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Fundamental groups of the spaces of regular orbits of affine Weyl groups of rank 2

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In [2], E. Brieskorn has proved that the fundamental group of the space of regular orbits of a finite real reflection group W is the Artin group of the same type as W. The purpose of this paper is to consider the fundamental groups of the space of regular orbits of affine Weyl groups.

Let W be an irreducible affine Weyl group of rank 2. Then W acts on $\mathbb{R}^2$ naturally. Let $\sum$ be the set consists of all the reflections in W, and for $s \in \sum$ let H_s' be the hyperplane in $\mathbb{R}^2$ which is fixed by s pointwisely. Let C be an open chamber, i.e., a connected component of $\mathbb{R}^2 - \bigcup_{s\in\sum} H_s'$ (see [1]). Let $\{H_s'\}_{s\in S}$ be the set of the walls of C. Then it is well known that $S = \{s_0, s_1, s_2\}$ and (W , S) is a Coxeter system, i. e., W has a presentation with generating set S and relations $(ss')^{m(ss')} = 1$, where m(ss') is the order of ss' and m(ss) = 1.

Now, let W acts on $\mathbb{C}^2$ naturally and let H_s, $s \in \sum$, be the hyperplane in $\mathbb{C}^2$ which is fixed by s pointwisely. Let us set $Y_W = \mathbb{C}^2 - \bigcup_{s\in\sum} H_s$. Then we have obtained the following:

<u>Theorem</u> The fundamental group of the space of regular orbits Y_W/W of an affine Weyl group of rank 2 is the Artin group of the same type as W, i.e., it has the following presentation with the generators g_s, $s \in S$, and the defining relations:

$$\underbrace{g_s g_t g_s \cdots}_{m(s,t)\text{ factors}} = \underbrace{g_t g_s g_t \cdots}_{m(s,t)\text{ factors}} .$$

<u>Remark</u> It is conjectured that the same result also holds for affine Weyl groups of rank > 2.

<u>Proposition 1</u>. Let W be the affine Weyl group of type $\widetilde{A}_2$ acting on $\mathbb{C}^2$, that is , W is generated by three reflections s_0, s_1, and s_2 defined by $s_i(\vec{u}) = \vec{u} - 2(\vec{u}, \vec{\alpha}_i)\vec{\alpha}_i$, $i = 1, 2$, and $s_0(\vec{u}) = \vec{u} - 2(\vec{u},\vec{\alpha}_0)\vec{\alpha}_0 + 2\vec{\alpha}_0$ for $\vec{u} \in \mathbb{C}^2$, where (,) denotes the ordinary inner product of $\mathbb{C}^2$ and $\vec{\alpha}_i$, $i = 1, 2$ and 0, are vectors in $\mathbb{C}^2$ defined by $\vec{\alpha}_1 = (\sqrt{3}/2, -1/2)$, $\vec{\alpha}_2 = (0,1)$ and $\vec{\alpha}_0 = \vec{\alpha}_1 + \vec{\alpha}_2$. Let us set

$$f_1 = \exp 2\sqrt{3}\pi i x_1/3 + \exp(-\sqrt{3}x_1+3x_2)\pi i/3 + \exp(-\sqrt{3}x_1-3x_2)\pi i/3,$$

$$f_2 = \exp(-2\sqrt{3}\pi i x_1)/3 + \exp(\sqrt{3}x_1-3x_2)\pi i/3 + \exp(\sqrt{3}x_1+3x_2)\pi i/3.$$

Then f_1 and f_2 are invariant under the action of W. Let $\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ define by $\Phi(u) = (f_1(u), f_2(u))$ for $u \in \mathbb{C}^2$. Then the map Φ induces a biholomorphism $\overline{\Phi}$ between $\mathbb{C}^2/W$ and $\mathbb{C}^2$ and the image of Y_W/W under this map is the complement of the algebraic curve in $\mathbb{C}^2$ defined by

$$h(z) = 4z_1^3 + 4z_2^3 - z_1^2 z_2^2 - 18z_1 z_2 + 27 = 0.$$

<u>Proof</u> It is clear that f_1 and f_2 are invariant under the action of W. We can also show that $\overline{\Phi}$ is onto and one to one map and $\Phi(\bigcup_{s\in\Sigma} H_s) = \{z \in \mathbb{C}^2 \mid h(z) = 0\}$. Since Y_W/W is dense in $\mathbb{C}^2/W$ and $\overline{\Phi}(Y_W/W)$ is dense in $\mathbb{C}^2$, $\overline{\Phi}$ is a biholomorphism between $\mathbb{C}^2/W$ and $\mathbb{C}^2$.

<u>Proposition 2</u>. Let W be the affine Weyl group of type $\widetilde{B}_2$ acting on $\mathbb{C}^2$, that is, W is generated by three reflections s_0, s_1 and s_2 defined by $s_1(\vec{u}) = \vec{u} - (\vec{u},\vec{\alpha}_1)\vec{\alpha}_1$, $s_2(\vec{u}) = \vec{u} - 2(\vec{u},\vec{\alpha}_2)\vec{\alpha}_2$

and $s_0(\vec{u}) = \vec{u} - (\vec{u},\vec{\alpha}_0)\vec{\alpha}_0 + \vec{\alpha}_0$ for $\vec{u} \in \mathbb{C}^2$, where $(\ ,\ )$ denotes the ordinary inner product of $\mathbb{C}^2$ and $\vec{\alpha}_i$, $i = 1, 2$ and 0 are vectors in $\mathbb{C}^2$ defined by $\vec{\alpha}_1 = (1, -1)$, $\vec{\alpha}_2 = (0,1)$ and $\vec{\alpha}_0 = \vec{\alpha}_1 + 2\vec{\alpha}_2$. Let us set

$$f_1 = \exp 2\pi i x_1 + \exp(-2\pi i x_1) + \exp 2\pi i x_2 + \exp(-2\pi i x_2)$$

$$f_2 = \exp(x_1+x_2)\pi i + \exp(-x_1-x_2)\pi i + \exp(x_1-x_2)\pi i + \exp(-x_1+x_2)\pi i.$$

Then f_1 and f_2 are invariant under the action of W. Let $\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ define by $\Phi(u) = (f_1(u), f_2(u))$ for $u \in \mathbb{C}^2$. Then the map induces a biholomorphism $\bar{\Phi}$ between $\mathbb{C}^2/W$ and $\mathbb{C}^2$ and the image of Y_W/W under this map is the complement of the algebraic curve in $\mathbb{C}^2$ defined by

$$h(z) = (z_2^2 - 4z_1)(z_2^2 - (z_2/2 + 2)^2) = 0.$$

<u>Proof</u> It is clear that f_1 and f_2 are invariant under the action of W. We can also show that $\bar{\Phi}$ is onto and one to one map and $\Phi(\bigcup_{s\in\Sigma} H_s) = \{z \in \mathbb{C}^2 \mid h(z) = 0\}$. Since Y_W/W is dense in $\mathbb{C}^2/W$ and $\bar{\Phi}(Y_W/W)$ is dense in $\mathbb{C}^2$, $\bar{\Phi}$ is a biholomorphism between $\mathbb{C}^2/W$ and $\mathbb{C}^2$.

<u>Proposition 3</u>. Let W be the affine Weyl group of type $\tilde{G}_2$ acting on $\mathbb{C}^2$, that is, W is generated by three reflections s_0, s_1 and s_2 defined by $s_1(\vec{u}) = \vec{u} - 2(\vec{u},\vec{\alpha}_1)/3\vec{\alpha}_1$, $s_2(\vec{u}) = \vec{u} - 2(\vec{u},\vec{\alpha}_2)\vec{\alpha}_2$ and $s_0(\vec{u}) = \vec{u} - 2(\vec{u},\vec{\alpha}_0)/3\vec{\alpha}_0 + 2/3\,\vec{\alpha}_0$ for $\vec{u} \in \mathbb{C}^2$, where $(\ ,\ )$ denotes the ordinary inner product of $\mathbb{C}^2$ and $\vec{\alpha}_i$, $i = 1, 2$ and 0, are vectors in $\mathbb{C}^2$ defined by $\vec{\alpha}_1 = (\sqrt{3}/2, -3/2)$, $\vec{\alpha}_2 = (0,1)$ and $\vec{\alpha}_0 = 2\vec{\alpha}_1 + 3\vec{\alpha}_2$. Let us set

$$f_1 = \exp 2\pi i x_2 + \exp(-2\pi i x_2) + \exp(\sqrt{3}x_1 - x_2)\pi i + \exp(-\sqrt{3}x_1 + x_2)\pi i + \exp(\sqrt{3}x_1 + x_2)\pi i + \exp(-\sqrt{3}x_1 - x_2)\pi i,$$

$$f_2 = \exp 2\sqrt{3}\pi i x_1 + \exp(-2\sqrt{3}\pi i x_1) + \exp(\sqrt{3}x_1+3x_2)\pi i$$
$$+ \exp(-\sqrt{3}x_1-3x_2)\pi i \quad \exp(\sqrt{3}x_1-3x_2)\pi i + \exp(-\sqrt{3}x_1+3x_2)\pi i .$$

Then f_1 and f_2 are invariant under the action of W. Let $\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ define by $\Phi(u) = (f_1(u), f_2(u))$ for $u \in \mathbb{C}^2$. Then the map induces a biholomorphism $\bar{\Phi}$ between $\mathbb{C}^2/W$ and $\mathbb{C}^2$ and the image of Y_W/W under this map is the complement of the algebraic curve in $\mathbb{C}^2$ defined by

$$h(z) = (z_2 - z_1^2/4 - 3)(z_2^2+12(2+z_1)z_2-4(z_1^3-9z_1-9)) = 0.$$

<u>Proof</u> It is clear that f_1 and f_2 are invariant under the action of W. We can also show that $\bar{\Phi}$ is onto and one to one map and $\Phi(\bigcup_{s\in\Sigma} H_s) = \{z \in \mathbb{C}^2 \mid h(z) = 0\}$. Since Y_W/W is dense in $\mathbb{C}^2/W$ and $\bar{\Phi}(Y_W/W)$ is dense in $\mathbb{C}^2$, $\bar{\Phi}$ is a biholomorphism between $\mathbb{C}^2/W$ and $\mathbb{C}^2$.

<u>Proof of the Theorem.</u>

<u>Case 1.</u> W = affine Weyl group of type $\tilde{A}_2$. Then $S = \{s_0, s_1, s_2\}$ and $m(s_0,s_1) = m(s_1,s_2) = m(s_0,s_2) = 3$. By setting $z_1 = z_1' - z_2'$ and $z_2 = z_1' + z_2'$, we can show that Y_W/W is isomorphic to the complement of the algebraic curve $z_2^4 - 2(z_1^2+12z_1+9)z_2^2 + z_1^4-8z_1^3+18z_1^2-27 = 0$. Then we can calculate $\pi_1(Y_W/W)$ by the method of Zariski [3]. In above equation z_2 has four distinct solution except for $z_1 = 3$, -1 and $-3/2$. As the generators of the fundamental group, we can choose g_0, g_1, g_2 and g in the $z_1 = 0$ plane as the figure 1 shows. Here the base point is taken far enough from the origin O.

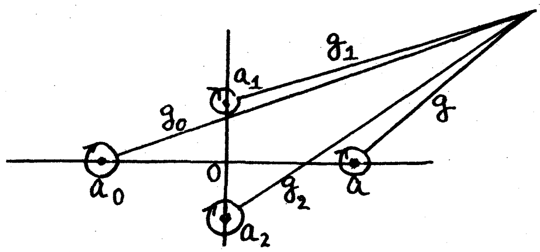

figure 1 (z_1=0 plane)

If we move z_1 along a closed path $z(t)$ ($t \in [0,1]$ and $z(0) = z(1) = 0$) in $\mathbb{C}$ which surrounds only the point 3, then we obtain the relation $g_1g_2g_1 = g_2g_1g_2$. If we move z_1 along a closed path $z(t)$ ($t \in [0,1]$ and $z(0) = z(1) = 0$) in $\mathbb{C}$ which surrounds only the point -1, then we obtain the relation $g_0 = g$. If we consider a path which surrounds only the point $-3/2$, then we obtain the relations $g_1gg_1 = gg_1g$ and $g_0g_2g_0 = g_2g_0g_2$. Therefore $\pi_1(Y_W/W)$ is the Artin group of the type $\tilde{A}_2$. Moreover $\Phi^{-1}(a_0)$, $\Phi^{-1}(a)$ $\in H_{s_0}/W$, $\Phi^{-1}(a_1) \in H_{s_1}/W$ and $\Phi^{-1}(a_2) \in H_{s_2}/W$. Therefore the correspondence between the generators of W and $\pi_1(Y_W/W)$ is natural. Thus we have proved the Theorem for the Case 1.

Case 2. W = affine Weyl group of type $\tilde{B}_2$. Then $S = \{s_0, s_1, s_2\}$ and $m(s_1,s_0) = 2$ and $m(s_1,s_2) = m(s_0,s_2) = 4$. By Proposition 2 we can calculate $\pi_1(Y_W/W)$ by the method of Zariski[3]. In the equation $h(z) = 0$ in Proposition 2, z_2 has four distinct solutions except for $z_1 = 0$, 4 and -4. As the generators of the fundamental group, we can choose g_0, g_1, g_2 and g in the $z_1 = i/4$ plane as the figure 2 shows. Here the base point is taken far enough from the origin 0.

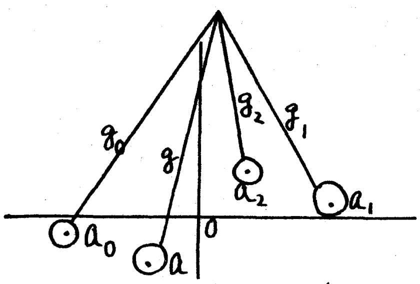

figure 2 (z_1= i/4 plane)

If we move z_1 along a closed path $z(t)$ ($t \in [0,1]$ and $z(0) = z(1) = i/4$) in $\mathbb{C}$ which surrounds only the point 0 , then we obtain the relation $g_2 = g$. If we move z_1 along a closed path $z(t)$ ($t \in [0,1]$ and $z(0) = z(1) = i/4$) in $\mathbb{C}$ which surrounds only the point 4, then we obtain the relations $g_1g_2g_1g_2 = g_2g_1g_2g_1$ and $g_0gg_0g = gg_0gg_0$. If we consider a path which surrounds only the point -4, then we obtain the relation $g_1g_0 = g_0g_1$. Therefore $\pi_1(Y_W/W)$ is the Artin group of the type $\widetilde{B}_2$. Moreover we have $\Phi^{-1}(a_0) \in H_{s_0}/W$, $\Phi^{-1}(a_1) \in H_{s_1}/W$ and $\Phi^{-1}(\{a, a_1\}) \in H_{s_2}/W$. Therefore the correspondence betweenthe generators of W and $\pi_1(Y_W/W)$ is natural. Thus we have proved the Theorem for the Case 2.

Case 3. W = affine Weyl group of type G_2. Then $S = \{s_0, s_1, s_2\}$ and $m(s_0,s_1) = 3$, $m(s_0,s_2) = 2$ and $m(s_1,s_2) = 6$. By Proposition 3 we can calculate $\pi_1(Y_W/W)$ by the method of Zariski [3]. In the equation $h(z) = 0$ in Proposition 3, z_2 has three distinct solutions except for $z_1 = -2, -3$ and 6. As the generators of the fundamental group, we can choose g_0, g_1, and g_2 in the $z_1 = 0$ plane as the figure 3 shows. Here the base point is taken far enough from the origin 0.

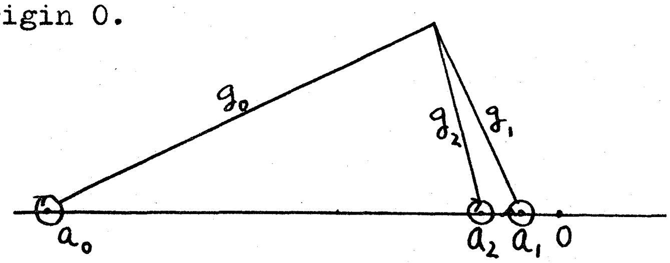

figure 3 ($z_1 = 0$ plane)

If we move z_1 along a closed paht $z(t)$ ($t \in [0,1]$ and $z(0) = z(1) = 0$) in $\mathbb{C}$ which surrounds only the point -2, then we obtain the relation $g_0g_2 = g_2g_0$. If we consider a path surrounding only the point -3, then we obtain the relation $g_0 g_1 g_0 = g_1 g_0 g_1$. If we consider a path surrounding only the point 6, then we obtain the relation $(g_1 g_2)^3 = (g_2 g_1)^3$. Therefore $\pi_1(Y_W/W)$ is the Artin group of the type G_2. Moreover we have $\Phi^{-1}(a_0) \in H_{s_0}/W$, $\Phi^{-1}(a_1) \in H_{s_1}/W$ and $\Phi^{-1}(a_2) \in H_{s_2}/W$. Therefore the correspondence between the generators of W and $\pi_1(Y_W/W)$ is natural. Thus we have proved the Theorem for the Case 3. This completes the proof of Theorem.

References

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