

On the Glauberman-Watanabe correspondence for p -blocks of a p -nilpotent group with a cyclic defect group

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Let p be a prime. Let $(\mathcal{K}, \mathcal{O}, k)$ be a p -modular system where $\mathcal{O}$ is a complete discrete valuation ring having an algebraically closed residue field k of characteristic p and having a quotient field $\mathcal{K}$ of characteristic zero which will be assumed to be large enough for any of finite groups we consider in this article. We use the notation $\bar{}$ for the reduction modulo $J(\mathcal{O})$. Let $\mathcal{R} \in \{\mathcal{K}, \mathcal{O}, k\}$. Below, for groups H_1 and H_2 , an $\mathcal{R}[H_1 \times H_2]$ -module X and an $(\mathcal{R}H_1, \mathcal{R}H_2)$ -bimodule X will be identified in the usual way, namely $(h_1, h_2) \cdot x = h_1 \cdot x \cdot h_2^{-1}$ where $h_1 \in H_1$, $h_2 \in H_2$ and $x \in X$. For a common subgroup D of H_1 and H_2 , denote by $\Delta D = \{(u, u) \mid u \in D\}$ a diagonal subgroup of $H_1 \times H_2$. Let $\mathcal{R}' \in \{\mathcal{O}, k\}$. For a p -group P , an $\mathcal{R}'$ -free $\mathcal{R}'P$ -module T is called an *endo-permutation* module if $\text{End}_{\mathcal{R}'}(T)$ has an P -invariant $\mathcal{R}'$ -basis ([1]).

Let q be a prime such that $q \neq p$. Let $S = \langle s \rangle$ be a cyclic group of order q . Let $\mu \in \mathcal{O}$ be a fixed non-trivial q -th root of unity.

Let G be a finite group such that $q \nmid |G|$. Assume that S acts on G . Then with this action, we can consider the semi-direct product of G and S , denoted by GS . Denote by G^S the centralizer $C_G(S)$ of S in G . When q is odd, for $\theta \in \text{Irr}(G)^S$, there is a unique extension $\hat{\theta} \in \text{Irr}(GS)$ of θ , a unique character $\pi(G, S)(\theta) \in \text{Irr}(G^S)$ and a unique sign ϵ_θ such that $\hat{\theta}(cs) = \epsilon_\theta \pi(G, S)(\theta)(c)$ where $c \in G^S$. When $q = 2$, for $\theta \in \text{Irr}(G)^S$ and a chosen sign ϵ_θ , there is a unique extension $\hat{\theta} \in \text{Irr}(GS)$ of θ and a unique character $\pi(G, S)(\theta) \in \text{Irr}(G^S)$ such that $\hat{\theta}(cs) = \epsilon_\theta \pi(G, S)(\theta)(c)$ for $c \in G^S$. The character $\pi(G, S)(\theta)$ is called the *Glauberman correspondence* of θ , see [3]. For $t \in \mathbb{Z}$, let $\lambda^t \hat{\theta} \in \text{Irr}(GS)$ be the extension of $\theta \in \text{Irr}(G)^S$ such that $\lambda^t \hat{\theta}(gs) = \mu^t \hat{\theta}(gs)$ where $g \in G$.

Let b be an S -invariant (p -)block of G having an S -centralized defect group D . Denote by $w(b)$ the *Glauberman-Watanabe corresponding block* of b , that is, the block of G^S with a defect group D such that $\text{Irr}(w(b)) = \{\pi(G, S)(\theta) \mid \theta \in \text{Irr}(b) = \text{Irr}(b)^S\}$. For $t \in \mathbb{Z}$, let $\hat{b}_t$ be the block of GS such that $\text{Irr}(\hat{b}_t) = \{\lambda^t \hat{\theta} \mid \theta \in \text{Irr}(b)\}$ (under appropriate choices of signs ϵ_θ when $q = 2$), and let e_t be the block of S corresponding to the representation of S determined by $s \mapsto \mu^t$. Let

$$b_r = \sum_{t=0}^{q-1} e_t \hat{b}_{t+r} \quad \text{for } 0 \leq r \leq q-1. \quad (1)$$

Then $b = \sum_{r=0}^{q-1} b_r$ is an orthogonal idempotent decomposition of b in $(\mathcal{O}Gb)^{G^S}$ and so $b_r \mathcal{O}G$ is a direct summand of the $\mathcal{O}[G^S \times G]$ -module $\mathcal{O}Gb$, and the following equation of the generalized characters of $G^S \times G$ holds, see [6] and [7]:

$$\chi_{b_0 \mathcal{O}G} - \chi_{b_l \mathcal{O}G} = \sum_{\theta \in \text{Irr}(b)} \epsilon_\theta \pi(G, S)(\theta) \otimes_{\mathcal{K}} \check{\theta} \quad \text{for } 1 \leq l \leq q-1, \quad (2)$$

where $\chi_{b_r \mathcal{O}G}$ is a character corresponding to a $\mathcal{K}[G^S \times G]$ -module $b_r \mathcal{K}G$ and $\check{\theta}$ is a $\mathcal{K}$ -dual of θ . (Below, denote by $\check{b}$ the block containing $\check{\theta}$ for $\theta \in \text{Irr}(b)$.) Equation (2) gives immediately the following Watanabe's result, see [9]:

The map determined by $\theta \mapsto \epsilon_\theta \pi(G, S)(\theta)$ where $\theta \in \text{Irr}(b)$, induces a perfect isometry $\mathbb{Z}\text{Irr}(b) \simeq \mathbb{Z}\text{Irr}(w(b))$ between the Glauberman-Watanabe corresponding blocks.

and, as noted by Okuyama in [6], raised the following question:

Is the left hand side of equation (2) is a “shadow” of a complex of $(\mathcal{O}G^S w(b), \mathcal{O}Gb)$ -bimodule which induces a derived equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$?

In fact, we have the following:

Theorem 1.1. *With the above notations, moreover assume that G is p -nilpotent and D is cyclic. Then there is a two term complex $C^\bullet$ of $(\mathcal{O}G^S w(b), \mathcal{O}Gb)$ -bimodule satisfying the following:*

(1) $b_0 \mathcal{O}G$ is in degree 0 and $b_l \mathcal{O}G$ is in degree 1 or -1 .

(2) $C^\bullet$ induces a derived equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$.

Further, $C^\bullet$ is quasi-isomorphic to a one term complex consisting of the bimodule M satisfying the following (M is in degree 0 if $\epsilon_b = 1$ and M is in degree 1 or -1 if $\epsilon_b = -1$ where $\epsilon_b = \epsilon_\theta$ for $\theta \in \text{Irr}(b)$, which depends only on b):

(a) M induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$.

(b) M has a vertex ΔD and an endo-permutation source.

$C^\bullet$ in Theorem 1.1 induces above Watanabe's perfect isometry, see the condition in Theorem 1.1(1) and quation (2), and M in Theorem 1.1 induces the Glauberman correspondence of characters belonging to b and $w(b)$. The existence of M as in Theorem 1.1 is a particular case of the result of Harris-Linckelman for p -solvable case and of Watanabe for p -nilpotent blocks, see [5] and [10]. See also [4] for the existence of a derived equivalence between blocks with cyclic defect groups inducing prescribed perfect isometry.

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Below, with the assumptions in Section 1, G and b are such that:

Condition 2.1. G is a p -nilpotent group with an S -centralized cyclic Sylow p -subgroup P of order p^α , that is, $G = KP = K \rtimes P$ where $K = O_{p'}(G)$. b is a P -invariant block of K , hence a block of G with a defect group $D = P$.

In fact, by the Fong's first reduction as described in [5, Section 5] and Theorem 2.2 and 2.3 below, Theorem 1.1 above can be shown.

Denote by P_i the unique subgroup of P with the order p^i for i such that $0 \leq i \leq \alpha$. Recall that the image $\text{Br}_{P_i}(b)$ of the Brauer homomorphism Br_{P_i} of b is primitive in $Z(kC_K(P_i))$ and hence is a block of $C_G(P_i) = C_K(P_i)P$, and let $\mathfrak{B}\mathfrak{r}_{P_i}(b)$ be the corresponding block over $\mathcal{O}$. Note that $b = \mathfrak{B}\mathfrak{r}_{P_0}(b)$. Idempotents $\mathfrak{B}\mathfrak{r}_{P_i}(b)_r \in (\mathcal{O}C_G(P_i)\mathfrak{B}\mathfrak{r}_{P_i}(b))^{C_{G^S}(P_i)}$ (see (1) in Section 1) are defined similarly. Denote by M_j^j the unique trivial source $\mathcal{O}[C_{G^S}(P_i) \times C_G(P_i)]$ -module in $w(\mathfrak{B}\mathfrak{r}_{P_i}(b)) \times \mathfrak{B}\mathfrak{r}_{P_i}(b)$ with vertex ΔP_j for j such that $0 \leq j \leq \alpha$. Let $M^j = M_0^j$. Let $\epsilon_{\mathfrak{B}\mathfrak{r}_{P_i}(b)} = \epsilon_{\chi_i}$ where $\chi_i \in \text{Irr}(C_G(P_i) | \mathfrak{B}\mathfrak{r}_{P_i}(b))$. Note that $\epsilon_{\mathfrak{B}\mathfrak{r}_{P_i}(b)}$ depends only on $\mathfrak{B}\mathfrak{r}_{P_i}(b)$.

Theorem 2.2. *The following are equivalent for a fixed i where $0 \leq i \leq \alpha$:*

- (1) $\epsilon_{\mathfrak{B}\mathfrak{r}_{P_h}(b)} = \epsilon_{\mathfrak{B}\mathfrak{r}_P(b)}$ for any h such that $i \leq h \leq \alpha$.
- (2) *The unique simple $k(C_{K^S}(P_i) \times C_K(P_i))\Delta P$ -module in $w(\text{Br}_{P_i}(b)) \times \tilde{\text{Br}}_{P_i}(b)$ is a trivial source module.*
- (3) M_i^α is a unique indecomposable direct summand of $\mathcal{O}C_G(P_i)\mathfrak{B}\mathfrak{r}_{P_i}(b) \downarrow_{C_{G^S}(P_i) \times C_G(P_i)}$ with a multiplicity not divisible by q .
- (4) (a) $\mathfrak{B}\mathfrak{r}_{P_i}(b)_0 \mathcal{O}C_G(P_i) \simeq M_i^\alpha \oplus \mathfrak{B}\mathfrak{r}_{P_i}(b)_l \mathcal{O}C_G(P_i)$ if $\epsilon_{\mathfrak{B}\mathfrak{r}_P(b)} = 1$.
 (b) $\mathfrak{B}\mathfrak{r}_{P_i}(b)_l \mathcal{O}C_G(P_i) \simeq M_i^\alpha \oplus \mathfrak{B}\mathfrak{r}_{P_i}(b)_0 \mathcal{O}C_G(P_i)$ if $\epsilon_{\mathfrak{B}\mathfrak{r}_P(b)} = -1$.
- (5) M_i^α induces a Morita equivalence between $\mathcal{O}C_G(P_i)\mathfrak{B}\mathfrak{r}_{P_i}(b)$ and $\mathcal{O}C_{G^S}(P_i)w(\mathfrak{B}\mathfrak{r}_{P_i}(b))$.
- (6) $\mathcal{O}C_G(P_i)\mathfrak{B}\mathfrak{r}_{P_i}(b)$ and $\mathcal{O}C_{G^S}(P_i)w(\mathfrak{B}\mathfrak{r}_{P_i}(b))$ are Puig equivalent.

The conditions of Theorem 2.2 above always holds for $i = \alpha$. If the conditions of Theorem 2.2 holds for $i = 0$, that is, $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$ are Puig equivalent, then, by the conditions of Theorem 2.2(4) and (5), we can construct a desired two term complex $C^\bullet$ as in Theorem 1.1 with $M = M^\alpha$.

Below, we consider the case where $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$ are not Puig equivalent. Then there is some β as in Theorem 2.3 below, see, for example, conditions of Theorem 2.2(1) and Theorem 2.3(1).

Since $(K^S \times K)\Delta P$ is p -nilpotent, sources of simple $k(K^S \times K)\Delta P$ -modules are endo-permutation modules (Dade [2]). Since ΔP is cyclic, indecomposable endo-permutation $k\Delta P$ -modules with vertex ΔP are the modules of the following form (Dade [2]):

$$\Omega_{\Delta P}^{a_0} \text{Inf}_{\Delta(P/P_1)}^{\Delta P} \Omega_{\Delta(P/P_1)}^{a_1} \text{Inf}_{\Delta(P/P_2)}^{\Delta(P/P_1)} \cdots \text{Inf}_{\Delta(P/P_{\alpha-2})}^{\Delta(P/P_{\alpha-3})} \Omega_{\Delta(P/P_{\alpha-2})}^{a_{\alpha-2}} \text{Inf}_{\Delta(P/P_{\alpha-1})}^{\Delta(P/P_{\alpha-2})} \Omega_{\Delta(P/P_{\alpha-1})}^{a_{\alpha-1}}(k),$$

where Ω means Heller translate and $a_i \in \{0, 1\}$.

Theorem 2.3. *Let β be such that $0 \leq \beta \leq \alpha - 1$. The following conditions on β are equivalent:*

- (1) $\epsilon_{\mathfrak{Br}_{P_\beta}(b)} \neq \epsilon_{\mathfrak{Br}_P(b)}$ and $\epsilon_{\mathfrak{Br}_{P_h}(b)} = \epsilon_{\mathfrak{Br}_P(b)}$ for any h such that $\beta + 1 \leq h \leq \alpha$.
- (2) $a_\beta = 1$ and $a_h = 0$ for any h such that $\beta + 1 \leq h \leq \alpha$ where a_i 's are 0 or 1 describing a source of the unique simple $k(K^S \times K)\Delta P$ -module in $w(\bar{b}) \times \bar{\bar{b}}$ as above (when $p = 2$, let $a_{\alpha-1} = 0$).
- (3) $\mathcal{O}C_G(P_\beta)\mathfrak{Br}_{P_\beta}(b)$ and $\mathcal{O}C_{G^S}(P_\beta)w(\mathfrak{Br}_{P_\beta}(b))$ are not Puig equivalent and $\mathcal{O}C_G(P_h)\mathfrak{Br}_{P_h}(b)$ and $\mathcal{O}C_{G^S}(P_h)w(\mathfrak{Br}_{P_h}(b))$ are Puig equivalent for any h such that $\beta + 1 \leq h \leq \alpha$.
- (4) The multiplicity of M^β in $\mathcal{O}Gb \downarrow_{G^S \times G}$ is not divisible by q .
- (5) M^α and M^β are only indecomposable direct summands of $\mathcal{O}Gb \downarrow_{G^S \times G}$ with multiplicities not divisible by q .
- (6) (a) $b_l \mathcal{O}G \oplus M^\alpha \simeq b_0 \mathcal{O}G \oplus M^\beta$ if $\epsilon_{\mathfrak{Br}_P(b)} = 1$.
(b) $b_0 \mathcal{O}G \oplus M^\alpha \simeq b_l \mathcal{O}G \oplus M^\beta$ if $\epsilon_{\mathfrak{Br}_P(b)} = -1$.
- (7) (a) When $\epsilon_b \epsilon_{\mathfrak{Br}_P(b)} = -1$, there is an epimorphism $\Phi : M^\beta \twoheadrightarrow M^\alpha$ such that $N = \text{Ker} \Phi$ induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$.
(b) When $\epsilon_b \epsilon_{\mathfrak{Br}_P(b)} = 1$, there is an epimorphism $\Phi : M^\alpha \twoheadrightarrow M^\beta$ such that $N = \text{Ker} \Phi$ induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$.

If $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$ are not Puig equivalent, then, by the conditions of Theorem 2.3(6) and (7), we can construct a desired two term complex $C^\bullet$ as in Theorem 1.1 with $M = N$. Note that a source of $\bar{N}$ is a source of the unique simple $k(K^S \times K)\Delta P$ -module in $w(\bar{b}) \times \bar{\bar{b}}$, and an $\mathcal{O}$ -lift of an endo-permutation module is an endo-permutation module.

In fact, a source of the module inducing the concerned Morita equivalence between $kG\bar{b}$ and $kG^S w(\bar{b})$ and “signs of the local blocks” $\epsilon_{\mathfrak{B}_{P_i}(b)}$ are related as follows:

Proposition 2.4. *The following conditions on α numbers $a_i \in \{0, 1\}$ ($0 \leq i \leq \alpha - 1$) are equivalent when p is odd:*

(1) *A source of the unique simple $k(K^S \times K)\Delta P$ -module in $w(\bar{b}) \times \check{\bar{b}}$ has the following form:*

$$\Omega_{\Delta P}^{a_0} \text{Inf}_{\Delta(P/P_1)}^{\Delta P} \Omega_{\Delta(P/P_1)}^{a_1} \text{Inf}_{\Delta(P/P_2)}^{\Delta(P/P_1)} \cdots \text{Inf}_{\Delta(P/P_{\alpha-2})}^{\Delta(P/P_{\alpha-3})} \Omega_{\Delta(P/P_{\alpha-2})}^{a_{\alpha-2}} \text{Inf}_{\Delta(P/P_{\alpha-1})}^{\Delta(P/P_{\alpha-2})} \Omega_{\Delta(P/P_{\alpha-1})}^{a_{\alpha-1}}(k).$$

(2) $\epsilon_{\mathfrak{B}_{P_i}(b)} = (-1)^{a_i} \epsilon_{\mathfrak{B}_{P_{i+1}}(b)}$ for any i such that $0 \leq i \leq \alpha - 1$.

Proposition 2.5. *The following conditions on $\alpha - 1$ numbers $a_i \in \{0, 1\}$ ($0 \leq i \leq \alpha - 2$) are equivalent when $p = 2$:*

(1) *A source of the unique simple $k(K^S \times K)\Delta P$ -module in $w(\bar{b}) \times \check{\bar{b}}$ has the following form:*

$$\Omega_{\Delta P}^{a_0} \text{Inf}_{\Delta(P/P_1)}^{\Delta P} \Omega_{\Delta(P/P_1)}^{a_1} \text{Inf}_{\Delta(P/P_2)}^{\Delta(P/P_1)} \cdots \text{Inf}_{\Delta(P/P_{\alpha-2})}^{\Delta(P/P_{\alpha-3})} \Omega_{\Delta(P/P_{\alpha-2})}^{a_{\alpha-2}}(k).$$

(2) $\epsilon_{\mathfrak{B}_{P_i}(b)} = (-1)^{a_i} \epsilon_{\mathfrak{B}_{P_{i+1}}(b)}$ for any i such that $0 \leq i \leq \alpha - 2$.

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