g > 0$ off of S_+ . We can find the generalization of this fact for the $\mathbb{R}^d$ case in [27]. Note that u^ε satisfies the boundary condition (2.12) on S_+ , but u does not in general. This discrepancy is corrected by z^ε , called the boundary layer term, and w^ε is the remainder term. The representation (3.1) implies that the width of the boundary layer at Γ_+ is of order $O(\varepsilon)$.

However, global $L^2(\Omega)$ estimates for $u^\varepsilon - u$ are generally available only for simple domains or under restrictive assumptions. For more general domains, estimates are often confined to subdomains away from the characteristic boundary Γ_0 . Near Γ_0 , different types of boundary layers, often called "parabolic" boundary layers, will appear with a larger width, typically $O(\varepsilon^{1/2})$ [10]. The interaction between these different layer types makes constructing uniform asymptotic expansions and deriving global estimates difficult. While qualitative convergence $u^\varepsilon \rightarrow u$ in $L^2(\Omega)$ for general domains was established in [2], obtaining precise rates globally is harder.

A typical case where the characteristic boundary layers appear is that the domain Ω is a rectangle and the vector field β is parallel to one of its edges. Here we consider the case $\Omega = (0, 1)^2 \subset \mathbb{R}^2$ and $\beta = (-1, 0)$. Then

$$\Gamma_+ = \{(0, y) \mid 0 \leq y \leq 1\} \quad \text{and} \quad \Gamma_0 = \{(x, 0) \mid 0 \leq x \leq 1\} \cup \{(x, 1) \mid 0 \leq x \leq 1\},$$

are the outflow boundary and the characteristic boundary, respectively. This setting can be found in numerous studies. For example, in [10], the boundary value problem

$$-\varepsilon \Delta u^\varepsilon - \frac{\partial u^\varepsilon}{\partial x} = f \quad \text{in } \Omega, \quad u^\varepsilon = 0 \quad \text{on } \partial\Omega$$

is considered. Here, it is assumed that $f = 0$ on $y = 1$. Then the solution u to the boundary value problem

$$-\frac{\partial u}{\partial x} = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } x = 1,$$

satisfies $u(x, 1) = 0$. In this case the characteristic boundary layers do not appear near $y = 1$ because u satisfies the same Dirichlet boundary condition as u^ε on $y = 1$. This situation is often referred to as a compatibility condition. On the other hand, a characteristic boundary layer typically appears near $y = 0$, and different behaviors called corner layers is observed near the corners $(0, 0)$ and $(1, 0)$ on $y = 0$. Therefore, to estimate $\|u^\varepsilon - u\|_{L^2(\Omega)}$ we need to construct not only the boundary layer terms near the Γ_+ and Γ_0 , but also the corner layer terms.

Some form of boundary layer analysis is typically required to obtain the global convergence estimates in most cases. Although the classical results mentioned above are obtained by boundary layer analysis, we will derive convergence estimates based on Proposition 2.4 without relying on such precise boundary layer analysis.

Let u and u^ε be solutions to the boundary value problem (2.1)-(2.2) and (2.11)-(2.12), respectively. Here we assume that u^ε belongs to $H^2(\Omega)$ and (2.11) holds in the sense of $L^2(\Omega)$. Then the difference $w^\varepsilon = u^\varepsilon - u$ satisfies

$$\int_{\Omega} (\beta \cdot \nabla w^\varepsilon + \mu w^\varepsilon) v dx = \varepsilon \int_{\Omega} (\Delta u^\varepsilon) v dx,$$

for all $v \in H_{\beta,+}(\Omega)$. Applying Proposition 2.4, we have

$$\|u^\varepsilon - u\|_{L^2(\Omega)} \leq \varepsilon C \|\Delta u^\varepsilon\|_{(H_{\beta,+}(\Omega))'}, \quad (3.2)$$

where C does not depend on ε . Unfortunately, we cannot derive convergence estimates from (3.2) in general because the test functions in $H_{\beta,+}(\Omega)$ may lack sufficient regularity in directions transversal to β , and integration by parts cannot be performed. However, in some special cases, we can obtain the convergence estimates.

Theorem 3.1. *Let $d = 1$ and $\Omega = (0, 1)$. In addition to (2.3) we assume that $\beta_0 = \inf_{0 \leq x \leq 1} \beta(x) > 0$. Then there exists a positive number C independent of ε such that*

$$\|u^\varepsilon - u\|_{L^2(0,1)} \leq C \varepsilon^{1/2} \quad \text{for all } \varepsilon > 0.$$

Proof. Note that $u^\varepsilon \in H^2(0, 1) \subset C^1[0, 1]$ and $u^\varepsilon(0) = u^\varepsilon(1) = 0$. From Rolle's theorem, there exists $x^* \in (0, 1)$ satisfying $u_x^\varepsilon(x^*) = 0$, for which we have

$$-\varepsilon \int_0^{x^*} u_{xx}^\varepsilon u_x^\varepsilon dx + \int_0^{x^*} \beta (u_x^\varepsilon)^2 dx = \int_0^{x^*} (f - \mu u^\varepsilon) u_x^\varepsilon dx,$$

and by using (2.16),

$$\varepsilon (u_x^\varepsilon(0))^2 + \beta_0 \|u_x^\varepsilon\|_{L^2(0,x^*)}^2 \leq \beta_0^{-1} (1 + \beta_0^{-1} \|\mu\|_\infty)^2 \|f\|_{L^2(0,x^*)}^2.$$

Thus it holds that $|u_x^\varepsilon(0)| \leq C \varepsilon^{-1/2}$. For $v \in H_\beta(0, 1) = H^1(0, 1)$ satisfying $v(1) = 0$, we have

$$\varepsilon |\langle u_{xx}^\varepsilon, v \rangle| \leq \varepsilon \left| \int_0^1 u_{xx}^\varepsilon v dx \right| + \varepsilon |u_x^\varepsilon(0)| |v(0)| \leq C \|v\|_{H_\beta(0,1)} \varepsilon^{1/2}.$$

Here we use the same notation as in Proposition 2.4. Therefore $\varepsilon \|u_{xx}^\varepsilon\|_{H'_{\beta,+}(0,1)} \leq C \varepsilon^{1/2}$. The conclusion follows from Proposition 2.4. $\square$

For the cases of $d \geq 2$, estimating the term $\varepsilon \|\Delta u^\varepsilon\|_{(H_{\beta,+}(\Omega))'}$ in (3.2) generally requires careful control the second-order derivatives of u^ε , including those in directions transversal to the advection field β . Obtaining such estimates is often challenging for arbitrary domains due to the complex behavior of boundary and internal layers. However, for certain domains with specific geometric structures that permit a more detailed analysis, it is possible to derive the necessary bounds, as we will now demonstrate for an annular domain in $\mathbb{R}^2$.

Theorem 3.2. *Let $\Omega = \{x \in \mathbb{R}^2 \mid R_1 < |x| < R_2\}$ be an annulus in $\mathbb{R}^2$, and let $\beta(x) = -x/|x|$ and $\mu = 0$. If $f \in H^1(\Omega)$, then there exists a positive number C independent of ε such that*

$$\|u^\varepsilon - u\|_{L^2(0,1)} \leq C\varepsilon^{1/2} \quad \text{for all } \varepsilon > 0.$$

Proof. As in the proof of Theorem 3.1, it is sufficient to estimate $\varepsilon \|u^\varepsilon\|_{(H_{\beta,+}(\Omega))'}$. By using the polar coordinate $x_1 = r \cos \theta, x_2 = r \sin \theta$, the solution $u^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ can be expanded into a Fourier series $\sum_n c_{\varepsilon,n}(r)e^{in\theta}$ and each $c_{\varepsilon,n}$ satisfies the boundary value problem:

$$\begin{aligned} -\varepsilon(c_{\varepsilon,n}'' + r^{-1}c_{\varepsilon,n}' - n^2r^{-2}c_{\varepsilon,n}) - c_{\varepsilon,n}' &= f_n, \quad \text{in } r \in (R_1, R_2), \\ c_{\varepsilon,n}(R_1) &= c_{\varepsilon,n}(R_2) = 0, \end{aligned} \quad (3.3)$$

where $f = \sum_n f_n(r)e^{in\theta}$. For $v \in H_{\beta,+}(\Omega)$,

$$\begin{aligned} \varepsilon \langle \Delta u^\varepsilon, v \rangle &= \varepsilon \int_0^{2\pi} \int_{R_1}^{R_2} (\partial_r^2 u^\varepsilon + r^{-1} \partial_r u^\varepsilon + r^{-2} \partial_\theta^2 u^\varepsilon) v r dr d\theta \\ &= \varepsilon R_2 \int_0^{2\pi} v \partial_r u^\varepsilon|_{r=R_2} d\theta \\ &\quad - \varepsilon \int_\Omega \partial_r u^\varepsilon (\partial_r v + r^{-1} v) dx + \varepsilon \int_\Omega (r^{-1} \partial_r u^\varepsilon + r^{-2} \partial_\theta^2 u^\varepsilon) v dx, \end{aligned}$$

and we have

$$\begin{aligned} \varepsilon |\langle \Delta u^\varepsilon, v \rangle| &\leq \varepsilon \|\partial_r u^\varepsilon\|_{L^2(\Gamma_-)} \|\partial_r v\|_{L^2(\Omega)} \\ &\quad + \varepsilon \|\partial_r u^\varepsilon\|_{L^2(\Omega)} (\|\partial_r v\|_{L^2(\Omega)} + R_1^{-1} \|v\|_{L^2(\Omega)}) \\ &\quad + \varepsilon (R_1^{-1} \|\partial_r u^\varepsilon\|_{L^2(\Omega)} + R_1^{-2} \|\partial_\theta^2 u^\varepsilon\|_{L^2(\Omega)}) \|v\|_{L^2(\Omega)}. \end{aligned} \quad (3.4)$$

Multiplying (3.3) by $r \overline{c_{\varepsilon,n}}$ and then integrating on $R_1 < r < R_2$ yields

$$\varepsilon R_1 \|c_{\varepsilon,n}'\|_{L^2(R_1, R_2)}^2 + \frac{\varepsilon n^2}{R_2^2} \|c_{\varepsilon,n}\|_{L^2(R_1, R_2)}^2 \leq \frac{1}{2} \|f_n\|_{L^2(R_1, R_2)}^2. \quad (3.5)$$

From (3.5) we have

$$\begin{aligned}
\|\partial_\theta^2 u^\varepsilon\|_{L^2(\Omega)}^2 &= \int_{R_1}^{R_2} \left(\int_0^{2\pi} \left| -\sum_n n^2 c_{\varepsilon,n} e^{in\theta} \right|^2 d\theta \right) r dr \\
&\leq R_2 \sum_n n^4 \|c_{\varepsilon,n}\|_{L^2(R_1, R_2)}^2 \\
&\leq \frac{\varepsilon^{-1} R_2^3}{2} \sum_n n^2 \|f_n\|_{L^2(R_1, R_2)}^2 \\
&\leq \frac{\varepsilon^{-1} R_2^3}{2R_1} \|\partial_\theta f\|_{L^2(\Omega)}^2, \tag{3.6}
\end{aligned}$$

$$\begin{aligned}
\|\partial_r u^\varepsilon\|_{L^2(\Omega)}^2 &\leq R_2 \sum_n \|c'_{\varepsilon,n}\|_{L^2(R_1, R_2)}^2 \\
&\leq \frac{\varepsilon^{-1} R_2}{2R_1} \sum_n \|f_n\|_{L^2(R_1, R_2)}^2 \\
&\leq \frac{\varepsilon^{-1} R_2}{2R_1^2} \|f\|_{L^2(\Omega)}^2. \tag{3.7}
\end{aligned}$$

We can take $R^* \in (R_1, R_2)$ satisfying $c'_{\varepsilon,n}(R^*) = 0$. Multiplying (3.3) by $-\overline{c_{\varepsilon,n}}$ and then integrating on $R^* < r < R_2$ yields

$$\begin{aligned}
&\frac{\varepsilon}{2} |c'_{\varepsilon,n}(R_2)|^2 + \left(\frac{\varepsilon}{R_2} + 1 \right) \|c'_{\varepsilon,n}\|_{L^2(R, R_2)}^2 \\
&\leq \|f_n\|_{L^2(R, R_2)} \|c'_{\varepsilon,n}\|_{L^2(R, R_2)} + \frac{\varepsilon n^2}{R_1^2} \|c_{\varepsilon,n}\|_{L^2(R, R_2)} \|c'_{\varepsilon,n}\|_{L^2(R, R_2)},
\end{aligned}$$

and we have

$$\begin{aligned}
\frac{\varepsilon}{2} |c'_{\varepsilon,n}(R_2)|^2 + \frac{\varepsilon}{R_2} \|c'_{\varepsilon,n}\|_{L^2(R^*, R_2)}^2 &\leq \frac{1}{2} \|f_n\|_{L^2(R^*, R_2)}^2 + \frac{\varepsilon^2 n^4}{R_1^4} \|c_{\varepsilon,n}\|_{L^2(R^*, R_2)}^2 \\
&\leq \frac{1}{2} \|f_n\|_{L^2(R^*, R_2)}^2 + \frac{\varepsilon n^2 R_2^2}{2R_1^4} \|f_n\|_{L^2(R^*, R_2)}^2.
\end{aligned}$$

Hence it holds that

$$\begin{aligned}
\|\partial_r u^\varepsilon\|_{L^2(\Gamma_-)}^2 &= \sum_n |c'_{\varepsilon,n}(R_2)|^2 \\
&\leq \varepsilon^{-1} \|f\|_{L^2(\Omega)}^2 + \frac{R_2^2}{R_1^4} \|\partial_\theta f\|_{L^2(\Omega)}^2. \tag{3.8}
\end{aligned}$$

The estimates (3.4), (3.6), (3.7) and (3.8) imply that there exists a positive number C depending on only R_1 , R_2 and f such that

$$\varepsilon |\langle \Delta u^\varepsilon, v \rangle| \leq C \varepsilon^{1/2} \|v\|_{H_\beta(\Omega)}$$

for all $\varepsilon \in (0, 1)$ and $v \in H_{\beta,+}(\Omega)$. This completes the proof. $\square$

Remark 3.1. The convergence rate $O(\varepsilon^{1/2})$ in Theorems 3.1 and 3.2 is obtained in specific geometries where Γ_0 is empty. The 1D case benefits from strong regularity and explicit derivative bounds. For the annulus with radial flow, the rotational symmetry and Fourier decomposition are the key to managing the derivative estimates. The $H^1(\Omega)$ regularity assumed for f in Theorem 3.2 ensures control over terms involving angular derivatives of f , which propagate to u^ε .

3.2 The case of the homogeneous Neumann boundary condition

In this subsection we discuss the convergence of the solutions u^ε of (2.11)-(2.13) to the solution of (2.1)-(2.2). In this case boundary layers at $\Gamma_0 \cup \Gamma_+$ are no longer visible (but higher-order derivatives still blow up) [28]. Therefore, when using u^ε as an approximation of u , it is optimal to choose the Neumann boundary condition on Γ_+ [19]. The convergence estimates of $\|u^\varepsilon - u\|_{L^2(\Omega)}$ were obtained for $u \in H^1(\Omega)$ in [19,26]. Although the solution u in general belongs to $H_\beta(\Omega)$, which is wider the Sobolev space $H^1(\Omega)$, we derive L^2 estimates of $u^\varepsilon - u$ assuming H^1 or H^2 regularity for u .

3.2.1 Convergence rate to H^1 solutions

Theorem 3.3. *Suppose that the solution u to the boundary value problem (2.1)-(2.2) belongs to $H^1(\Omega)$. Then there exists a positive number C independent of u and ε such that*

$$\varepsilon^{1/2} \|\nabla u^\varepsilon - \nabla u\|_{L^2(\Omega)} + \|u^\varepsilon - u\|_{L^2(\Omega)} + \|u^\varepsilon - u\|_{L^2(\Gamma_+; \beta \cdot n)} \leq C \|\nabla u\|_{L^2(\Omega)} \varepsilon^{1/2} \quad (3.9)$$

for all $\varepsilon > 0$, where u^ε is the (weak) solution to the boundary value problem (2.11)-(2.13).

Similar arguments to this theorem can be seen in several references. For example, see [26].

Proof. From the assumption, it holds that

$$\varepsilon \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} (\beta \cdot \nabla u + \mu u) v dx = \int_{\Omega} f v dx + \varepsilon \int_{\Omega} \nabla u \cdot \nabla v dx$$

and

$$\varepsilon \int_{\Omega} \nabla u^\varepsilon \cdot \nabla v dx + \int_{\Omega} (\beta \cdot \nabla u^\varepsilon + \mu u^\varepsilon) v dx = \int_{\Omega} f v dx$$

for all $v \in H_{\Gamma_-}^1(\Omega)$. Hence, the difference $w^\varepsilon = u^\varepsilon - u$ satisfies

$$\varepsilon \int_{\Omega} \nabla w^\varepsilon \cdot \nabla v dx + \int_{\Omega} (\beta \cdot \nabla w^\varepsilon + \mu w^\varepsilon) v dx = \varepsilon \int_{\Omega} \nabla u \cdot \nabla v dx \quad (3.10)$$

for all $v \in H_{\Gamma_-}^1(\Omega)$. In particular we can choose $v = w^\varepsilon$, and the same argument in (2.15) leads us to

$$\varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \leq \varepsilon \|\nabla u\|_{L^2(\Omega)} \|\nabla w^\varepsilon\|_{L^2(\Omega)}. \quad (3.11)$$

This inequality yields

$$\varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 \leq \varepsilon \|\nabla u\|_{L^2(\Omega)} \|\nabla w^\varepsilon\|_{L^2(\Omega)},$$

that is,

$$\|\nabla w^\varepsilon\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}. \quad (3.12)$$

From this and (3.11) we have

$$\varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \leq \varepsilon \|\nabla u\|_{L^2(\Omega)}^2,$$

which establishes the proof. $\square$

Corollary 3.4. *Under the assumption of Theorem (3.3), the sequence of solutions $\{u^\varepsilon\}_{\varepsilon>0}$ converges to u weakly in $H^1(\Omega)$ as $\varepsilon \rightarrow +0$.*

Proof. From (3.12), we have $\|\nabla u^\varepsilon\|_{L^2(\Omega)} \leq 2\|\nabla u\|_{L^2(\Omega)}$, which implies that the family $\{u^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $H^1(\Omega)$. There exists a subsequence of $\{u^\varepsilon\}_{\varepsilon>0}$ which converges to $\tilde{u}$ weakly in $H^1(\Omega)$. This $\tilde{u}$ must coincide with u because u^ε converges to u strongly in $L^2(\Omega)$ by Theorem 3.3. From the same argument, any subsequence of $\{u^\varepsilon\}_{\varepsilon>0}$ has a subsequence which converges to u weakly in $H^1(\Omega)$. This proves that $\{u^\varepsilon\}_{\varepsilon>0}$ itself converges to u weakly in $H^1(\Omega)$. $\square$

As we mentioned in Section 3.1, for the cases that the Dirichlet boundary condition is imposed for u^ε , the widths of boundary layers near Γ_+ and Γ_0 are of order $O(\varepsilon)$ and $O(\varepsilon^{1/2})$ in general, respectively. This suggests that the boundary layer appearing near Γ_0 has worse effect on convergence of u^ε to u than that appearing near Γ_+ . The same may be true for the case of the Neumann boundary condition. Indeed, the convergence rate of the trace of u^ε on Γ_0 obtained in the following theorem is worse than that on Γ_+ (with weight $\beta \cdot n$).

Corollary 3.5. *Suppose that the surface measure of the characteristic boundary Γ_0 is positive. Then, under the assumption of Theorem 3.3, there exists a constant C independent of u and ε such that*

$$\|u^\varepsilon\|_{L^2(\Gamma_0)} \leq C \|\nabla u\|_{L^2(\Gamma_0)} \varepsilon^{1/4}$$

for all $0 < \varepsilon < 1$.

Proof. Proposition 2.6 yields

$$\|w^\varepsilon\|_{L^2(\Gamma_0)} \leq \|w^\varepsilon\|_{L^2(\partial\Omega)} \leq C(\eta\|\nabla w^\varepsilon\|_{L^2(\Omega)} + \eta^{-1}\|w^\varepsilon\|_{L^2(\Omega)}) \quad (3.13)$$

for all $0 < \eta < 1$. We use (3.9) and (3.12) and obtain

$$\|w^\varepsilon\|_{L^2(\Gamma_0)} \leq C\|\nabla u\|_{L^2(\Omega)}(\eta + \eta^{-1}\varepsilon^{1/2})$$

for all $0 < \eta < 1$. It is clear that the best choice is $\eta = \varepsilon^{1/4}$, which proves the corollary. $\square$

Before ending this subsection, we will show the strong convergence u^ε to u in $L^2(\Omega)$ without the regularity assumptions for u^ε based on Theorem 3.3. The same assertion can be found in [2].

Corollary 3.6. *Suppose $u \in H_\beta(\Omega)$ and $u^\varepsilon \in H^1(\Omega)$ are solutions to (2.1)-(2.2) and (2.11)-(2.13), respectively. Then it holds that the sequence $\{u^\varepsilon\}_{\varepsilon>0}$ converges to u strongly in $L^2(\Omega)$ together with its trace $\{u^\varepsilon|_{\Gamma_+}\}_{\varepsilon>0}$ to $u|_{\Gamma_+}$ strongly in $L^2(\Gamma_+; \beta \cdot n)$.*

Proof. We first take an arbitrary positive number δ . There exists $u_\delta \in H_{\Gamma_-}^1(\Omega)$ such that

$$\|u_\delta - u\|_{H_\beta(\Omega)} \leq \delta.$$

We define $f_\delta = \beta \cdot \nabla u_\delta + \mu u_\delta$, which belongs to $L^2(\Omega)$. We estimate $f_\delta - f$ such as

$$\|f_\delta - f\|_{L^2(\Omega)} = \|\beta(u_\delta - u) + \mu(u_\delta - u)\|_{L^2(\Omega)} \leq C\|u_\delta - u\|_{H_\beta(\Omega)} \leq C\delta.$$

From Proposition 2.5, there exists a unique solution $u_\delta^\varepsilon \in H_{\Gamma_-}^1(\Omega)$ to the boundary value problem

$$\begin{cases} -\varepsilon \Delta u_\delta^\varepsilon + \beta \cdot \nabla u_\delta^\varepsilon + \mu u_\delta^\varepsilon = f_\delta & \text{in } \Omega, \\ u_\delta^\varepsilon = 0 & \text{on } \Gamma_-, \\ \frac{\partial u_\delta^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma_-. \end{cases}$$

Using (2.15), we have

$$\varepsilon\|\nabla(u^\varepsilon - u_\delta^\varepsilon)\|_{L^2(\Omega)}^2 + \mu_0\|u^\varepsilon - u_\delta^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u^\varepsilon - u_\delta^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \leq \mu_0^{-1}\|f - f_\delta\|_{L^2(\Omega)}^2.$$

Because u_δ belongs to $H^1(\Omega)$, we can apply Theorem 3.3 and we have

$$\|u_\delta^\varepsilon - u_\delta\|_{L^2(\Omega)} + \|u_\delta^\varepsilon - u_\delta\|_{L^2(\Gamma_+; \beta \cdot n)} \leq C_\delta \varepsilon^{1/2},$$

where the number C_δ is independent of ε . From this inequalities we have

$$\begin{aligned} \|u^\varepsilon - u\|_{L^2(\Omega)} &\leq \|u^\varepsilon - u_\delta^\varepsilon\|_{L^2(\Omega)} + \|u_\delta^\varepsilon - u_\delta\|_{L^2(\Omega)} + \|u_\delta - u\|_{L^2(\Omega)} \\ &\leq (\mu_0^{-1}C + 1)\delta + C_\delta \varepsilon^{1/2}. \end{aligned}$$

Taking the limit $\sup \varepsilon \rightarrow +0$, since δ is arbitrary, we have

$$\lim_{\varepsilon \rightarrow +0} \|u^\varepsilon - u\|_{L^2(\Omega)} = 0.$$

The same argument can be applied to $\{u^\varepsilon|_{\Gamma_+}\}_{\varepsilon>0}$. $\square$

3.2.2 Convergence rate to H^2 solutions

In this subsection, we discuss convergence rates of u^ε assuming that the solution u to the boundary value problem (2.1)-(2.2) has H^2 regularity.

Assuming that $u \in H^2(\Omega)$, we can apply integration by parts to the right hand side of (3.10), which results in

$$\varepsilon \int_{\Omega} \nabla w^\varepsilon \cdot \nabla v dx + \int_{\Omega} (\beta \cdot \nabla w^\varepsilon + \mu w^\varepsilon) v dx = \varepsilon \int_{\partial\Omega \setminus \Gamma_-} \frac{\partial u}{\partial n} v d\sigma - \varepsilon \int_{\Omega} (\Delta u) v dx$$

for all $v \in H_{\Gamma_-}^1(\Omega)$. Choosing $v = w^\varepsilon \in H_{\Gamma_-}^1(\Omega)$,

$$\begin{aligned} \varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \\ \leq \varepsilon \int_{\partial\Omega \setminus \Gamma_-} \frac{\partial u}{\partial n} w^\varepsilon d\sigma - \varepsilon \int_{\Omega} (\Delta u) w^\varepsilon dx. \end{aligned} \quad (3.14)$$

The improved rates depend on controlling the boundary integral term.

Theorem 3.7. *Suppose that the solution u to the boundary value problem (2.1)-(2.2) belongs to $H^2(\Omega)$. Also, suppose that the surface measure of the characteristic boundary Γ_0 is positive. Then, there exists a positive number C independent of u and ε such that*

$$\|u^\varepsilon - u\|_{L^2(\Omega)} + \|u^\varepsilon - u\|_{L^2(\Gamma_+; \beta \cdot n)} \leq C \|u\|_{H^2(\Omega)} \varepsilon^{3/4}$$

and

$$\|\nabla(u^\varepsilon - u)\|_{L^2(\Omega)} \leq C \|u\|_{H^2(\Omega)} \varepsilon^{1/4}$$

for all $0 < \varepsilon < 1$, where u^ε is the solution to the boundary value problem (2.11)-(2.13).

Proof. Starting from (3.14). Now we have

$$\begin{aligned} \varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \\ \leq C \varepsilon \|u\|_{H^2(\Omega)} (\|\nabla w^\varepsilon\|_{L^2(\Omega)} + \|w^\varepsilon\|_{L^2(\Omega)}) \\ \leq C \varepsilon \|u\|_{H^2(\Omega)} (\varepsilon^{1/4} \|\nabla w^\varepsilon\|_{L^2(\Omega)} + \varepsilon^{-1/4} \|w^\varepsilon\|_{L^2(\Omega)} + \|w^\varepsilon\|_{L^2(\Omega)}). \end{aligned}$$

Here we use Proposition 2.6 with $\eta = \varepsilon^{1/4}$ so that, for $\varepsilon \in (0, 1)$, we obtain

$$\begin{aligned} & \varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \\ & \leq C \|u\|_{H^2(\Omega)} (\varepsilon^{5/4} \|\nabla w^\varepsilon\|_{L^2(\Omega)} + \varepsilon^{3/4} \|w^\varepsilon\|_{L^2(\Omega)}) \\ & \leq C \|u\|_{H^2(\Omega)}^2 \varepsilon^{3/2} + \frac{\varepsilon}{2} \|\nabla w^\varepsilon\|_{L^2(\Omega)} + \frac{\mu_0}{2} \|w^\varepsilon\|_{L^2(\Omega)}. \end{aligned}$$

Hence we have

$$\varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \leq C \|u\|_{H^2(\Omega)}^2 \varepsilon^{3/2},$$

and we have both

$$\|w^\varepsilon\|_{L^2(\Omega)} + \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)} \leq C \|u\|_{H^2(\Omega)} \varepsilon^{3/4}$$

and

$$\|\nabla w^\varepsilon\|_{L^2(\Omega)} \leq C \|u\|_{H^2(\Omega)} \varepsilon^{1/4}$$

for all $0 < \varepsilon < 1$. □

Observing this proof, we can see that the right hand side of (3.14) improves the convergence rate compared to Theorem 3.3. Indeed, the term $\varepsilon \int_\Omega (\Delta u) w^\varepsilon dx$ converges faster than $\varepsilon \int_\Omega \nabla u \cdot \nabla w^\varepsilon dx$ because $\varepsilon^{1/2} \|\nabla w^\varepsilon\|_{L^2(\Omega)}$ and $\|w^\varepsilon\|_{L^2(\Omega)}$ are of the same order as $\varepsilon \rightarrow +0$. The difference in convergence rate in the cases of $u \in H^2(\Omega)$ arises from the boundary integral terms in the right hand side of (3.14).

Combining this theorem and (3.13), the following estimate of convergence rate on Γ_0 immediately follows.

Corollary 3.8. *Under the assumption of Theorem 3.7, there exists a positive number C independent of u and ε such that*

$$\|u^\varepsilon - u\|_{L^2(\Gamma_0)} \leq \|\nabla u\|_{L^2(\Omega)} \varepsilon^{1/2}$$

for all $0 < \varepsilon < 1$.

From what was said immediately after Theorem 3.7, the best convergence rate is obtained when no boundary integral terms appear in (3.14).

Theorem 3.9. *Suppose that the solution u to the boundary value problem (2.1)–(2.2) belongs to $H^2(\Omega)$. Also, suppose that the surface measures of the characteristic boundary Γ_0 and the outflow boundary Γ_+ are 0. Then, there exists a positive number C independent of u and ε such that*

$$\|u^\varepsilon - u\|_{L^2(\Omega)} \leq C \|u\|_{H^2(\Omega)} \varepsilon$$

and

$$\|\nabla(u^\varepsilon - u)\|_{L^2(\Omega)} \leq C \|u\|_{H^2(\Omega)} \varepsilon^{1/2}$$

for all $\varepsilon > 0$, where u^ε is the solution to the boundary value problem (2.11)–(2.13).

Proof. By the assumption, the boundary integral term does not appear in (3.14). From (3.10) it follows that

$$\begin{aligned} & \varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \\ & \leq C \varepsilon \|u\|_{H^2(\Omega)} \|w^\varepsilon\|_{L^2(\Omega)} \leq C \|u\|_{H^2(\Omega)}^2 \varepsilon^2 + \frac{\mu_0}{2} \|w^\varepsilon\|_{L^2(\Omega)}^2, \end{aligned}$$

that is,

$$2\varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|w^\varepsilon\|_{L^2(\Omega)}^2 + \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \leq C \|u\|_{H^2(\Omega)}^2 \varepsilon^2.$$

□

Let us consider the case where the surface measure of Γ_0 is 0 but the surface measure of Γ_+ is positive. The difficulty in this case is that the weighted L^2 norm $\|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}$ appearing in the left hand side of (3.14) is not equivalent to the L^2 norm $\|w^\varepsilon\|_{L^2(\Gamma_+)}$ and, as a result, the term $\|w^\varepsilon\|_{L^2(\Gamma_+)}$ that appear by applying Schwartz's inequality to the first term on the right hand side of (3.14) cannot be absorbed into the left hand side. Conversely, convergence estimate can be obtained when these norms are equivalent, and one of the sufficient conditions for this is the uniform positivity of $\beta \cdot n$ on Γ_+ .

Theorem 3.10. *Suppose that the solution u to the boundary value problem (2.1)-(2.2) belongs to $H^2(\Omega)$. Also, suppose that the surface measure of the characteristic boundary Γ_0 is 0. Furthermore, we assume $\lambda := \inf_{\Gamma_+} \beta \cdot n > 0$. Then, there exists a positive number C independent of u and ε such that*

$$\|u^\varepsilon - u\|_{L^2(\Omega)} \leq C \|u\|_{H^2(\Omega)} \varepsilon$$

and

$$\|\nabla(u^\varepsilon - u)\|_{L^2(\Omega)} \leq C \|u\|_{H^2(\Omega)} \varepsilon^{1/2}$$

for all $\varepsilon > 0$, where u^ε is the solution to the boundary value problem (2.11)-(2.13).

Proof. Since the surface measure of Γ_0 is 0, (3.14) yields

$$\begin{aligned} & \varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \\ & \leq \varepsilon \int_{\Gamma_+} \left| \frac{\partial u}{\partial n} \right| |w^\varepsilon| d\sigma + \varepsilon \int_{\Omega} |\Delta u| |w^\varepsilon| dx \\ & \leq \varepsilon C \|u\|_{H^2(\Omega)} (\|w^\varepsilon\|_{L^2(\Gamma_+)} + \|w^\varepsilon\|_{L^2(\Omega)}). \end{aligned}$$

By the positivity of $\lambda > 0$, it holds that

$$\|w^\varepsilon\|_{L^2(\Gamma_+)}^2 = \int_{\Gamma_+} (\beta \cdot n)^{-1} |w^\varepsilon|^2 (\beta \cdot n) d\sigma \leq \lambda^{-1} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2.$$

Thus, we have

$$\varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\mu_0}{2} \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{4} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \leq C \|u\|_{H^2(\Omega)}^2 \varepsilon^2.$$

We can estimate each term in the left-hand side of this inequality as follows:

$$\begin{aligned} \|\nabla w^\varepsilon\|_{L^2(\Omega)} &\leq C \|u\|_{H^2(\Omega)} \varepsilon^{1/2}, \\ \|w^\varepsilon\|_{L^2(\Omega)} &\leq C \|u\|_{H^2(\Omega)} \varepsilon, \\ \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)} &\leq C \|u\|_{H^2(\Omega)} \varepsilon. \end{aligned}$$

These complete the proof. $\square$

We finally discuss convergence rates if the measure of Γ_0 is 0 and the inner product $\beta \cdot n$ can degenerate on Γ_+ . Because one-dimensional case is covered by Theorem 3.7, Theorem 3.9 and Theorem 3.10, in what follows we discuss the case $d \geq 2$.

Theorem 3.11. *Suppose that the solution u to the boundary value problem (2.1)-(2.2) belongs to $H^2(\Omega)$. Also suppose that the surface measure of the characteristic boundary Γ_0 is 0, and that there exists a positive number α such that $(\beta \cdot n)^{-\alpha} \in L^1(\Gamma_+)$. Then, we have the following convergence estimates:*

1. *When $d = 2$, for any $q \geq 2$, there exists a positive number C_q independent of u and ε such that*

$$\|u^\varepsilon - u\|_{L^2(\Omega)} + \|u^\varepsilon - u\|_{L^2(\Gamma_+; \beta \cdot n)} \leq C_q \|u\|_{H^2(\Omega)} \varepsilon^{\min\{1, \frac{3}{4} + R_1(\alpha, q)\}}$$

and

$$\|\nabla(u^\varepsilon - u)\|_{L^2(\Omega)} \leq C_q \|u\|_{H^2(\Omega)} \varepsilon^{\min\{\frac{1}{2}, \frac{1}{4} + R_1(\varepsilon, q)\}}$$

for all $0 < \varepsilon < 1$, where

$$R_1(\alpha, q) := \frac{\alpha(q-2)}{4q}.$$

2. *When $d \geq 3$, there exists a positive number C independent of u and ε such that*

$$\|u^\varepsilon\|_{L^2(\Omega)} + \|u^\varepsilon - u\|_{L^2(\Gamma_+; \beta \cdot n)} \leq C \|u\|_{H^2(\Omega)} \varepsilon^{\min\{1, \frac{3}{4} + \frac{\alpha}{4(d-1)}\}}$$

and

$$\|\nabla(u^\varepsilon - u)\|_{L^2(\Omega)} \leq C \|u\|_{H^2(\Omega)} \varepsilon^{\min\{\frac{1}{2}, \frac{1}{4} + \frac{\alpha}{4(d-1)}\}}$$

for all $0 < \varepsilon < 1$.

Remark 3.2. Theorem 3.11 quantifies how the rate of degeneracy of $\beta \cdot n$ on Γ_+ (parameterized by the largest α such that $(\beta \cdot n)^{-\alpha}$ is integrable) influences the convergence rates when Γ_0 is absent. A larger α (slower degeneracy) leads to a better exponent (closer to 1 for L^2 -error and 1/2 for H^1 -seminorm error). For instance, in $d = 2$, as $q \rightarrow \infty$, the term $\frac{\alpha(q-2)}{4q}$ approaches $\alpha/4$. If α is large enough such that $\frac{3}{4} + \frac{\alpha}{4} \geq 1$ (i.e., $\alpha \geq 1$), the rate saturates at $O(\varepsilon)$ for the L^2 -error, consistent with the non-degenerate case (Theorem 3.10).

Proof of Theorem 3.11. Proposition 2.7 leads

$$\left\| \frac{\partial u}{\partial n} \right\|_{L^q(\partial\Omega)} \leq \|\gamma_{0,q}\| \sum_{i=1}^d \|u_{x_i}\|_{H^1(\Omega)} \leq C_q \|u\|_{H^2(\Omega)}$$

for all $q \geq 2$ when $d = 2$, and for all $2 \leq q \leq 2(d-1)/(d-2)$ when $d \geq 3$.

We first prove the case $d = 2$. Since the surface measure of Γ_0 is 0, for $q \geq 2$ (3.14) yields

$$\begin{aligned} & \varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \\ & \leq \varepsilon \left\| \frac{\partial u}{\partial n} \right\|_{L^q(\Gamma_+)} \|w^\varepsilon\|_{L^{q'}(\Gamma_+)} + \varepsilon \|\Delta u\|_{L^2(\Omega)} \|w^\varepsilon\|_{L^2(\Omega)} \\ & \leq \varepsilon \|\gamma_{0,q}\| \|u\|_{H^2(\Omega)} \|w^\varepsilon\|_{L^{q'}(\Gamma_+)} + \varepsilon \|\Delta u\|_{L^2(\Omega)} \|w^\varepsilon\|_{L^2(\Omega)} \\ & \leq \varepsilon C_q \|u\|_{H^2(\Omega)} \left(\|w^\varepsilon\|_{L^{q'}(\Gamma_+)} + \|w^\varepsilon\|_{L^2(\Omega)} \right), \end{aligned} \quad (3.15)$$

where q' is the Hölder conjugate of $q \leq 2$, namely, $1/q + 1/q' = 1$. To obtain the convergence estimate of w^ε , we give an estimate for the boundary integral term $\|w^\varepsilon\|_{L^{q'}(\Gamma_+)}$.

By the Hölder inequality, we have

$$\begin{aligned} \int_{\Gamma_+} |w^\varepsilon|^{q'} d\sigma &= \int_{\Gamma_+} (\beta \cdot n)^{-\frac{\alpha(2-q')}{2}} \cdot |w^\varepsilon|^{q'} (\beta \cdot n)^{\frac{\alpha(2-q')}{2}} d\sigma \\ &\leq \left(\int_{\Gamma_+} (\beta \cdot n)^{-\alpha} d\sigma \right)^{1-q'/2} \left(\int_{\Gamma_+} |w^\varepsilon|^2 (\beta \cdot n)^{\frac{\alpha(2-q')}{q'}} d\sigma \right)^{q'/2}, \end{aligned}$$

and hence we have

$$\|w^\varepsilon\|_{L^{q'}(\Gamma_+)} \leq C_q \left(\int_{\Gamma_+} |w^\varepsilon|^2 (\beta \cdot n)^{\frac{\alpha(q-2)}{q}} d\sigma \right)^{1/2}. \quad (3.16)$$

If $\alpha(q-2)/q \geq 1$, (3.16) leads to $\|w^\varepsilon\|_{L^{q'}(\Gamma_+)} \leq C_q \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}$ because

$$\int_{\Gamma_+} |w^\varepsilon|^2 (\beta \cdot n)^{\frac{\alpha(q-2)}{q}} d\sigma \leq (\|\beta\|_{L^\infty(\Omega)})^{\frac{(\alpha-1)q-2\alpha}{q}} \int_{\Gamma_+} |w^\varepsilon|^2 (\beta \cdot n) d\sigma.$$

Thus (3.15) yields

$$\begin{aligned} & \varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \mu_0 \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \\ & \leq \varepsilon C_q \|u\|_{H^2(\Omega)} (\|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)} + \|w^\varepsilon\|_{L^2(\Omega)}), \end{aligned}$$

and a similar argument as in the proof of Theorem 3.10 gives

$$\|w^\varepsilon\|_{L^2(\Omega)} + \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)} \leq C_q \|u\|_{H^2(\Omega)} \varepsilon$$

and

$$\|\nabla w^\varepsilon\|_{L^2(\Omega)} \leq C_q \|u\|_{H^2(\Omega)} \varepsilon^{1/2}.$$

In what follows, we assume that $\alpha(q-2)/q < 1$. We take a positive number δ , which is determined later, and split the integral in (3.16) into two parts:

$$\int_{\Gamma_+} |w^\varepsilon|^2 (\beta \cdot n)^{\frac{\alpha(q-2)}{q}} d\sigma = \int_{\Gamma_+ \setminus \Gamma_{+, \delta}} |w^\varepsilon|^2 (\beta \cdot n)^{\frac{\alpha(q-2)}{q}} d\sigma + \int_{\Gamma_{+, \delta}} |w^\varepsilon|^2 (\beta \cdot n)^{\frac{\alpha(q-2)}{q}} d\sigma, \quad (3.17)$$

where the set $\Gamma_{+, \delta}$ is defined by

$$\Gamma_{+, \delta} = \{x \in \Gamma_+ \mid \beta(x) \cdot n(x) < \delta\}.$$

We can estimate each term in the right hand side of (3.17) as follows:

$$\begin{aligned} \int_{\Gamma_+ \setminus \Gamma_{+, \delta}} |w^\varepsilon|^2 (\beta \cdot n)^{\frac{\alpha(q-2)}{q}} d\sigma &= \delta^{\frac{\alpha(q-2)}{q}-1} \int_{\partial\Omega \setminus \Gamma_{+, \delta}} |w^\varepsilon|^2 (\beta \cdot n) d\sigma \\ &\leq \delta^{\frac{\alpha(q-2)}{q}-1} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2, \\ \int_{\Gamma_{+, \delta}} |w^\varepsilon|^2 (\beta \cdot n)^{\frac{\alpha(q-2)}{q}} d\sigma &\leq \delta^{\frac{\alpha(q-2)}{q}} \|w^\varepsilon\|_{L^2(\Gamma_+)}^2 \\ &\leq C_q \delta^{\frac{\alpha(q-2)}{q}} \left(\varepsilon^{1/4} \|\nabla w^\varepsilon\|_{L^2(\Omega)} + \varepsilon^{-1/4} \|w^\varepsilon\|_{L^2(\Omega)} \right)^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|w^\varepsilon\|_{L^{q'}(\Gamma_+)} &\leq C_q \left(\int_{\Gamma_+} |w^\varepsilon|^2 (\beta \cdot n)^{\frac{\alpha(q-2)}{q}} d\sigma \right)^{1/2} \\ &\leq C_q \delta^{\frac{\alpha(q-2)}{2q}} \left(\delta^{-1/2} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)} + \varepsilon^{1/4} \|\nabla w^\varepsilon\|_{L^2(\Omega)} + \varepsilon^{-1/4} \|w^\varepsilon\|_{L^2(\Omega)} \right). \end{aligned}$$

From this, (3.15) yields

$$\varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\mu_0}{2} \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{4} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \leq C_q \|u\|_{H^2(\Omega)} \delta^{\frac{\alpha(q-2)}{q}} (\varepsilon^2 \delta^{-1} + \varepsilon^{3/2})$$

for all $0 < \varepsilon < 1$. Choosing $\delta = \varepsilon^{1/2}$, we have

$$\varepsilon \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\mu_0}{2} \|w^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{4} \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)}^2 \leq C_q \varepsilon^{\frac{3}{2} + \frac{\alpha(q-2)}{2q}}.$$

Therefore, it holds that

$$\|w^\varepsilon\|_{L^2(\Omega)} + \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)} \leq C_q \varepsilon^{\frac{3}{4} + \frac{\alpha(q-2)}{4q}},$$

and

$$\|\nabla w^\varepsilon\|_{L^2(\Omega)} \leq C_q \varepsilon^{\frac{1}{4} + \frac{\alpha(q-2)}{4q}}$$

for all $q \geq 2$ and $0 < \varepsilon < 1$.

For the case when $d \geq 3$, we replace the parameter q in the above argument by its upper bound $2(d-1)/(d-2)$. Then, we have

$$\frac{\alpha(q-2)}{q} = \frac{\alpha}{d-1}.$$

When $\alpha \geq d-1$, it holds that

$$\|w^\varepsilon\|_{L^2(\Omega)} + \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)} \leq C\varepsilon$$

and

$$\|\nabla w^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{1/2}$$

for all $0 < \varepsilon < 1$, and when $\alpha < d-1$ it holds that

$$\|w^\varepsilon\|_{L^2(\Omega)} + \|w^\varepsilon\|_{L^2(\Gamma_+; \beta \cdot n)} \leq C\varepsilon^{\frac{3}{4} + \frac{\alpha}{4(d-1)}}$$

and

$$\|\nabla w^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{\frac{1}{4} + \frac{\alpha}{4(d-1)}}$$

for all $0 < \varepsilon < 1$. □

4 Optimality of Convergence Rate

In this section, we numerically verify the optimality of the $L^2(\Omega)$ -norm convergence rates for $u^\varepsilon - u$ derived in Section 3.2, using the finite element method (FEM). The numerical examples presented are for two-dimensional cases, and they confirm the optimality of the theoretical convergence estimates.

We provide a brief overview of the FEM (For more detail, see, e.g., [8]). The FEM is a numerical method well-suited for finding approximate solutions to weak formulations of partial differential equations, such as those discussed in Proposition 2.5. Let V be a real Hilbert space, and let $a(\cdot, \cdot)$ be a real-valued bilinear form on $V \times V$ that is bounded and coercive. Given

a bounded linear functional F on V , the abstract variational problem is to find $v \in V$ such that

$$a(v, w) = F(w) \quad \text{for all } w \in V. \quad (4.1)$$

Let V_h be a finite-dimensional subspace of V . If $\{\varphi_i\}_{i=1}^N$ is a basis for V_h (so $V_h = \text{Span}\{\varphi_i\}_{i=1}^N$), the discrete problem is to find $v_h \in V_h$ such that

$$a(v_h, \varphi_i) = F(\varphi_i) \quad \text{for } 1 \leq i \leq N. \quad (4.2)$$

We write $v_h = \sum_{j=1}^N v^{(j)} \varphi_j$, and the equation (4.2) becomes a system of linear equations:

$$\sum_{j=1}^N a(\varphi_j, \varphi_i) v^{(j)} = F(\varphi_i) \quad \text{for } 1 \leq i \leq N.$$

If we let A be the matrix with entries $A_{ij} = a(\varphi_j, \varphi_i)$, $\mathbf{v} = (v^{(j)})$, and $\mathbf{F} = (F(\varphi_i))$, this system is $A\mathbf{v} = \mathbf{F}$. The FEM is a numerical method that constructs the subspace V_h using finite elements to find an approximate solution to (4.1).

For the problems considered in this study, we set $V = H_{\Gamma_-}^1(\Omega)$ and use the bilinear form $B^\varepsilon(\cdot, \cdot)$ defined in (2.14). We discretize the domain $\bar{\Omega} \subset \mathbb{R}^2$ using a triangulation $\mathcal{T}_h = \{T_m\}_{m=1}^M$, which is a collection of closed triangles such that $\bar{\Omega} = \bigcup_{m=1}^M T_m$. For any two distinct triangles $T_m, T_{m'}$, their interiors $\mathring{T}_m, \mathring{T}_{m'}$ are disjoint, and their intersection $T_m \cap T_{m'}$ is either empty, a common vertex, or a common edge. The finite element space V_h is then defined using continuous piecewise linear functions (P_1 elements) on this triangulation:

$$V_h = \left\{ w_h \in C^0(\bar{\Omega}) \mid w_h|_{\Gamma_-} = 0 \quad \text{and} \quad w_h|_{T_m} \text{ is linear for all } T_m \in \mathcal{T}_h \right\}.$$

Thus, V_h is a finite-dimensional subspace of $H_{\Gamma_-}^1(\Omega)$. The mesh parameter h is defined as $h = \max_{T_m \in \mathcal{T}_h} \text{diam}(T_m)$. We assume that the family of triangulations $\{\mathcal{T}_h\}_h$ is shape-regular, that is, there exists a constant $\sigma_0 > 0$ such that for all h and all $T_m \in \mathcal{T}_h$, the ratio $\text{diam}(T_m)/\rho_{T_m} \leq \sigma_0$, where ρ_{T_m} is the diameter of the inscribed circle of T_m .

The continuous problem for $u^\varepsilon \in H_{\Gamma_-}^1(\Omega)$ and the discrete problem for $u_h^\varepsilon \in V_h$ are, respectively:

$$\begin{aligned} B^\varepsilon(u^\varepsilon, v) &= (f, v)_{L^2(\Omega)}, & \text{for all } v &\in H_{\Gamma_-}^1(\Omega), \\ B^\varepsilon(u_h^\varepsilon, v_h) &= (f, v_h)_{L^2(\Omega)}, & \text{for all } v_h &\in V_h, \end{aligned}$$

where $f \in L^2(\Omega)$ is the source term. For second-order elliptic boundary value problems, FEM error analysis is well-established. Under appropriate

regularity assumptions on u^ε and shape-regularity of the mesh, a standard L^2 -error estimate is

$$\|u^\varepsilon - u_h^\varepsilon\|_{L^2(\Omega)} \leq C_\varepsilon h^2. \quad (4.3)$$

It is important to note that the constant C_ε typically depends on ε and may tend to infinity as $\varepsilon \rightarrow +0$, especially for singularly perturbed problems.

This study focuses on the convergence of u^ε to u (the solution of the limit advection problem) as $\varepsilon \rightarrow +0$. We are interested in the error $\|u^\varepsilon - u\|_{L^2(\Omega)}$. In the numerical experiments, we compute u_h^ε for a fixed (small) h and varying ε . Let $\Pi_h : H_{\Gamma_-}^1(\Omega) \rightarrow V_h$ be the projection operator. The error can be decomposed using the triangle inequality:

$$\|u^\varepsilon - u\|_{L^2(\Omega)} \leq \|u^\varepsilon - u_h^\varepsilon\|_{L^2(\Omega)} + \|u_h^\varepsilon - \Pi_h u\|_{L^2(\Omega)} + \|\Pi_h u - u\|_{L^2(\Omega)}. \quad (4.4)$$

The first term on the right-hand side of (4.4) is the FEM discretization error for u^ε , estimated by (4.3). The third term is estimated by the interpolation inequality: For a shape-regular family of triangulation $\{\mathcal{T}_h\}_h$, there exists a positive number c independent of h such that, for all h and $v \in H^{k+1}(\Omega)$,

$$\|v - \Pi_h v\|_{L^2(\Omega)} \leq ch^{k+1} \|\nabla^{k+1} v\|_{L^2(\Omega)}.$$

The middle term, $\|u_h^\varepsilon - \Pi_h u\|_{L^2(\Omega)}$, is what is often computed and analyzed in numerical studies to infer the behavior of $\|u^\varepsilon - u\|_{L^2(\Omega)}$ when u^ε itself is not known. In our numerical experiments, the domain Ω is the unit square $(0,1)^2$. The triangulation $\mathcal{T}_h$ is fixed with $h \approx 0.0020$, where the number of triangles in $\mathcal{T}_h$ is 2,391,180. We then find the FEM solution u_h^ε for $\varepsilon = 1.6^{-k}$, $k = 0, 1, \dots, 14$ (so ε ranges from 1 down to approximately 0.0014) using FreeFEM [14]. All computations are carried out in IEEE 754 double precision. We directly compute $\|u_h^\varepsilon - \Pi_h u\|_{L^2(\Omega)}$ where u is the known exact solution to the advection problem. By ensuring h is sufficiently small, we aim for the FEM error $\|u_h^\varepsilon - u^\varepsilon\|_{L^2(\Omega)}$ to be negligible relative to the regularization error $\|u^\varepsilon - u\|_{L^2(\Omega)}$, so that the observed behavior of $\|u_h^\varepsilon - \Pi_h u\|_{L^2(\Omega)}$ as ε varies reflects the true convergence rate of u^ε to u .

4.1 Example 1: corresponding to Theorem 3.3

According to Theorem 3.3 and their corollaries, when the solution u to the boundary value problem (2.1)-(2.2) belongs to $H_{\Gamma_-}^1(\Omega)$, it follows that

$$\begin{aligned} \|u^\varepsilon - u\|_{L^2(\Omega)} + \|u^\varepsilon - u\|_{L^2(\Gamma_+; \beta \cdot n)} &\leq C\varepsilon^{\frac{1}{2}}, \\ \|\nabla(u^\varepsilon - u)\|_{L^2(\Omega)} &\leq C, \\ \|u^\varepsilon - u\|_{L^2(\Gamma_0)} &\leq C\varepsilon^{\frac{1}{4}}. \end{aligned}$$

Let

$$\beta(x_1, x_2) = (x_1, 1), \quad \mu(x_1, x_2) = 1, \quad f(x_1, x_2) = (s+1)x_1^s x_2 + x_1^s + x_2 + 1,$$

for a positive number s , and the exact solution to (2.1)-(2.2) is

$$u(x_1, x_2) = (1 + x_1^s)x_2.$$

Since $\Omega = (0, 1)^2 \subset \mathbb{R}^2$, the solution u belongs to $H_{\beta,-}(\Omega)$ for all $s > 0$ and also to $H_{\Gamma_-}^1(\Omega)$ for $s > 1/2$. In particular, we should remark that u belongs to $H_{\Gamma_-}^1(\Omega) \setminus H^2(\Omega)$ for $s \in (1/2, 1)$. The L^2 error estimation of $u - u^\varepsilon$ for $s = 0.51$ is shown in Figure 4.1, where the circles are errors obtained by numerical computations and the solid lines are obtained by the least square methods to results with $\varepsilon = 1.6^{-k}$, $8 \leq k \leq 14$ (The same shall apply hereinafter). These graphs clarify the validity of our estimate.

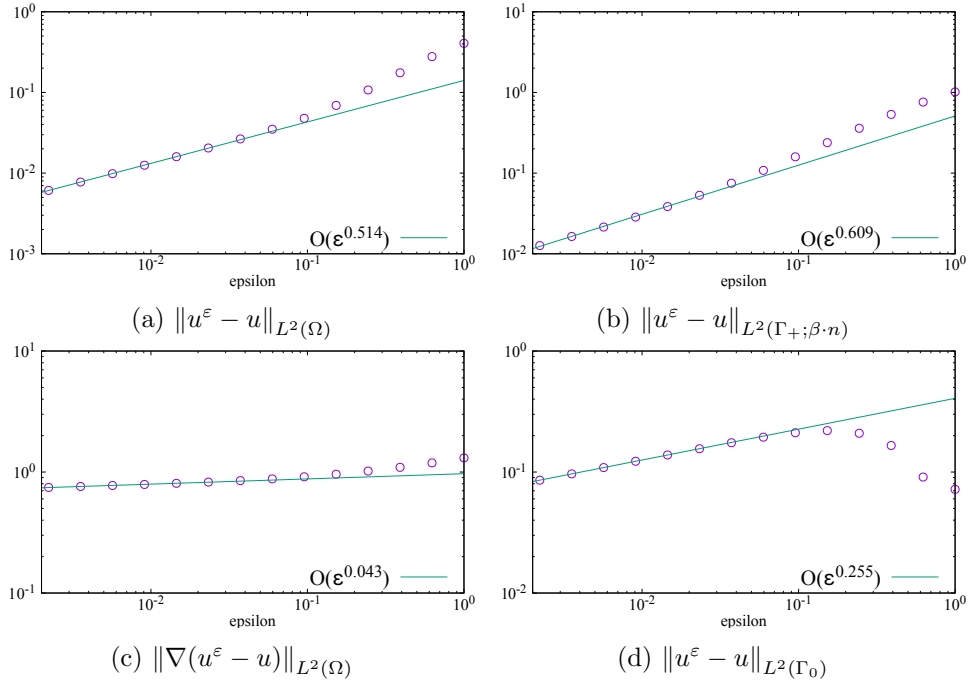


Figure 4.1: Numerical results for Example 1 with $s = 0.51$: $u \in H_{\Gamma_-}^1(\Omega) \setminus H^2(\Omega)$. The circles are numerical results and the solid lines are given by the least square method.

If we take $s \in (0, 1/2)$, then u no longer belongs to $H_{\Gamma_-}^1(\Omega)$. The L^2 error estimation of $u^\varepsilon - u$ for $s = 0.3$ are depicted in Figure 4.2. These graphs strongly suggests that the convergence rate becomes worse if u does not have H^1 regularity. From Figure 4.2a and 4.2b, the convergence rate of $\|u^\varepsilon - u\|_{L^2(\Omega)} + \|u^\varepsilon - u\|_{L^2(\Gamma_+; \beta \cdot n)}$ is significantly less than 0.5, which indicates that H^1 regularity is essential to achieve convergence rate $\varepsilon^{0.5}$.

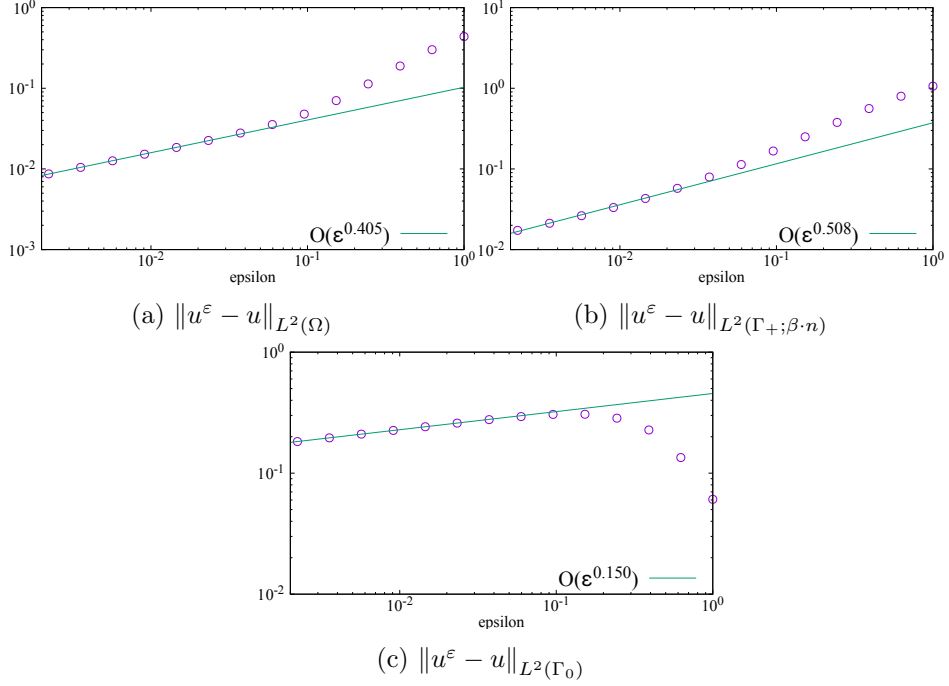


Figure 4.2: Numerical results for Example 1 with $s = 0.3$: $u \in H_{\Gamma_-}^1(\Omega)$. The convergence rates are worse than those shown Theorem 3.3

4.2 Example 2: corresponding to Theorem 3.7

We proved in Theorem 3.7 and Corollary 3.8 that, if $u \in H_{\beta,-}(\Omega) \cap H^2(\Omega)$ solves (2.11), (2.13) and if the surface measure of $\Gamma_0 > 0$, then it holds that

$$\begin{aligned} \|u^\varepsilon - u\|_{L^2(\Omega)} + \|u^\varepsilon - u\|_{L^2(\Gamma_{+;\beta \cdot n})} &\leq C\varepsilon^{\frac{3}{4}}, \\ \|\nabla(u^\varepsilon - u)\|_{L^2(\Omega)} &\leq C\varepsilon^{\frac{1}{4}}, \\ \|u^\varepsilon - u\|_{L^2(\Gamma_0)} &\leq C\varepsilon^{\frac{1}{2}}. \end{aligned}$$

For

$$\beta(x_1, x_2) = (1, 0), \quad \mu(x_1, x_2) = 1, \quad f(x_1, x_2) = x_1x_2 + x_2,$$

the exact solution u to (2.1)-(2.2) is

$$u(x_1, x_2) = x_1x_2,$$

which belongs to $H_{\beta,-}(\Omega) \cap H^2(\Omega)$. Figure 4.3 illustrates numerical results, and convergence rates agree well with the theoretical results:

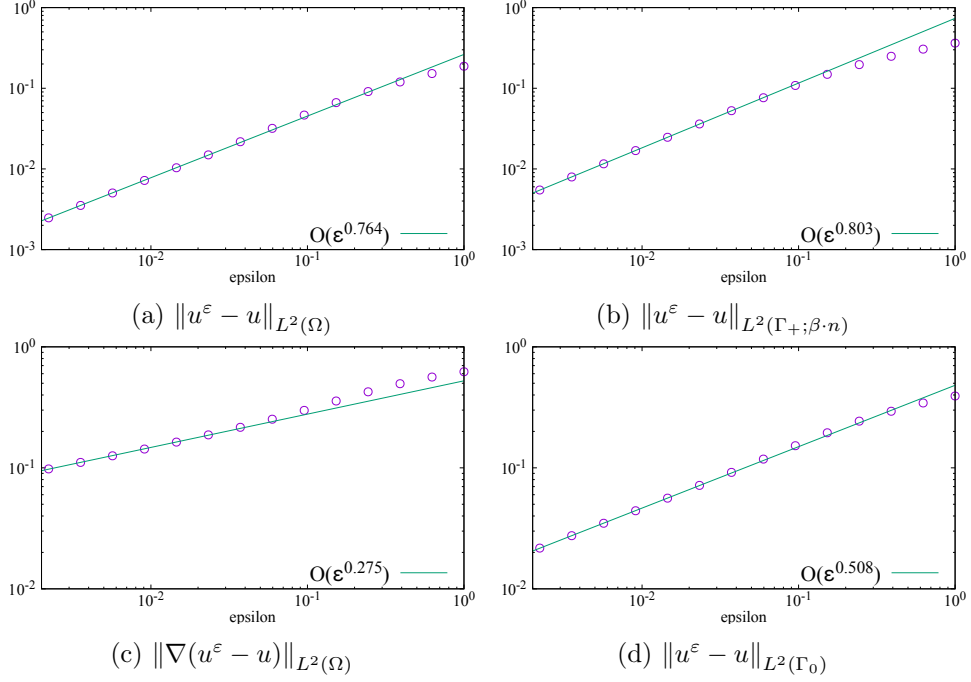


Figure 4.3: Numerical results for Example 2: $u \in H_{\beta,-}(\Omega) \cap H^2(\Omega)$ and the surface measure of Γ_0 is positive

4.3 Example 3: corresponding to Theorem 3.10

By Theorem 3.10, in addition to the hypothesis of Example 2, if the surface measure of Γ_0 is 0 and $\beta \cdot n$ is uniformly positive on Γ_+ , then we have

$$\begin{aligned} \|u^\varepsilon - u\|_{L^2(\Omega)} + \|u^\varepsilon - u\|_{L^2(\Gamma_+; \beta \cdot n)} &\leq C\varepsilon, \\ \|\nabla(u^\varepsilon - u)\|_{L^2(\Omega)} &\leq C\varepsilon^{\frac{1}{2}}. \end{aligned}$$

In this example, we take

$$\beta(x_1, x_2) = (1, 1), \quad \mu(x_1, x_2) = 1, \quad f(x_1, x_2) = (4x_2 + 1) \sin(4x_1) + \cos(4x_1),$$

which satisfy (2.3) and $f \in L^2(\Omega)$. The exact solution to (2.1)-(2.2) is

$$u(x_1, x_2) = x_2 \sin(4x_1),$$

which belongs to $H_{\beta,-}(\Omega) \cap H^2(\Omega)$. Figure 4.4 shows that the rates of convergence of L^2 error estimates is approximately equal to 1, which is expected by the Theorem 3.10

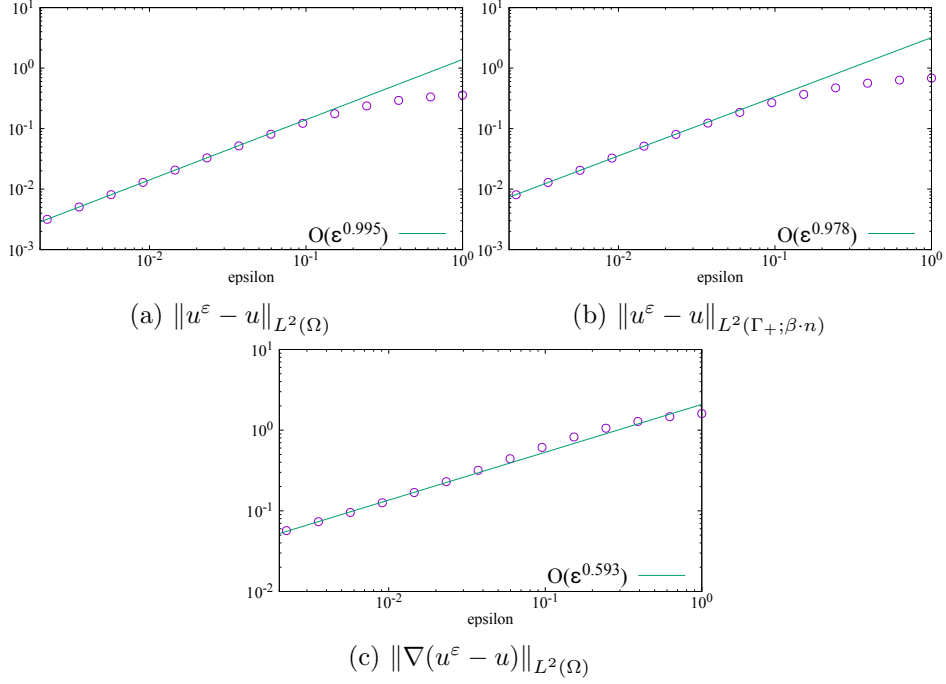


Figure 4.4: Numerical results for Example 3: $u \in H_{\beta,-}(\Omega) \cap H^2(\Omega)$, the surface measure of Γ_0 vanishes, and $\beta \cdot n$ is uniformly positive on Γ_+ .

4.4 Example 4: corresponding to Theorem 3.11

Finally we investigate the influence of degeneracy to the convergence rate. Theorem 3.11 says that if $u \in H_{\beta,-}(\Omega) \cap H^2(\Omega)$ solves (2.1)-(2.2) and the surface measure of Γ_0 vanishes, then the convergence rate depends on the maximum exponent α satisfying $(\beta \cdot n)^{-\alpha} \in L^1(\Gamma_+)$. In order to check this, we consider the following family of vector fields parameterized by $s > 0$:

$$\beta_s = (1 - x_1 + (1 - x_2)^s, 1 + x_2).$$

For each β_s , $\Gamma_+ = \Gamma_t \cup \Gamma_r$, where $\Gamma_t = (0, 1) \times \{1\}$ and $\Gamma_r = \{1\} \times (0, 1)$. On Γ_t and Γ_r , we have

$$\begin{aligned} \int_{\Gamma_t} (\beta \cdot n)^{-\alpha} d\sigma_x &= \int_0^1 (\beta_s(x_2, 1) \cdot n(x_1, 1))^{-\alpha} dx_1 = 2^{-\alpha}, \\ \int_{\Gamma_r} (\beta \cdot n)^{-\alpha} d\sigma_x &= \int_0^1 (\beta_s(1, x_2) \cdot n(1, x_2))^{-\alpha} dx_2 = \int_0^1 (1 - x_2)^{-s\alpha} dx_2, \end{aligned}$$

respectively. So $(n \cdot \beta_s)^{-\alpha} \in L^1(\Gamma_+)$ if and only if $s\alpha < 1$. For $s > 0$, we take

$$f_s(x_1, x_2) = [\{2 - x_1 + (1 - x_2)^s\}e^{x_1} + 1] \sin(x_2) + (1 + x_2)(e^{x_1} - 1) \cos(x_2),$$

$$\mu = 1,$$

then the exact solution is

$$u(x_1, x_2) = (e^{x_1} - 1) \sin(x_2).$$

We numerically compute the convergence rate of $\|u^\varepsilon - u\|_{L^2(\Omega)}$ for f_s which is exhibited in Figure 4.5 where the relation between the convergence rate and $\alpha = 1/s$ is shown. In the figure, the solid line is $(3 + \alpha)/4$, which is the expected convergence rate from our theoretical result. The convergence rate seems to be worse than expected when α is close to 0, and one possible reason for this is the effect of numerical errors as s becomes large.

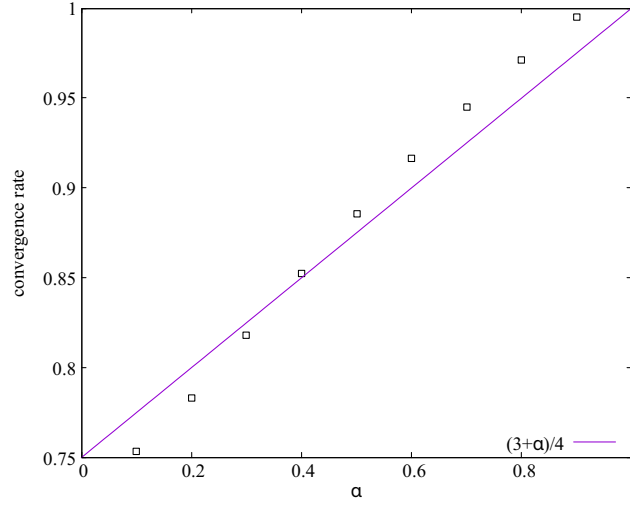


Figure 4.5: Convergence rate of $\|u^\varepsilon - u\|_{L^2(\Omega)}$.

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