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On the theory of higher rank Euler, Kolyvagin and Stark systems: a research announcement

By

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Abstract

This is an announcement of the joint paper [1] with David Burns and Takamichi Sano. In the present paper, we report on the recent progress on the theory of higher rank Euler, Kolyvagin and Stark systems. We modify the definitions of the higher rank Euler, Kolyvagin and Stark systems by using the exterior bi-dual, and improve the theory of higher rank Kolyvagin and Stark systems studied by Mazur and Rubin. In particular, we prove the existence of a canonical higher Kolyvagin derivative homomorphism.

Notations

Let p be an odd prime, and K a number field. For any field L, we fix a separable closure $\overline{L}$ of L and put $G_L := \mathrm{Gal}(\overline{L}/L)$.

Throughout this paper, R denotes a complete noetherian local ring with a finite residue field of characteristic p. Let T denote a free R-module of finite rank with an R-linear continuous G_K-action which is unramified outside a finite set of places of K. Let Σ be the finite set of all primes of K where T is ramified and all primes of K above p. Fix a power $q \in p^{\mathbb{Z}_{>0}}$ of the prime p and then set $A := T/qT$.

For any finite set Σ' of primes of K, we denote by $K_{\Sigma'}$ the maximal extension of K that is unramified outside Σ' and all infinite places of K.

For any integer $i \geq 0$ and Galois extension L'/L, let $H^i(L, -)$ and $H^i(L'/L, -)$ denote the i-th continuous Galois cohomology groups of G_L and $\mathrm{Gal}(L'/L)$, respectively.

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§ 1. Introduction

In this section, we will explain the theory of Euler, Kolyvagin and Stark systems studied by Mazur and Rubin in [5, 6]. We will also explain some issues on higher rank theory of Kolyvagin and Stark systems studied by Mazur and Rubin (see Remark (3) after Theorem 1.2), and introduce an exterior bi-dual in order to solve such issues.

Suppose that $R = \mathbb{Z}_p$. Let r be a non-negative integer. An Euler system of rank r in Perrin-Riou's terminology [7] is an element

$$c = (c_{\mathfrak{n}})_{\mathfrak{n}} \in \prod_{\mathfrak{n}} \bigwedge\nolimits^{r}_{\mathbb{Z}_p[\mathrm{Gal}(K(\mathfrak{n})/K)]} H^1(K(\mathfrak{n}), T)$$

which satisfies "norm relations" (see [9, Definition 2.1.1] or [7, (1.2.3)]). Here

- $\mathfrak{n}$ runs over a certain set of integral ideals of K,
- the field $K(1)$ is the maximal p-subextension of the Hilbert class field of K,
- for any prime $\mathfrak{q}$ of K and positive integer m, the field $K(\mathfrak{q}^m)$ denote the maximal p-subextension of the ray class field modulo $\mathfrak{q}^m$, and
- put $K(\mathfrak{n}) := K(\mathfrak{q}_1^{m_1}) \cdots K(\mathfrak{q}_n^{m_n})$ if $\mathfrak{n} = \mathfrak{q}_1^{m_1} \cdots \mathfrak{q}_n^{m_n}$ and $\mathfrak{q}_i$ is a prime of K.

Let $\mathrm{ES}_r^{\mathrm{PR}}(T)$ denote the module of Euler systems of rank r.

First, we review the Euler system argument introduced by Kolyvagin. Assume that $r = 1$. The Euler system argument can be separated into two steps.

(Step 1) Applying a Kolyvagin derivative operator $D_{\mathfrak{n}} \in \mathbb{Z}_p[\mathrm{Gal}(K(\mathfrak{n})/K)]$ to $c_{\mathfrak{n}}$, we get an element

$$\kappa_{\mathfrak{n}} := D_{\mathfrak{n}} c_{\mathfrak{n}} \bmod q \in H^1(K(\mathfrak{n}), A)^{\mathrm{Gal}(K(\mathfrak{n})/K)} \xleftarrow{\sim} H^1(K, A).$$

(See [9, Lemma 4.4.2].) Here we only consider the ideals $\mathfrak{n}$ for which the restriction map

$$H^1(K, A) \to H^1(K(\mathfrak{n}), A)^{\mathrm{Gal}(K(\mathfrak{n})/K)}$$

is an isomorphism. We see that the element $(\kappa_{\mathfrak{n}})_{\mathfrak{n}} \in \prod_{\mathfrak{n}} H^1(K, A)$ satisfies "finite-singular relations" (see [9, Theorem 4.5.4] and Definition 3.8).

(Step 2) Controlling a (strict) Selmer group $\mathrm{Sel}(A)$ by using finite-singular relations (see [9, Theorem 2.2.2]).

Remark. If we have an Euler system of rank 1 that is related to the special values of L-series, then we get a relation between the Selmer group $\mathrm{Sel}(A)$ and the L-values by the Euler system argument.

Remark. Roughly speaking, in [1], we established a higher rank version of the Kolyvagin's Euler system argument (see Theorems 2.1 and 2.2).

Remark. Let

$$\chi(T) := \sum_{v \mid \infty} \operatorname{rank}_{\mathbb{Z}_p}\bigl(H^0(K_v, T^*(1))\bigr) + \sum_{\mathfrak{p} \mid p} \operatorname{rank}_{\mathbb{Z}_p}\bigl(H^2(K_{\mathfrak{p}}, T)\bigr),$$

where $T^*(1) := \operatorname{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$. We refer to this non-negative integer as the core rank of the G_K-module T (see [5, Theorem 5.2.15]). For example, if T is the p-adic Tate module of an abelian variety A over K, then $\chi(T) = \dim(A) \cdot [K \colon \mathbb{Q}]$.

To get an Euler system which relates to the L-values, in [7], Perrin-Riou proposed that we should need to consider an Euler system of rank $\chi(T)$.

Next, we will review the theory of Kolyvagin and Stark systems introduced by Mazur and Rubin in [5, 6].

The step 2 is complicated. The theory of Kolyvagin systems constructed by Mazur and Rubin in [5] makes arguments in the step 2 clearer. A Kolyvagin system is defined as an element of $\prod_{\mathfrak{n}} H^1(K, A)$ which satisfies finite-singular relations. (In § 3.3, we will give the precise definition of a Kolyvagin system.) We denote by $\mathrm{KS}_1^{\mathrm{MR}}(A)$ the module of Kolyvagin systems. Note that $\mathrm{KS}_1^{\mathrm{MR}}(A) \subseteq \prod_{\mathfrak{n}} H^1(K, A)$. The theory of Kolyvagin systems is a refinement of the step 2. In [5], Mazur and Rubin also proved the following theorem which corresponds to the step 1.

Theorem 1.1 ([5, Theorem 3.2.4])**.** *There is a canonical 'Kolyvagin derivative' homomorphism*

$$\mathcal{D}_1 \colon \mathrm{ES}_1^{\mathrm{PR}}(T) \to \mathrm{KS}_1^{\mathrm{MR}}(A).$$

Remark. In Theorem 1.1, we do not need to assume that $\chi(T) = 1$.

In [6], Mazur and Rubin also constructed the theories of higher rank Kolyvagin and Stark systems. The module $\mathrm{KS}_r^{\mathrm{MR}}(A)$ of Kolyvagin systems of rank r is defined as an R-submodule of $\prod_{\mathfrak{n}} \bigwedge^r H^1(K_{\Sigma_{\mathfrak{n}}}/K, A)$ and the module $\mathrm{SS}_r^{\mathrm{MR}}(A)$ of Stark systems of rank r is defined as an R-submodule of $\prod_{\mathfrak{n}} \bigwedge^{r+\nu(\mathfrak{n})} H^1(K_{\Sigma_{\mathfrak{n}}}/K, A)$. (In § 3.3, we will give the precise definition of a Kolyvagin and Stark system.) Here $\Sigma_{\mathfrak{n}} := \Sigma \cup \{\mathfrak{q} \mid \mathfrak{n}\}$ and $\nu(\mathfrak{n})$ denotes the number of prime factors of $\mathfrak{n}$. In [6], Mazur and Rubin proved the following

Theorem 1.2 ([6, Theorems 6.7, 8.5, and 12.4])**.** *Let $r = \chi(T)$ denote the core rank of T. Under certain assumptions (see Theorem 2.1), we have the following:*

(1) The $\mathbb{Z}_p/q\mathbb{Z}_p$-module $\mathrm{SS}_r^{\mathrm{MR}}(A)$ is free of rank one.

(2) *A basis of* $\mathrm{SS}_r^{\mathrm{MR}}(A)$ *determines the* $\mathbb{Z}_p/q\mathbb{Z}_p$*-module structure of* $\mathrm{Sel}(A)$.

(3) *There is a canonical injective* $\mathbb{Z}_p/q\mathbb{Z}_p$*-homomorphism*

$$\mathrm{Reg}_r^{\mathrm{MR}} \colon \mathrm{SS}_r^{\mathrm{MR}}(A) \hookrightarrow \mathrm{KS}_r^{\mathrm{MR}}(A).$$

Remark.

(1) In [6], Mazur and Rubin expected that there exists a canonical homomorphism

$$\mathrm{ES}_r^{\mathrm{PR}}(T) \to \mathrm{KS}_r^{\mathrm{MR}}(A).$$

Under auxiliary strong assumptions, in [4], Büyükboduk constructed a non-canonical homomorphism from $\mathrm{ES}_r^{\mathrm{PR}}(T)$ to the module of $\mathbb{L}$-restricted Kolyvagin systems (of rank one) and proved that these systems control the module $\mathrm{Sel}(A)$.

(2) In [5, Theorem 4.4.1], Mazur and Rubin proved that the map

$$\mathrm{Reg}_1^{\mathrm{MR}} \colon \mathrm{SS}_1^{\mathrm{MR}}(A) \hookrightarrow \mathrm{KS}_1^{\mathrm{MR}}(A)$$

is an isomorphism (see [6, Remark 11.5 and Theorem 12.4]).

(3) In [6, Remark 11.9], Mazur and Rubin observed that the map $\mathrm{Reg}_r^{\mathrm{MR}}$ is not surjective when $r > 1$ and

$$\dim_{\mathbb{F}_p}(\mathrm{Sel}(A) \otimes \mathbb{F}_p) - \dim_{\mathbb{F}_p}(q' \cdot \mathrm{Sel}(A)) \geq r.$$

Here $q' := q/p \in p^{\mathbb{Z}_{\geq 0}}$. Only higher Kolyvagin systems in $\mathrm{im}\left(\mathrm{Reg}_r^{\mathrm{MR}}\right)$ control the module $\mathrm{Sel}(A)$ while Mazur and Rubin expected higher rank Euler systems to give rise to higher rank Kolyvagin (rather than Stark) systems. Hence the non-surjectivity of the map $\mathrm{Reg}_r^{\mathrm{MR}}$ was a key problem and we would need to understand when such higher rank Kolyvagin systems came from higher rank Stark systems (see Theorem 2.2 (1)).

In the rank one case, Mazur-Rubin's Kolyvagin and Stark system theories have been very successful. However, as remarked above, the higher rank versions of these theories have some problems. To overcome the problems, we will consider where an element which relates to the L-values exists.

In order to consider this, we introduce the notion of an exterior bi-dual which plays a crucial role in this paper. Let r be a non-negative integer. For any (commutative) ring S and S-module M, we denote the S-dual of M by $M^* := \mathrm{Hom}_S(M, S)$. We define an r-th exterior bi-dual ${\bigcap}_S^r M$ of M to be

$${\bigcap}_S^r M := \left({\bigwedge}_S^r (M^*)\right)^* = \mathrm{Hom}_S\left({\bigwedge}_S^r (M^*), S\right).$$

We usually regard an element of ${\bigcap}_S^r M$ as an S-homomorphism ${\bigwedge}_S^r (M^*) \to S$.

Remark. There is a natural homomorphism of S-modules

$$\xi_M^r \colon \bigwedge\nolimits_S^r M \to \bigcap\nolimits_S^r M \ ; \ m_1 \wedge \cdots \wedge m_r \mapsto (\phi_1 \wedge \cdots \wedge \phi_r \mapsto \det(\phi_i(m_j))_{i,j \le r}) \, .$$

Note that the map ξ_M^r is, in general, neither injective nor surjective.

Remark. There is a canonical isomorphism

$$\left\{ x \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge\nolimits_{\mathbb{Z}_p[G]}^r M \mid \Phi(x) \in \mathbb{Z}_p[G] \text{ for any } \Phi \in \bigwedge\nolimits_{\mathbb{Z}_p[G]}^r (M^*) \right\} \xrightarrow{\sim} \bigcap\nolimits_{\mathbb{Z}_p[G]}^r M$$

for any finite abelian group G and finitely generated $\mathbb{Z}_p[G]$-module M. The left hand side is the usual definition of the Rubin's lattice. This fact tells us that the notion of exterior bi-dual is a generalization of the notion of Rubin's lattice.

Rubin conjectured that there is an element which relates to the values at zero of the r-th derivative of Dedekind zeta functions in the lattice

$$\bigcap\nolimits_{\mathbb{Z}_p[\mathrm{Gal}(K(\mathfrak{n})/K)]}^r H^1(K_{\Sigma_\mathfrak{n}}/K(\mathfrak{n}), \mathbb{Z}_p(1))$$

of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r H^1(K_{\Sigma_\mathfrak{n}}/K(\mathfrak{n}), \mathbb{Z}_p(1))$ (see [8, Conjecture B']). This conjecture is known as the Rubin-Stark Conjecture, and is proved when $K = \mathbb{Q}$.

For general T, in [2], Burns and Sano proved that, under the equivariant Tamagawa number conjecture, the canonical "zeta element" naturally gives rise to an element of the lattice $\bigcap^r_{\mathbb{Z}_p[\mathrm{Gal}(K(\mathfrak{n})/K)]} H^1(K_{\Sigma_\mathfrak{n}}/K(\mathfrak{n}), T)$.

For the above reasons, we should use $\bigcap^r$ instead of $\bigwedge^r$ to define higher rank Euler, Kolyvagin and Stark systems.

From now on, we change the definition of higher rank Euler systems to the following

Definition 1.3. We say that $c = (c_\mathfrak{n})_\mathfrak{n}$ is an Euler system of rank r if

$$c_\mathfrak{n} \in \bigcap\nolimits_{\mathbb{Z}_p[\mathrm{Gal}(K(\mathfrak{n})/K)]}^r H^1(K(\mathfrak{n}), T)$$

and c satisfies "norm relations". We denote by $\mathrm{ES}_r(T)$ the module of Euler systems of rank r. Note that the natural transformations $\xi_{(-)}^r \colon \bigwedge^r(-) \to \bigcap^r(-)$ induce a homomorphism

$$\mathrm{ES}_r^{\mathrm{PR}}(T) \to \mathrm{ES}_r(T)$$

and the kernel of this homomorphism is $\bigcup_n \ker \left(\mathrm{ES}_r^{\mathrm{PR}}(T) \xrightarrow{p^n \times} \mathrm{ES}_r^{\mathrm{PR}}(T) \right)$. Hence, our definition of higher rank Euler systems is a generalization of Perrin-Riou's one since the torsion part of the Euler systems does not contribute to control the Selmer groups.

Remark. In [2], Burns and Sano constructed an Euler system of rank $\chi(T)$ in the sense of Definition 1.3 algebraically and proved that it relates to the L-values if we assume the equivariant Tamagawa number conjecture.

§ 2. Main Results

In this section, we explain the main results of the paper [1]. Let $r = \chi(T)$ denote the core rank of T. The following theorem is proved by Burns and Sano in [2, Theorems 3.11 and 3.19] and by the author in [10, Theorems 4.7 and 4.10], independently.

Theorem 2.1. *Assume that R/qR is a 0-dimensional Gorenstein local ring. We defined the module*

$$\mathrm{SS}_r(A) \subseteq \prod_{\mathfrak{n}} \bigcap\nolimits_{R/qR}^{r+\nu(\mathfrak{n})} H^1(K_{\Sigma_{\mathfrak{n}}}/K, A)$$

of Stark systems of rank r (see Definition 3.6). Under Hypothesis 3.9 below (which are precisely the same kind of the assumptions in Theorem 1.2), we have the following:

(1) The R/qR-module $\mathrm{SS}_r(A)$ is free of rank one.

(2) Let $\epsilon = (\epsilon_{\mathfrak{n}})_{\mathfrak{n}} \in \mathrm{SS}_r(A)$ be a basis and i a non-negative integer. Then we have an equality

$$\mathrm{Fitt}^i_{R/qR}(\mathrm{Sel}(A)^*) = \sum_{\nu(\mathfrak{n})=i} \mathrm{im}(\epsilon_{\mathfrak{n}}).$$

Here we regard $\epsilon_{\mathfrak{n}} \in \bigcap_{R/qR}^{r+\nu(\mathfrak{n})} H^1(K_{\Sigma_{\mathfrak{n}}}/K, A)$ as an R/qR-homomorphism

$$\epsilon_{\mathfrak{n}} \colon \bigwedge\nolimits_{R/qR}^{r+\nu(\mathfrak{n})} (H^1(K_{\Sigma_{\mathfrak{n}}}/K, A)^*) \to R/qR.$$

Remark.

(1) If R/qR is a 0-dimensional principal ideal local ring, then the natural transformations $\xi_{(-)}^{r+\nu(\mathfrak{n})} \colon \bigwedge_{R/qR}^{r+\nu(\mathfrak{n})}(-) \to \bigcap_{R/qR}^{r+\nu(\mathfrak{n})}(-)$ induce an isomorphism

$$\mathrm{SS}_r^{\mathrm{MR}}(A) \xrightarrow{\sim} \mathrm{SS}_r(A).$$

Hence Theorem 2.1 is a generalization of Theorem 1.2 to a 0-dimensional Gorenstein local ring.

(2) Assume that R/qR is a 0-dimensional Gorenstein local ring. In [2], Burns and Sano defined a Kolyvagin system of rank r as an element of $\prod_{\mathfrak{n}} \bigcap_{R/qR}^{r} H^1(K_{\Sigma_{\mathfrak{n}}}/K, A)$ which satisfies finite-singular relations (see Definition 3.8).

(3) In Definitions 3.6 and 3.8, we will give a precise definitions of the modules $\mathrm{SS}_r(A)$ and $\mathrm{KS}_r(A)$.

Let $\mathcal{P}$ denote the set of all primes $\mathfrak{q}$ of K not contained in Σ such that $\mathfrak{q}$ splits completely in $K_q := \mathcal{H}((\mathcal{O}_K^\times)^{1/q})$, and $A/(\mathrm{Fr}_{\mathfrak{q}} - 1)A \simeq R$ as R-modules. Here

- $\mathcal{H}$ is the Hilbert class field of K,
- $\mathrm{Fr}_{\mathfrak{q}}$ denotes the Frobenius element at $\mathfrak{q}$,
- $\mathcal{O}_K$ denotes the ring of integers in K,
- $(\mathcal{O}_K^\times)^{1/q} := \{x \in \overline{K} \mid x^q \in \mathcal{O}_K^\times\}$.

In the paper [1], we proved the following

Theorem 2.2 ([1]).

(1) Suppose that R/qR is a 0-dimensional Gorenstein local ring, and Hypotheses 3.9 below. Then there is a natural isomorphism

$$\mathrm{Reg}_r \colon \mathrm{SS}_r(A) \xrightarrow{\sim} \mathrm{KS}_r(A).$$

(2) Assume that R is finite flat over $\mathbb{Z}_p$ and Gorenstein. Let $\Bbbk$ denote the residue field of R. Consider the following conditions:

(i) $(A \otimes_R \Bbbk)^{G_K}$ vanishes,

(ii) $\mathrm{Fr}_{\mathfrak{q}}^{p^k} - 1$ is injective on T for every $k \geq 0$ and prime $\mathfrak{q}$ in $\mathcal{P}$,

(iii) $A \otimes_R \Bbbk$ is an irreducible $\Bbbk[G_K]$-module,

(iv) there is an element $\tau \in G_{K_q}$ such that $A/(\tau - 1)A \simeq R$,

(v) $H^1(K_q(A)/K, A) = H^1(K_q(A)/K, A^*(1)) = 0$,

(vi) $p > 3$,

(vii) Hypothesis 3.9 (1), which is introduced in § 3.3, is satisfied,

(viii) $H^0(K_{\mathfrak{q}}, A^*(1))$ vanishes for each prime $\mathfrak{q} \in \Sigma$,

(ix) $\bigoplus_{v\mid\infty} H^0(K_v, T^*(1))$ is a free R-module of rank $r > 0$.

Here $A^*(1) := A^* \otimes \mu_q$, and $K_q(A)$ denotes the minimal Galois extension of K_q such that $G_{K_q(A)}$ acts trivially on A.

Under the conditions (i) and (ii), there exists a canonical 'higher Kolyvagin derivative' homomorphism

$$\mathcal{D}_r \colon \mathrm{ES}_r(T) \to \mathrm{KS}_r(A).$$

(When $r = 1$ and R is a principal ideal local ring, the higher Kolyvagin derivative homomorphism coincides with the homomorphism that occurs in Theorem 1.1.) Furthermore, if the conditions (i)–(ix) are satisfied, then the homomorphism $\mathcal{D}_r$ is surjective.

(3) Suppose that R is finite flat over $\mathbb{Z}_p$ and Gorenstein. Under the conditions (i)–(ix), Euler systems of rank r determine all higher Fitting ideals of $\mathrm{Sel}(A)^*$.

Remark.

(1) In this paper, we only give the proof of Theorem 2.2 (1) in § 3.4.

(2) The higher Kolyvagin derivative homomorphism

$$\mathcal{D}_r \colon \mathrm{ES}_r(T) \to \prod_{\mathfrak{n}} \bigcap\nolimits_{R/qR}^{r} H^1(K, A)$$

is defined by Burns and Sano (see [2, Definition 4.13]). However, it was not known the inclusion

$$\mathrm{im}(\mathcal{D}_r) \subseteq \mathrm{KS}_r(A).$$

To prove the inclusion $\mathrm{im}(\mathcal{D}_r) \subseteq \mathrm{KS}_r(A)$, in [1], we developed a "rank-reduction" method introduced by Rubin and Perrin-Riou (see [8, § 6] and [7, § 1]). We can reduce to the rank one case by using the "rank-reduction" method and, in this case, the inclusion is already known by Theorem 1.1. The surjectivity of the map $\mathcal{D}_r$ follows from [2, Theorems 2.26 and 3.11] and Theorem 2.2 (1).

(3) Theorem 2.2 (3) follows from Theorems 2.1, 2.2 (1), and 2.2 (2).

As an immediate application of Theorems 2.1 and 2.2, one can show the following

Corollary 2.3. *Let $\chi \colon G_K \to \overline{\mathbb{Q}}_p^{\times}$ be a non-trivial finite prime-to-p order character and L the field corresponds to* $\ker(\chi)$. *Assume that every real place of K splits completely in L, no primes above p split completely in L/K, the character χ is not equal to the Teichmüllar character, and the Rubin-Stark conjecture holds (see [8, Conjecture B']). Then we have*

$$\mathrm{im}(\epsilon_{L/K}^{\chi}) \subseteq \mathrm{Fitt}^0(A_L^{\chi}).$$

Here $\epsilon_{L/K}^{\chi} \in \bigcap^r H^1(L, \mathbb{Z}_p(1))^{\chi}$ is the (p,χ)-part of the Rubin-Stark element and A_L^{χ} is the (p,χ)-part of the ideal class group of L.

Remark. If K and L are totally real fields, then, in [3], Büyükboduk proved Corollary 2.3 under the Leopoldt's conjecture for L.

§ 3. Higher rank Kolyvagin and Stark systems

In this section, we will give definitions of Kolyvagin and Stark systems and prove Theorem 2.2 (1).

Throughout this section, we assume that R is a 0-dimensional Gorenstein local ring and $q = 0$ in R. Then the functor $(-)^* = \mathrm{Hom}_R(-, R)$ is exact by the definition of Gorenstein ring. Let r be a positive integer and let $\mathfrak{q}$ denote a prime of K.

§ 3.1. Algebraic preliminaries

Let X be an R-module. Let s and t denote non-negative integers with $s \leq t$. For any element $\Phi \in \bigwedge_R^s(X^*)$, we have an R-homomorphism

$$\bigwedge\nolimits_R^{t-s}(X^*) \to \bigwedge\nolimits_R^{t}(X^*) \ ; \ \Psi \mapsto \Phi \wedge \Psi.$$

By taking the R-dual $(-)^*$, we get an R-homomorphism (also denoted by Φ)

$$\Phi \colon \bigcap\nolimits_R^t X \to \bigcap\nolimits_R^{t-s} X.$$

The following lemma is easy to prove by using the exactness of the functor $(-)^*$.

Lemma 3.1. *Let X be an R-module. Let s and t denote non-negative integers with $s \leq t$.*

(1) Let Y be an R-submodule of X. Then the R-homomorphism

$$\bigcap\nolimits_R^t Y \to \bigcap\nolimits_R^t X$$

induced by the inclusion map $Y \hookrightarrow X$ is injective, and so we regard $\bigcap_R^t Y$ as an R-submodule of $\bigcap_R^t X$.

(2) Let $\phi_1, \ldots, \phi_s \in X^*$. Put $\Phi := \phi_1 \wedge \cdots \wedge \phi_s \in \bigwedge_R^s(X^*)$ and $Y := \bigcap_{i=1}^s \ker(\phi_i)$. Then we have

$$\mathrm{im}\left(\Phi \colon \bigcap\nolimits_R^t X \to \bigcap\nolimits_R^{t-s} X\right) \subseteq \bigcap\nolimits_R^{t-s} Y,$$

and so we get an R-homomorphism (also denoted by Φ)

$$\Phi \colon \bigcap\nolimits_R^t X \to \bigcap\nolimits_R^{t-s} Y.$$

§ 3.2. Selmer structures

In this subsection, we will recall some basic facts of local Galois cohomology groups in [5, § 1] and the definition of a Selmer structure on A.

Recall that $\mathcal{P}$ is the set of all primes $\mathfrak{q}$ of K not contained in Σ such that $\mathfrak{q}$ splits completely in $\mathcal{H}((\mathcal{O}_K^\times)^{1/q})$, and $A/(\mathrm{Fr}_\mathfrak{q} - 1)A \simeq R$ as R-modules. Here $\mathcal{H}$ denotes the Hilbert class field of K. Let $\mathcal{N}$ denote the set of all square-free products of primes in $\mathcal{P}$.

For any prime $\mathfrak{q} \in \mathcal{P}$, let

$$H^1_f(K_\mathfrak{q}, A) := \ker\left(H^1(K_\mathfrak{q}, A) \to H^1(K_\mathfrak{q}^{\mathrm{unr}}, A)\right),$$
$$H^1_{\mathrm{tr}}(K_\mathfrak{q}, A) := \ker\left(H^1(K_\mathfrak{q}, A) \to H^1(K(\mathfrak{q})_\mathfrak{q}, A)\right),$$

where $K_\mathfrak{q}^{\mathrm{unr}}$ is the maximal unramified extension of $K_\mathfrak{q}$, and $K(\mathfrak{q})_\mathfrak{q}$ is the completion of $K(\mathfrak{q})$ at a prime above $\mathfrak{q}$. For any prime $\mathfrak{q} \in \mathcal{P}$, let $G_\mathfrak{q} := \mathrm{Gal}(K(\mathfrak{q})_\mathfrak{q}/K_\mathfrak{q})$. Note that $G_\mathfrak{q}$ is a cyclic p-group with $q \leq |G_\mathfrak{q}|$ by the definition of $\mathcal{P}$.

Lemma 3.2 ([5, Lemmas 1.2.3 and 1.2.4])**.**

(1) For any prime $\mathfrak{q} \in \mathcal{P}$, the R-modules $H^1_f(K_\mathfrak{q}, A)$ and $H^1_{\mathrm{tr}}(K_\mathfrak{q}, A)$ are free of rank one.

(2) For any prime $\mathfrak{q} \in \mathcal{P}$, we have

$$H^1(K_\mathfrak{q}, A) = H^1_f(K_\mathfrak{q}, A) \oplus H^1_{\mathrm{tr}}(K_\mathfrak{q}, A).$$

Let $\mathfrak{q} \in \mathcal{P}$. We denote by

$$\psi_\mathfrak{q} \colon H^1_f(K_\mathfrak{q}, A) \xrightarrow{\sim} H^1_{\mathrm{tr}}(K_\mathfrak{q}, A) \otimes G_\mathfrak{q}$$

the finite-singular comparison isomorphism at $\mathfrak{q}$ defined in [5, Definition 1.2.2 and Lemma 1.2.3]. For simplicity, we fix a generator of $G_\mathfrak{q}$ and an R-basis of $H^1_{\mathrm{tr}}(K_\mathfrak{q}, A)$ for each prime $\mathfrak{q} \in \mathcal{P}$, and so we regard $H^1_{\mathrm{tr}}(K_\mathfrak{q}, A)$ and $G_\mathfrak{q}$ as R and $\mathbb{Z}/|G_\mathfrak{q}|\mathbb{Z}$, respectively. Then we get an identification $H^1_{\mathrm{tr}}(K_\mathfrak{q}, A) \otimes G_\mathfrak{q} = R$ since $q \leq |G_\mathfrak{q}|$.

Definition 3.3. A Selmer structure $\mathcal{F}$ on A is a collection of the following data:

- a finite set $\Sigma(\mathcal{F})$ of primes of K containing Σ;
- a choice of an R-submodule $H^1_\mathcal{F}(K_\mathfrak{q}, A)$ of $H^1(K_\mathfrak{q}, A)$ for each prime $\mathfrak{q} \in \Sigma(\mathcal{F})$.

We put $H^1_\mathcal{F}(K_\mathfrak{q}, A) := H^1_f(K_\mathfrak{q}, A)$ for each prime $\mathfrak{q}$ of K not contained in $\Sigma(\mathcal{F})$. We define a Selmer module $H^1_\mathcal{F}(K, A)$ to be the kernel

$$H^1_\mathcal{F}(K, A) := \ker\left(H^1(K_{\Sigma(\mathcal{F})}/K, A) \to \bigoplus_{\mathfrak{q} \in \Sigma(\mathcal{F})} H^1(K_\mathfrak{q}, A)/H^1_\mathcal{F}(K_\mathfrak{q}, A)\right).$$

Remark. Let $\mathcal{F}$ be a Selmer structure on A and $\Sigma(\mathcal{F}^*) := \Sigma(\mathcal{F})$. For any prime $\mathfrak{q} \in \Sigma(\mathcal{F}^*)$, we put

$$H^1_{\mathcal{F}^*}(K_\mathfrak{q}, A^*(1)) := \ker\left(H^1(K_\mathfrak{q}, A^*(1)) \simeq H^1(K_\mathfrak{q}, A)^* \to H^1_\mathcal{F}(K_\mathfrak{q}, A)^*\right),$$

where the isomorphism is defined by the local Tate pairing. In this manner, the Selmer structure $\mathcal{F}$ on A gives rise to a Selmer structure $\mathcal{F}^*$ on $A^*(1)$.

Remark. The notation $\mathrm{Sel}(A)$ used in §§ 1 and 2 means the Selmer module $H^1_{(\mathcal{F}_{\mathrm{can}})^*}(K, A^*(1))$. Here $\mathcal{F}_{\mathrm{can}}$ is the canonical Selmer structure defined in [5, Definition 3.2.1].

By using the Poitou-Tate global duality, we deduce the following theorem (proved by Mazur and Rubin in [5, Theorem 2.3.4]). In this paper, we refer to the following theorem as the global duality.

Theorem 3.4 (Global duality)**.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Selmer structures on A. If $H^1_{\mathcal{F}_1}(K_{\mathfrak{q}}, A) \subseteq H^1_{\mathcal{F}_2}(K_{\mathfrak{q}}, A)$ for any prime $\mathfrak{q}$, then there exists the following exact sequence of R-modules:*

$$\begin{aligned} 0 \to H^1_{\mathcal{F}_1}(K, A) \to H^1_{\mathcal{F}_2}(K, A) \to & \bigoplus_{\mathfrak{q}} H^1_{\mathcal{F}_2}(K_{\mathfrak{q}}, A)/H^1_{\mathcal{F}_1}(K_{\mathfrak{q}}, A) \\ & \to H^1_{\mathcal{F}_1^*}(K, A^*(1))^* \to H^1_{\mathcal{F}_2^*}(K, A^*(1))^* \to 0, \end{aligned}$$

where $\mathfrak{q}$ runs over all primes of K which satisfies $H^1_{\mathcal{F}_1}(K_{\mathfrak{q}}, A) \neq H^1_{\mathcal{F}_2}(K_{\mathfrak{q}}, A)$.

Definition 3.5. Let $\mathcal{F}$ be a Selmer structure on A, and let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ be pairwise relatively prime integral ideals in $\mathcal{N}$. Define a Selmer structure $\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})$ on A by the following data:

- $\Sigma(\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})) := \Sigma(\mathcal{F}) \cup \{\mathfrak{q} \mid \mathfrak{abc}\}$,
- $H^1_{\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c})}(K_{\mathfrak{q}}, A) := \begin{cases} H^1(K_{\mathfrak{q}}, A) & \text{if } \mathfrak{q} \mid \mathfrak{a}, \\ 0 & \text{if } \mathfrak{q} \mid \mathfrak{b}, \\ H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, A) & \text{if } \mathfrak{q} \mid \mathfrak{c}, \\ H^1_{\mathcal{F}}(K_{\mathfrak{q}}, A) & \text{otherwise.} \end{cases}$

Note that $(\mathcal{F}^{\mathfrak{a}}_{\mathfrak{b}}(\mathfrak{c}))^* = (\mathcal{F}^*)^{\mathfrak{b}}_{\mathfrak{a}}(\mathfrak{c})$ by [5, Proposition 1.3.2]. For simplicity, we will write $\mathcal{F}^{\mathfrak{a}}$, $\mathcal{F}(\mathfrak{c})$, and $\mathcal{F}_{\mathfrak{b}}(\mathfrak{c})$ instead of $\mathcal{F}^{\mathfrak{a}}_1(1)$, $\mathcal{F}^1_1(\mathfrak{c})$, and $\mathcal{F}^1_{\mathfrak{b}}(\mathfrak{c})$, respectively.

§ 3.3. Definitions of Kolyvagin and Stark systems

Let $\mathcal{F}$ be a Selmer structure on A. First, we give the definition of Stark systems for $(A, \mathcal{F})$.

Definition 3.6. Let r be a non-negative integer. Let $\mathfrak{n} \in \mathcal{N}$ be any element, and $\mathfrak{q} \in \mathcal{P}$ any prime divisor of $\mathfrak{n}$. By definition, we have an exact sequence

$$0 \to H^1_{\mathcal{F}^{\mathfrak{n}/\mathfrak{q}}}(K, A) \to H^1_{\mathcal{F}^{\mathfrak{n}}}(K, A) \to H^1(K_{\mathfrak{q}}, A)/H^1_f(K_{\mathfrak{q}}, A) \xleftarrow{\sim} H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, A) = R. \tag{3.1}$$

By Lemma 3.1 (2), the exact sequence (3.1) induces an R-homomorphism

$$\bigcap\nolimits_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^{\mathfrak{n}}}(K, A) \to \bigcap\nolimits_R^{r+\nu(\mathfrak{n}/\mathfrak{q})} H^1_{\mathcal{F}^{\mathfrak{n}/\mathfrak{q}}}(K, A),$$

where $\nu(\mathfrak{n})$ is the number of prime divisors of $\mathfrak{n}$. We define the module $\mathrm{SS}_r(A, \mathcal{F})$ of Stark systems of rank r for $(A, \mathcal{F})$ to be the inverse limit

$$\mathrm{SS}_r(A, \mathcal{F}) := \varprojlim_{\mathfrak{n}\in\mathcal{N}} \bigcap\nolimits_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^{\mathfrak{n}}}(K, A).$$

Next, we define a finite-singular comparison map which plays an important role in the definition of a Kolyvagin system.

Definition 3.7. Let $\mathfrak{n} \in \mathcal{N}$ be any element, and $\mathfrak{q} \in \mathcal{P}$ any prime divisor of $\mathfrak{n}$. We denote by

$$v_{\mathfrak{q}} \colon H^1_{\mathcal{F}(\mathfrak{n})}(K, A) \to H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, A) = R$$

the localization map at $\mathfrak{q}$. Then the map $v_{\mathfrak{q}}$ induces an R-homomorphism

$$v_{\mathfrak{q}} \colon \bigcap\nolimits_R^{r} H^1_{\mathcal{F}(\mathfrak{n})}(K, A) \to \bigcap\nolimits_R^{r-1} H^1_{\mathcal{F}_{\mathfrak{q}}(\mathfrak{n}/\mathfrak{q})}(K, A)$$

by Lemma 3.1 (2). Let

$$\varphi^{\mathrm{fs}}_{\mathfrak{q}} \colon H^1_{\mathcal{F}(\mathfrak{n}/\mathfrak{q})}(K, A) \to H^1_f(K_{\mathfrak{q}}, A) \xrightarrow{\psi^{\mathrm{fs}}_{\mathfrak{q}}} H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, A) \otimes G_{\mathfrak{q}} = R$$

denote the composition of the localization map at $\mathfrak{q}$ and the isomorphism $\psi^{\mathrm{fs}}_{\mathfrak{q}}$. We call the map $\varphi^{\mathrm{fs}}_{\mathfrak{q}}$ the finite-singular comparison map at $\mathfrak{q}$.

Note that, by Lemma 3.1 (2), the map $\varphi^{\mathrm{fs}}_{\mathfrak{q}}$ induces an R-homomorphism

$$\varphi^{\mathrm{fs}}_{\mathfrak{q}} \colon \bigcap\nolimits_R^{r} H^1_{\mathcal{F}(\mathfrak{n}/\mathfrak{q})}(K, A) \to \bigcap\nolimits_R^{r-1} H^1_{\mathcal{F}_{\mathfrak{q}}(\mathfrak{n}/\mathfrak{q})}(K, A).$$

By using this homomorphism, we define Kolyvagin systems for $(A, \mathcal{F})$.

Definition 3.8. Let r be a positive integer. We define the module $\mathrm{KS}_r(A, \mathcal{F})$ of Kolyvagin systems of rank r for $(A, \mathcal{F})$ to be the set of all elements

$$(\kappa_{\mathfrak{n}})_{\mathfrak{n}} \in \prod_{\mathfrak{n}\in\mathcal{N}} \bigcap\nolimits_R^{r} H^1_{\mathcal{F}(\mathfrak{n})}(K, A)$$

which satisfies the finite-singular relation

$$v_{\mathfrak{q}}(\kappa_{\mathfrak{n}}) = \varphi^{\mathrm{fs}}_{\mathfrak{q}}(\kappa_{\mathfrak{n}/\mathfrak{q}})$$

for any $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{q} \in \mathcal{P}$ with $\mathfrak{q} \mid \mathfrak{n}$.

Remark.

(1) The notations $\mathrm{KS}_r(A)$ and $\mathrm{SS}_r(A)$ used in § 2 mean the modules $\mathrm{KS}_r(A, \mathcal{F}_{\mathrm{can}})$ and $\mathrm{SS}_r(A, \mathcal{F}_{\mathrm{can}})$, respectively.

(2) Assume that R is a principal ideal local ring. Let s and t be positive integers with $s \le t$. Applying [6, Proposition A.2] with $C_1 = 0$ and $C_2 = R^s$, one can show that the exact sequence of R-modules

$$0 \to N \to M \xrightarrow{x \mapsto (h_i(x))_{i \le s}} R^s$$

induces an R-homomorphism

$$\bigwedge\nolimits_R^t M \to \bigwedge\nolimits_R^{t-s} N$$

such that the diagram

$$\begin{array}{ccc} \bigwedge_R^t M & \longrightarrow & \bigwedge_R^{t-s} N \\ \Big\downarrow \xi_M^t & & \Big\downarrow \xi_N^{t-s} \\ \bigcap_R^t M & \xrightarrow{\bigwedge_{i \le s} h_i} & \bigcap_R^{t-s} N. \end{array}$$

commutes. Hence, in the same way as in Definitions 3.6 and 3.8, we can define the modules

$$\mathrm{KS}_r^{\mathrm{MR}}(A, \mathcal{F}) \subseteq \prod_{\mathfrak{n} \in \mathcal{N}} \bigwedge\nolimits_R^r H^1_{\mathcal{F}(\mathfrak{n})}(K, A),$$

$$\mathrm{SS}_r^{\mathrm{MR}}(A, \mathcal{F}) \subseteq \prod_{\mathfrak{n} \in \mathcal{N}} \bigwedge\nolimits_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^{\mathfrak{n}}}(K, A),$$

for any Selmer structure $\mathcal{F}$ on A and we have natural R-homomorphisms

$$\mathrm{KS}_r^{\mathrm{MR}}(A, \mathcal{F}) \to \mathrm{KS}_r(A, \mathcal{F}),$$

$$\mathrm{SS}_r^{\mathrm{MR}}(A, \mathcal{F}) \to \mathrm{SS}_r(A, \mathcal{F}).$$

The notations $\mathrm{KS}_r^{\mathrm{MR}}(A)$ and $\mathrm{SS}_r^{\mathrm{MR}}(A)$ used in §§ 1 and 2 mean the modules $\mathrm{KS}_r^{\mathrm{MR}}(A, \mathcal{F}_{\mathrm{can}})$ and $\mathrm{SS}_r^{\mathrm{MR}}(A, \mathcal{F}_{\mathrm{can}})$, respectively.

Let $\mathfrak{n} \in \mathcal{N}$ be an ideal and $\mathfrak{q} \in \mathcal{P}$ a prime satisfying $\mathfrak{q} \mid \mathfrak{n}$. We denote the composition of the R-homomorphisms

$$\begin{aligned} H^1_{\mathcal{F}^{\mathfrak{n}}}(K, A) &\to H^1(K_{\mathfrak{q}}, A)/H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, A) \\ &\xleftarrow{\sim} H^1_f(K_{\mathfrak{q}}, A) \\ &\xrightarrow{\psi_{\mathfrak{q}}^{\mathrm{fs}}} H^1_{\mathrm{tr}}(K_{\mathfrak{q}}, A) \otimes G_{\mathfrak{q}} = R \end{aligned}$$

by $r_{\mathfrak{q}}$. Since the sequence

$$0 \to H^1_{\mathcal{F}(\mathfrak{n})}(K, A) \to H^1_{\mathcal{F}^{\mathfrak{n}}}(K, A) \xrightarrow{x \mapsto (r_{\mathfrak{q}}(x))_{\mathfrak{q}}} \bigoplus_{\mathfrak{q} \mid \mathfrak{n}} R$$

is exact, by Lemma 3.1 (2), we obtain an R-homomorphism

$$\bigwedge\nolimits_{\mathfrak{q} \mid \mathfrak{n}} r_{\mathfrak{q}} \colon \bigcap\nolimits_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^{\mathfrak{n}}}(K, A) \to \bigcap\nolimits_R^{r} H^1_{\mathcal{F}(\mathfrak{n})}(K, A).$$

The R-homomorphisms $\bigwedge_{\mathfrak{q} \mid \mathfrak{n}} r_{\mathfrak{q}}$ induce an R-homomorphism

$$\mathrm{Reg}_r \colon \mathrm{SS}_r(A, \mathcal{F}) \to \mathrm{KS}_r(A, \mathcal{F}).$$

Hypothesis 3.9. *Let $\mathcal{F}$ be a Selmer structure on A, and r a positive integer.*

(1) There is an ideal $\mathfrak{d} \in \mathcal{N}$ such that $H^1_{\mathcal{F}(\mathfrak{d})}(K, A) \simeq R^r$ and $H^1_{\mathcal{F}^*(\mathfrak{d})}(K, A^*(1))$ vanishes.

(2) For any non-zero elements $c_1, c_2 \in H^1(K, A)$ and $c_3, c_4 \in H^1(K, A^*(1))$, there are infinitely many primes $\mathfrak{q} \in \mathcal{P}$ such that, for any $1 \leq i \leq 4$, the image of c_i under the localization map at $\mathfrak{q}$ is non-zero.

Remark. The conditions (iii)–(vi) in Theorem 2.2 imply Hypothesis 3.9 (2) (see [5, Lemma 3.6.1]).

Remark. Assume Hypothesis 3.9 (1). Let $\mathfrak{n} \in \mathcal{N}$. Then, by using the global duality, one can show that $H^1_{\mathcal{F}(\mathfrak{n})}(K, A) \simeq R^r$ if $H^1_{\mathcal{F}^*(\mathfrak{n})}(K, A^*(1))$ vanishes and that $H^1_{\mathcal{F}^{\mathfrak{n}}}(K, A) \simeq R^{r+\nu(\mathfrak{n})}$ if $H^1_{(\mathcal{F}^{\mathfrak{n}})^*}(K, A^*(1))$ vanishes.

Theorem 2.2 (1) is restated as follows.

Theorem 3.10 (Theorem 2.2 (1))**.** *Let $\mathcal{F}$ be a Selmer structure on A. Suppose Hypothesis 3.9. Let r be as in Hypothesis 3.9. Then the map*

$$\mathrm{Reg}_r \colon \mathrm{SS}_r(A, \mathcal{F}) \to \mathrm{KS}_r(A, \mathcal{F})$$

is an isomorphism.

Remark. If $\mathcal{F} = \mathcal{F}_{\mathrm{can}}$, then the integer r that appears in Hypothesis 3.9 coincides with the core rank $\chi(T)$ of T (see [5, Theorem 5.2.15]).

We will give the proof of Theorem 3.10 in § 3.4.

§ 3.4. Proof of Theorem 2.2 (1)

Let $\mathcal{F}$ be a Selmer structure on A and assume Hypothesis 3.9. We also fix an ideal $\mathfrak{d} \in \mathcal{N}$ as in Hypothesis 3.9 (1). Then we have a commutative diagram

$$\begin{tikzcd} \mathrm{SS}_r(A,\mathcal{F}) \arrow[r] \arrow[d, "\mathrm{Reg}_r"] & \bigcap_R^{r+\nu(\mathfrak{d})} H^1_{\mathcal{F}^{\mathfrak{d}}}(K,A) \arrow[d, "\bigwedge_{\mathfrak{q}\mid\mathfrak{d}} r_{\mathfrak{q}}"] \\ \mathrm{KS}_r(A,\mathcal{F}) \arrow[r] & \bigcap_R^r H^1_{\mathcal{F}(\mathfrak{d})}(K,A), \end{tikzcd} \tag{3.2}$$

where the horizontal maps are natural projection maps.

Lemma 3.11. *Under Hypothesis 3.9, the projection map*

$$\mathrm{SS}_r(A,\mathcal{F}) \to \bigcap_R^{r+\nu(\mathfrak{d})} H^1_{\mathcal{F}^{\mathfrak{d}}}(K,A)$$

is an isomorphism. In particular, $\mathrm{SS}_r(A,\mathcal{F})$ *is free of rank* 1 *(see Theorem 2.1 (1)).*

Proof. Let $\mathfrak{n} \in \mathcal{N}$ be an ideal such that $H^1_{(\mathcal{F}^{\mathfrak{n}})^*}(K, A^*(1))$ vanishes, and $\mathfrak{q} \in \mathcal{P}$ a prime not dividing $\mathfrak{n}$. Then we have $H^1_{\mathcal{F}^{\mathfrak{n}}}(K,A) \simeq R^{r+\nu(\mathfrak{n})}$ and $H^1_{\mathcal{F}^{\mathfrak{nq}}}(K,A) \simeq R^{r+\nu(\mathfrak{nq})}$ by Hypothesis 3.9 (1). Since $H^1_{(\mathcal{F}^{\mathfrak{n}})^*}(K, A^*(1))$ vanishes, we have a split exact sequence of free R-modules

$$0 \to H^1_{\mathcal{F}^{\mathfrak{n}}}(K,A) \to H^1_{\mathcal{F}^{\mathfrak{nq}}}(K,A) \to H^1(K_{\mathfrak{q}},A)/H^1_f(K_{\mathfrak{q}},A) \to 0$$

by the global duality. Since $H^1_{\mathcal{F}^{\mathfrak{n}}}(K,A) \simeq R^{r+\nu(\mathfrak{n})}$ and $H^1_{\mathcal{F}^{\mathfrak{nq}}}(K,A) \simeq R^{r+\nu(\mathfrak{nq})}$, we see that the transition map

$$\bigcap_R^{r+\nu(\mathfrak{nq})} H^1_{\mathcal{F}^{\mathfrak{nq}}}(K,A) \to \bigcap_R^{r+\nu(\mathfrak{n})} H^1_{\mathcal{F}^{\mathfrak{n}}}(K,A)$$

is an isomorphism. This completes the proof since the set

$$\{\mathfrak{n} \in \mathcal{N} \mid H^1_{(\mathcal{F}^{\mathfrak{n}})^*}(K, A^*(1)) \text{ vanishes } \}$$

is a cofinal subset of $\mathcal{N}$ by Hypothesis 3.9 (2). □

Lemma 3.12. *The map*

$$\bigwedge_{\mathfrak{q}\mid\mathfrak{d}} r_{\mathfrak{q}} \colon \bigcap_R^{r+\nu(\mathfrak{d})} H^1_{\mathcal{F}^{\mathfrak{d}}}(K,A) \to \bigcap_R^r H^1_{\mathcal{F}(\mathfrak{d})}(K,A)$$

is an isomorphism.

Proof. By the global duality and Hypothesis 3.9 (1), we have a split exact sequence of free R-modules

$$0 \to H^1_{\mathcal{F}(\mathfrak{d})}(K,A) \to H^1_{\mathcal{F}^{\mathfrak{d}}}(K,A) \xrightarrow{x \mapsto (r_{\mathfrak{q}}(x))_{\mathfrak{q}}} \bigoplus_{\mathfrak{q}\mid\mathfrak{d}} R \to 0.$$

Since $H^1_{\mathcal{F}(\mathfrak{d})}(K,A) \simeq R^r$ and $H^1_{\mathcal{F}^{\mathfrak{d}}}(K,A) \simeq R^{r+\nu(\mathfrak{d})}$ by Hypothesis 3.9 (1), we see that the map $\bigwedge_{\mathfrak{q}\mid\mathfrak{d}} r_{\mathfrak{q}}$ is an isomorphism. $\square$

By the commutative diagram (3.2), and Lemmas 3.11 and 3.12, we deduce the following

Corollary 3.13. *Under Hypothesis 3.9, the projection map*

$$\mathrm{KS}_r(A,\mathcal{F}) \to \bigcap\nolimits^r_R H^1_{\mathcal{F}(\mathfrak{d})}(K,A)$$

is surjective.

Definition 3.14. We define a graph $\mathcal{X}^0$ as follows.

- The set of vertices of $\mathcal{X}^0$ is all ideals $\mathfrak{n} \in \mathcal{N}$ such that $H^1_{\mathcal{F}^*(\mathfrak{n})}(K,A^*(1))$ vanishes.
- Let $\mathfrak{n} \in \mathcal{X}^0$, and $\mathfrak{q}$ a prime divisor of $\mathfrak{n}$ satisfying $\mathfrak{n}/\mathfrak{q} \in \mathcal{X}^0$. We join $\mathfrak{n}/\mathfrak{q}$ and $\mathfrak{n}$ by an edge in $\mathcal{X}^0$ if and only if the localization map

  $$H^1_{\mathcal{F}(\mathfrak{n}/\mathfrak{q})}(K,A) \to H^1_f(K_{\mathfrak{q}},A)$$

  is surjective.

The following theorem is proved by Howard, Mazur, and Rubin in [5, Theorem 4.3.12] and by Mazur and Rubin in [6, Theorem 14.4].

Theorem 3.15. *Under Hypothesis 3.9, the graph $\mathcal{X}^0$ is connected.*

By using the global duality, one can show the following

Lemma 3.16. *Let $\mathfrak{n} \in \mathcal{X}^0$, and $\mathfrak{q}$ a prime divisor of $\mathfrak{n}$ satisfying $\mathfrak{n}/\mathfrak{q} \in \mathcal{X}^0$. If $\mathfrak{n}/\mathfrak{q}$ and $\mathfrak{n}$ are joined by an edge in $\mathcal{X}^0$, then the maps*

$$\varphi^{\mathrm{fs}}_{\mathfrak{q}} \colon \bigcap\nolimits^r_R H^1_{\mathcal{F}(\mathfrak{n}/\mathfrak{q})}(K,A) \to \bigcap\nolimits^{r-1}_R H^1_{\mathcal{F}_{\mathfrak{q}}(\mathfrak{n}/\mathfrak{q})}(K,A),$$
$$v_{\mathfrak{q}} \colon \bigcap\nolimits^r_R H^1_{\mathcal{F}(\mathfrak{n})}(K,A) \to \bigcap\nolimits^{r-1}_R H^1_{\mathcal{F}_{\mathfrak{q}}(\mathfrak{n}/\mathfrak{q})}(K,A),$$

are isomorphisms.

Theorem 3.17. *Under Hypothesis 3.9, the projection map*

$$\mathrm{KS}_r(A, \mathcal{F}) \to \bigcap\nolimits_R^r H^1_{\mathcal{F}(\mathfrak{d})}(K, A)$$

is an isomorphism.

Proof. By Corollary 3.13, we only need to show the injectivity of the projection map. Let $\kappa = (\kappa_\mathfrak{n})_\mathfrak{n} \in \mathrm{KS}_r(K, A)$ with $\kappa_\mathfrak{d} = 0$. We will show that $\kappa_\mathfrak{n} = 0$ for any ideal $\mathfrak{n} \in \mathcal{N}$ by induction on $\lambda^*(\mathfrak{n}) := \dim_{R/I_R}\left(H^1_{\mathcal{F}^*(\mathfrak{n})}(K, A)[I_R]\right)$. Here I_R is the maximal ideal of R and

$$H^1_{\mathcal{F}^*(\mathfrak{n})}(K, A)[I_R] := \{x \in H^1_{\mathcal{F}^*(\mathfrak{n})}(K, A) \mid I_R x = 0\}.$$

If $\lambda^*(\mathfrak{n}) = 0$, then $\mathfrak{n}$ is a vertex of $\mathcal{X}^0$. Hence by Theorem 3.15, Lemma 3.16, and the fact that $\kappa_\mathfrak{d} = 0$, we conclude that $\kappa_\mathfrak{n} = 0$.

Suppose that $\lambda^*(\mathfrak{n}) > 0$ and $\kappa_\mathfrak{n} \neq 0$. By Hypothesis 3.9 (2), we can take an ideal $\mathfrak{m} \in \mathcal{N}$ coprime to $\mathfrak{n}$ such that the localization maps

$$H^1_{\mathcal{F}(\mathfrak{n})}(K, A)[I_R] \to H^1_f(K_\mathfrak{q}, A), \tag{3.3}$$

$$H^1_{\mathcal{F}^*(\mathfrak{n})}(K, A^*(1))[I_R] \to H^1_f(K_\mathfrak{q}, A^*(1)), \tag{3.4}$$

are non-zero for any prime $\mathfrak{q} \mid \mathfrak{m}$, and that the map

$$H^1_{\mathcal{F}(\mathfrak{n})}(K, A) \xrightarrow{x \mapsto (\varphi^{\mathrm{fs}}_\mathfrak{q}(x))_\mathfrak{q}} \bigoplus_{\mathfrak{q} \mid \mathfrak{m}} R \tag{3.5}$$

is injective. Since the contravariant functor $(-)^*$ is exact, by (3.5), the R-module $H^1_{\mathcal{F}(\mathfrak{n})}(K, A)^*$ is generated by the set $\{\varphi^{\mathrm{fs}}_\mathfrak{q}\}_{\mathfrak{q} \mid \mathfrak{m}}$. Hence there is a prime $\mathfrak{q} \mid \mathfrak{m}$ such that $\varphi^{\mathrm{fs}}_\mathfrak{q}(\kappa_\mathfrak{n}) \neq 0$ since $\kappa_\mathfrak{n}$ is an R-homomorphism $\bigwedge^r_R(H^1_{\mathcal{F}(\mathfrak{n})}(K, A)^*) \to R$. By the definition of Kolyvagin systems, we have

$$v_\mathfrak{q}(\kappa_{\mathfrak{n}\mathfrak{q}}) = \varphi^{\mathrm{fs}}_\mathfrak{q}(\kappa_\mathfrak{n}) \neq 0.$$

By using the global duality and the fact that the maps (3.3) and (3.4) are non-zero, one can show that $\lambda^*(\mathfrak{n}\mathfrak{q}) = \lambda^*(\mathfrak{n}) - 1$. Hence we conclude that $\kappa_{\mathfrak{n}\mathfrak{q}} = 0$ by the induction hypothesis. This is a contradiction with $v_\mathfrak{q}(\kappa_{\mathfrak{n}\mathfrak{q}}) \neq 0$. □

Proof of Theorem 3.10. This theorem follows from the commutative diagram (3.2), Lemmas 3.11 and 3.12, and Theorem 3.17. □

Remark. We will explain why the map $\mathrm{Reg}^{\mathrm{MR}}_r$ in Theorem 1.2 (3) is not surjective in general, and why such problem is avoided after replacing $\bigwedge$ by $\bigcap$.

Suppose that R is a principal ideal local ring with the maximal ideal I_R. Let $\Bbbk := R/I_R$ denote the residue field of R. We also assume that Hypothesis 3.9 holds.

Take an ideal $\mathfrak{n} \in \mathcal{N}$. Note that we have an equality

$$H^1_{\mathcal{F}(\mathfrak{n})}(K, A)^* = \sum_{\mathfrak{q} \nmid \mathfrak{n}} R\varphi^{\mathrm{fs}}_{\mathfrak{q}} \tag{3.6}$$

by Hypothesis 3.9(2). Let $\kappa_{\mathfrak{n}} \in \bigwedge^r_R H^1_{\mathcal{F}(\mathfrak{n})}(K, A)$ be a non-zero element such that

$$\widehat{\varphi}^{\mathrm{fs}}_{\mathfrak{q}}(\kappa_{\mathfrak{n}}) = 0 \text{ for any } \mathfrak{q} \in \mathcal{P} \text{ with } \mathfrak{q} \nmid \mathfrak{n}. \tag{3.7}$$

Here

$$\widehat{\varphi}^{\mathrm{fs}}_{\mathfrak{q}} \colon \bigwedge\nolimits^r_R H^1_{\mathcal{F}(\mathfrak{n})}(K, A) \to \bigwedge\nolimits^{r-1}_R H^1_{\mathcal{F}_{\mathfrak{q}}(\mathfrak{n})}(K, A)$$

is the map defined in [6, Proposition A.1]. Note that there exists such an element $\kappa_{\mathfrak{n}}$ when $r > 1$ and

$$\dim_{\Bbbk}\left(H^1_{\mathcal{F}^*(\mathfrak{n})}(K, A^*(1)) \otimes_R \Bbbk\right) - \dim_{\Bbbk}\left(H^1_{\mathcal{F}^*(\mathfrak{n})}(K, A^*(1)[I_R])\right) \geq r$$

(see [6, Remark 11.9]). Then we see that $\kappa := (\kappa_{\mathfrak{n}})_{\mathfrak{n}} \in \mathrm{KS}^{\mathrm{MR}}_r(A, \mathcal{F})$, where we define $\kappa_{\mathfrak{m}} = 0$ if $\mathfrak{m} \neq \mathfrak{n}$. Furthermore, we have

$$\kappa \notin \mathrm{im}\left(\mathrm{Reg}^{\mathrm{MR}}_r \colon \mathrm{SS}^{\mathrm{MR}}_r(A, \mathcal{F}) \hookrightarrow \mathrm{KS}^{\mathrm{MR}}(A, \mathcal{F})\right).$$

In fact, by the definition of the map $\mathrm{Reg}^{\mathrm{MR}}_r$, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{SS}^{\mathrm{MR}}_r(A, \mathcal{F}) \arrow[r, "\mathrm{Reg}^{\mathrm{MR}}_r"] \arrow[d] & \mathrm{KS}^{\mathrm{MR}}_r(A, \mathcal{F}) \arrow[d] \\
\bigwedge\nolimits^{r+\nu(\mathfrak{d})}_R H^1_{\mathcal{F}^{\mathfrak{d}}}(K, A) \arrow[r] & \bigwedge\nolimits^r_R H^1_{\mathcal{F}(\mathfrak{d})}(K, A)
\end{tikzcd}$$

for any ideal $\mathfrak{d} \in \mathcal{N}$. Here the vertical maps are the projection maps. If $H^1_{\mathcal{F}^*(\mathfrak{d})}(K, A)$ vanishes, in the same way as in the proof of Lemmas 3.11 and 3.12, one can show that the left vertical map and the bottom horizontal map are isomorphisms. Thus for any non-zero element $\kappa' \in \mathrm{im}\left(\mathrm{Reg}^{\mathrm{MR}}_r\right)$ and ideal $\mathfrak{d} \in \mathcal{N}$ with $H^1_{\mathcal{F}^*(\mathfrak{d})}(K, A)$ vanishes, we have $\kappa'_{\mathfrak{d}} \neq 0$. This fact implies $\kappa \notin \mathrm{im}\left(\mathrm{Reg}^{\mathrm{MR}}_r\right)$.

Hence elements with the property (3.7) is one of the obstruction of the surjectivity of the map $\mathrm{Reg}^{\mathrm{MR}}_r$.

However, there are no elements with the property (3.7) in $\bigcap^r_R H^1_{\mathcal{F}(\mathfrak{n})}(K, A)$ by definition and the equality (3.6). Namely, if $\kappa_{\mathfrak{n}} \in \bigcap^r_R H^1_{\mathcal{F}(\mathfrak{n})}(K, A)$ is non-zero, then there is a prime $\mathfrak{q} \in \mathcal{P}$ with $\mathfrak{q} \nmid \mathfrak{n}$ such that $\varphi^{\mathrm{fs}}_{\mathfrak{q}}(\kappa_{\mathfrak{n}}) \neq 0$ since $\kappa_{\mathfrak{n}}$ is an R-homomorphism $\bigwedge^r_R(H^1_{\mathcal{F}}(K, A)^*) \to R$. This is one of the key points of the proof of Theorem 2.2 (1).

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