

FROM THE DELIGNE-IHARA CONJECTURE TO MULTIPLE MODULAR VALUES.

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1. INTRODUCTION

I first met Professor Ihara at a conference in Kyoto on Galois-Teichmüller theory in 2010. My talk was one of the very last of the conference and had a particularly uninspiring title. I was greatly honoured that Professor Ihara patiently attended my talk, in which I chaotically sketched a proof of the conjecture attributed to him and Deligne. It has been a huge privilege, and a great pleasure, to return to Kyoto in 2018 and participate in the celebrations for his 80th birthday conference.

Whilst already mindful of the profound impact of Ihara's ideas on my own work, I was particularly struck at the conference by the scale of his influence over number theory as a whole, especially in Japan. In this talk, I shall only focus on a tiny fraction of Ihara's impressive legacy, namely his work on the projective line minus three points. It has two, related, strands:

- (1) (Genus 0). I will report on recent progress related to the Deligne-Ihara conjecture and raise some new questions inspired by it (§2,3).
- (2) (Genus 1). I will describe in §4 the main features of a nascent theory in genus one, and explain how it relates to the Ihara-Takao relation, where modular forms make their first unexpected appearance. Paragraphs §§5-8 are devoted to numerical examples to illustrate the main features of this somewhat abstract theory and make it more widely accessible.

In the sequel to this talk, which is logically independent from it, I explain how the periods associated to (1) and (2), namely multiple zeta values, and multiple modular values, can be subsumed into a more general definition of multiple L -values.

2. A BRIEF HISTORY OF THE DELIGNE-IHARA CONJECTURE

2.1. The Deligne-Ihara conjecture (pro- ℓ version). In 1984, Grothendieck [28] proposed studying the profinite completion of the fundamental group of the projective line minus three points $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ with its outer action of $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Shortly afterwards, in a series of influential papers [34, 35, 36, 37], Ihara initiated the study of the Galois action on the pro- ℓ completion:

$$\rho_{\ell} : G_{\mathbb{Q}} \longrightarrow \text{Out } \pi_1^{(\ell)}(X, x)$$

where $\pi_1^{(\ell)}(X, x)$ is the inverse limit of all finite quotients of $\pi_1(X(\mathbb{C}), x)$ whose order is a power of any prime ℓ . This action does not depend on the base point x . Ihara used it to define, for every ℓ , a decreasing filtration on the absolute Galois group itself:

$$G_{\mathbb{Q}} = I_{\ell}^0 G_{\mathbb{Q}} \supseteq I_{\ell}^1 G_{\mathbb{Q}} \supseteq I_{\ell}^2 G_{\mathbb{Q}} \supseteq \dots$$

Elements in the subgroup I_{ℓ}^k are those which act as inner automorphisms on the $(k+1)^{\text{th}}$ -step in the lower central series filtration on $\pi_1^{(\ell)}(X, x)$. The first graded quotient in this filtration gives back the ℓ -adic cyclotomic character:

$$\text{gr}_{I_{\ell}}^0 G_{\mathbb{Q}} \cong \mathbb{Z}_{\ell}^{\times}.$$

Ihara showed that $[I_\ell^p, I_\ell^q] \subset I_\ell^{p+q}$ and defined the associated graded object

$$\mathfrak{g}^\ell = \bigoplus_{m \geq 1} \mathrm{gr}_{I_\ell}^m G_{\mathbb{Q}} ,$$

which is a Lie algebra over $\mathbb{Z}_\ell$. He proved that the degree 1 part of this Lie algebra contains the images of Soulé characters $\sigma_3^\ell, \sigma_5^\ell, \dots$ indexed by every odd integer ≥ 3 , and made the following conjecture, jointly attributed to Deligne ([36], p. 600) see also ([27] Conj. 2.1, [21], p. 859):

Conjecture 1. (Deligne-Ihara.) The Lie algebra $\mathfrak{g}^\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is freely generated by the elements σ_{2n+1}^ℓ , for $n \geq 1$.

What makes this subject so compelling is a tension between the freeness in the previous conjecture, and the existence of infinitely many ‘near-relations’, the first of which occurs in weight 12 and is due to Ihara and Takao:

$$(2.1) \quad \left[\sigma_3^\ell, \sigma_9^\ell \right] - 3 \left[\sigma_5^\ell, \sigma_7^\ell \right] = \frac{691}{144} \delta_{12}^\ell .$$

It turns out that δ_{12}^ℓ , which can be represented as an element in a free Lie algebra in two non-commuting elements, is nearly zero: almost all its coefficients vanish (see [38] and [10], §8 for a precise statement). This clearly makes life difficult if one wants to prove conjecture 1, which implies that δ_{12}^ℓ is non-zero. Furthermore, (2.1) becomes an actual relation in two different ways: firstly modulo the prime 691, and secondly modulo the depth filtration, since δ_{12}^ℓ vanishes, unexpectedly, in depths two (and even three). Therefore the modulo ℓ and depth-graded analogues of conjecture 1 are false: neither $\mathfrak{g}^\ell \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell$, nor $\mathrm{gr}_{\mathcal{D}}(\mathfrak{g}^\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$ is freely generated by the σ_{2n+1}^ℓ . The element δ_{12}^ℓ seems to be related in a fundamental way to the existence of the first non-zero cusp form for $\mathrm{SL}_2(\mathbb{Z})$ in weight 12. Much of the work described in this talk has been devoted to gaining a better understanding of this fascinating phenomenon.

2.2. Motivic fundamental group. A solution to the Deligne-Ihara conjecture came from a different part of mathematics: namely the theory of periods and multiple zeta values. The first step is to formulate a motivic generalisation of conjecture 1 and recast it in a different realisation.

Based on Deligne’s foundational work [18] on realisations of the motivic fundamental group of $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and subsequent advances in the theory of motives, Deligne and Goncharov [20] defined a motivic fundamental groupoid:

$$\pi_1^{\mathrm{m}}(X, \overset{\rightarrow}{1}_0, -\overset{\rightarrow}{1}_1)$$

as a pro-object in the abelian category of mixed Tate motives over $\mathbb{Z}$, denoted $\mathcal{MT}(\mathbb{Z})$. The basepoints are tangent vectors of length 1 (respectively -1) at the point 0 (respectively 1). One reason for these apparently obscure basepoints is that X has no ordinary points over $\mathbb{Z}$. Another reason is that they formalise the logarithmic regularisation of divergent integrals. As a first approximation, the reader can pretend that they are the points 0 and 1 respectively, although these points lie in $\mathbb{P}^1$ and not in fact in X .

The motivic fundamental groupoid admits the following classical realisations:

- (Betti). Its Betti realisation is the scheme over $\mathbb{Q}$ defined by the unipotent completion of the topological fundamental torsor of paths:

$$\pi_1(X, \overset{\rightarrow}{1}_0, -\overset{\rightarrow}{1}_1) .$$

It contains a distinguished path dch which is given by the straight line from 0 to 1 along the real axis. The k -points of the Betti fundamental groupoid are given by the set of group-like formal power series $S \in k\langle\langle x_0, x_1 \rangle\rangle$ in two non-commuting variables x_0, x_1 , corresponding to loops around 0, 1.

- (de Rham). The de Rham realisation is dual to the $\mathbb{Q}$ -vector space of iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. More precisely, its affine ring is isomorphic to the graded tensor coalgebra over $\mathbb{Q}$ on generators denoted by

$$\omega_0 = \frac{dz}{z} \quad , \quad \omega_1 = \frac{dz}{1-z} \quad \in \quad \Gamma(\mathbb{P}^1, \Omega_{\mathbb{P}^1/\mathbb{Q}}^1(\log\{0, 1, \infty\})) \quad .$$

Its k -points are also isomorphic to the set of group-like formal power series $S \in k\langle\langle e_0, e_1 \rangle\rangle$ in two non-commuting variables e_0, e_1 dual to ω_0, ω_1 .

- (ℓ -adic). The set of $\mathbb{Q}_\ell$ -points of its ℓ -adic realisation $\pi_1^\ell(X, \vec{1}_0, -\vec{1}_1)(\mathbb{Q}_\ell)$ is isomorphic to the group-like formal power series in two variables $\mathbb{Q}_\ell\langle\langle x_0, x_1 \rangle\rangle$ with a continuous action of $G_{\mathbb{Q}}$. There is a natural isomorphism of $\mathbb{Q}_\ell$ -schemes ([31], §A):

$$\pi_1^\ell(X, \vec{1}_0, -\vec{1}_1) \cong \left(\pi_1^{(\ell)}(X, \vec{1}_0, -\vec{1}_1) \right)^\text{un}$$

where the right-hand side is the (continuous) unipotent completion of the pro- ℓ completion. The pro- ℓ completion of the topological fundamental groupoid is Zariski-dense in the $\mathbb{Q}_\ell$ -points of the latter.

The Betti and de Rham realisations are related via the period isomorphism, and the theory of iterated integrals. More precisely, the periods of the motivic fundamental groupoid include the regularised iterated integrals from 0 to 1:

$$\int_{\text{dch}} \omega_{i_1} \dots \omega_{i_n} \quad \text{for} \quad i_1, \dots, i_n \in \{0, 1\} \quad .$$

By a remark due to Kontsevich, these can be expressed as $\mathbb{Q}$ -linear combinations of multiple zeta values, which are defined for $n_1, \dots, n_{r-1} \geq 1$ and $n_r \geq 2$ by

$$(2.2) \quad \zeta(n_1, \dots, n_r) = \sum_{1 \leq k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}}$$

and go back to Euler, at least for $r = 2$. The ℓ -adic theory, related to Betti via the Betti- ℓ -adic comparison isomorphism, enables us to make the connection with conjecture 1 after taking x to be one of the above tangential basepoints. There is also a crystalline theory which is related to the de Rham realisation via p -adic periods. The motivic structure of the fundamental groupoid is reflected in all of these realisations, but the theory of periods is perhaps the most accessible. For example, the Ihara-Takao relation is related to (but not in fact directly equivalent to) the following elementary relation between double zeta values which first occurs in weight 12

$$(2.3) \quad 28\zeta(3, 9) + 150\zeta(5, 7) + 168\zeta(7, 5) = \frac{5197}{691}\zeta(12) \quad .$$

This relation is also directly related to the Ramanujan cusp form of weight 12. It could have been discovered by Euler, but was found much more recently [24].

2.3. Motivic reformulation of the Deligne-Ihara conjecture. Since mixed Tate motives form a Tannakian category, the information contained in the motivic fundamental groupoid is entirely encoded by any one of its realisations, together with the action of the corresponding Tannaka group. For example,

$$\pi_1^\bullet(X, \vec{1}_0, -\vec{1}_1) \quad \text{with its action of} \quad G_{\mathcal{MT}(\mathbb{Z})}^\bullet = \text{Aut}_{\mathcal{MT}(\mathbb{Z})}^\otimes(\omega_\bullet)$$

where $\bullet \in \{B, dR, \ell\}$. It is known by the general theory, and most crucially Borel's theorems [3, 4] on the algebraic K -theory of $\mathbb{Q}$, that $G_{\mathcal{MT}(\mathbb{Z})}^\bullet$ is an extension

$$1 \longrightarrow U_{\mathcal{MT}(\mathbb{Z})}^\bullet \longrightarrow G_{\mathcal{MT}(\mathbb{Z})}^\bullet \longrightarrow \mathbb{G}_m \longrightarrow 1$$

of the multiplicative group by a pro-unipotent group whose graded Lie algebra is the free Lie algebra on non-canonical generators $\sigma_3^\bullet, \sigma_5^\bullet, \dots$ in every odd degree $-3, -5, \dots$. In the ℓ -adic case, there is a canonical continuous Zariski-dense homomorphism

$$G_{\mathbb{Q}} \longrightarrow G_{\mathcal{MT}(\mathbb{Z})}^{\ell}(\mathbb{Q}_{\ell}).$$

It is easier to work in the de Rham setting, where the above exact sequence is canonically split. The following theorem was conjectured by Goncharov and proved in [6].

Theorem 2.1. (*Motivic version of the Deligne-Ihara conjecture*). *The affine group scheme $G_{\mathcal{MT}(\mathbb{Z})}^{dR}$ acts faithfully on $\pi_1^{dR}(X, \vec{1}_0, -\vec{1}_1)$.*

Equivalently, the $\{\sigma_{2n+1}^{dR}\}$ act freely on $\pi_1^{dR}(X, \vec{1}_0, -\vec{1}_1)$. It follows that

$$G_{\mathcal{MT}(\mathbb{Z})}^{\bullet} \text{ acts faithfully on the group scheme } \pi_1^{\bullet}(X, \vec{1}_0)$$

in any realisation $\bullet$. Taking $x = \vec{1}_0$ and $\bullet = \ell$ proves conjecture 1.

2.4. Classification of mixed Tate motives over $\mathbb{Z}$. The previous theorem implies that every mixed Tate motive over $\mathbb{Z}$ is classified in the sense that it appears inside the motivic fundamental groupoid of the projective line minus 3 points.

Corollary 2.2. *The motivic fundamental groupoid $\pi_1^m(X, \vec{1}_0, -\vec{1}_1)$ generates the category $\mathcal{MT}(\mathbb{Z})$. In other words, every mixed Tate motive over $\mathbb{Z}$ is isomorphic to a direct sum of Tate twists of subquotients of its affine ring.*

Such a classification theorem has applications. For instance:

Corollary 2.3. *The periods of any mixed Tate motive over $\mathbb{Z}$ are $\mathbb{Q}[(2\pi i)^{-1}]$ -linear combinations of multiple zeta values.*

There are many examples of integrals, notably in high-energy physics, which are extremely hard to compute. When the underlying geometry is mixed Tate over $\mathbb{Z}$, one can deduce without computation that the integral is a multiple zeta value.

2.5. Questions. In this talk, we describe a precise relationship between cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ and the fundamental group of the projective line minus three points. This is directly inspired by the Ihara-Takao relation.

In the sequel to this talk, we ask whether modular forms are intrinsically related to mixed Tate motives at all, or whether this is merely an artefact of the way in which they are generated via $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. More specifically, we consider whether there exists a more natural way to construct the periods of mixed Tate motives without reference to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, which is seemingly pulled out of thin air.

3. AN UPDATE ON THE DELIGNE-IHARA CONJECTURE

Much of the combinatorial difficulty in understanding the objects of the previous paragraph is that the indexation is poorly adapted to the underlying motivic structure. I explain how this can be rectified, indicate some of the ingredients in the proof of conjecture 1, and then turn to the depth-defect and Ihara-Takao relation. Some of the work described here has occurred since the paper [6] and the surveys [19, 8] appeared.

3.1. Decomposition of multiple zeta values. One of the crucial ingredients in the proof of conjecture 1 is the notion of motivic period (or motivic multiple zeta value). We shall not spend much time motivating or defining this concept here, referring instead to [8], [9]. In short, a motivic multiple zeta value is formally defined for $n_1, \dots, n_{r-1} \geq 1$, $n_r \geq 2$ to be an equivalence class

$$\zeta^{\mathfrak{m}}(n_1, \dots, n_r) = [\mathcal{O}(\pi_1^{\mathfrak{m}}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \vec{1}_0, -\vec{1}_1)), \omega_{\underline{n}}, \text{dch}]^{\mathfrak{m}}$$

which cuts out a ‘piece’ of the affine ring of the fundamental groupoid. The form $\omega_{\underline{n}}$ is the integrand of the expression for $\zeta(n_1, \dots, n_r)$ as an integral along dch . These symbols satisfy the standard algebraic relations for multiple zeta values. They refine an earlier notion due to Goncharov, for which the analogue of $\zeta(2)$ vanishes.

We only need to know that motivic multiple zeta values form a $\mathbb{Q}$ -algebra $\mathcal{H}$, which is graded by the weight $n_1 + \dots + n_r \geq 0$, and admits a period homomorphism:

$$\begin{aligned} \text{per} : \mathcal{H} &\longrightarrow \mathbb{R} \\ \zeta^{\mathfrak{m}}(n_1, \dots, n_r) &\mapsto \zeta(n_1, \dots, n_r) . \end{aligned}$$

It is conjectured to be injective. Motivic multiple zeta values come equipped with a certain ‘motivic’ coaction, which is an analogue of Ihara’s formula for the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the pro- ℓ fundamental group. It takes the form

$$\begin{aligned} \Delta : \mathcal{H} &\longrightarrow \mathcal{H} \otimes_{\mathbb{Q}} \mathcal{A} \\ \Delta(\zeta^{\mathfrak{m}}(n_1, \dots, n_r)) &= \sum_I c_I \zeta^{\mathfrak{m}}(a'_I) \otimes \zeta^{\text{dr}}(a_I) \end{aligned}$$

where the sum is over some indexing set I , the $c_I \in \mathbb{Q}$, and ζ^{dr} denote ‘de Rham’ zeta values, which are defined in a similar manner to motivic multiple zeta values and can be thought of as ‘motivic multiple zeta’s modulo $\zeta^{\mathfrak{m}}(2)$ ’. They generate an algebra $\mathcal{A}$ isomorphic to $\mathcal{H}/\zeta^{\mathfrak{m}}(2)\mathcal{H}$. There exists a formula for this coaction, generalising an earlier one due to Goncharov, which *a posteriori* turns out to be exactly dual to a formula due to Ihara (see [8] for a quick derivation of the coaction formula from Ihara’s). One can compute the coaction (or parts of it) in practice [11].

As a rough first approximation, one can think of a motivic multiple zeta value as a *matrix of multiple zeta values* whose entries satisfy certain conditions, together with a distinguished entry. The period homomorphism picks out the distinguished entry; the coaction formula is obtained by deleting rows and columns from the matrix.

Using the coaction, together with some structure theorems on the category of mixed Tate motives, we can decompose a motivic multiple zeta value into simpler pieces.

Theorem 3.1. *Let $\mathbb{Q}\langle f_3, f_5, \dots \rangle$ denote the graded $\mathbb{Q}$ -vector space generated by words in non-commuting symbols $f_3, f_5, \dots$ of weights $3, 5, \dots$ with the shuffle product. Let f_2 denote a symbol of weight 2 which commutes with the symbols f_{2n+1} with odd indices. There is a non-canonical injective homomorphism*

$$(3.1) \quad \phi : \mathcal{H} \longrightarrow \mathbb{Q}\langle f_3, f_5, \dots \rangle \otimes_{\mathbb{Q}} \mathbb{Q}[f_2]$$

which satisfies $\phi(\zeta^{\mathfrak{m}}(2)) = f_2$ and

$$\phi(\zeta^{\mathfrak{m}}(2n+1)) = f_{2n+1} \quad \text{for all } n \geq 1 .$$

In particular, every motivic multiple zeta value admits, via ϕ , a decomposition into the alphabet of letters f_{2n+1} which respects all algebraic relations between the motivic multiple zeta values, and determines it uniquely.

As far as I know, an f -alphabet representation is presently the only clear way to navigate the complicated relations satisfied by multiple zeta values. This language will be used in an essential way when we discuss multiple modular values.

The number of letters of odd weight f_{odd} in a word defines an increasing filtration called the *coradical* or *unipotency* filtration on $\mathcal{H}$ which we denote by C . It is well-defined (a conceptual and more general definition is given in [9], §2.5).

Example 3.2. The map ϕ depends on some choices. For some such choice,

$$\begin{aligned}\phi(\zeta^{\mathfrak{m}}(3, 3)) &= f_3 f_3 - \frac{4}{35} f_2^3 \\ \phi(\zeta^{\mathfrak{m}}(3, 5)) &= -5 f_3 f_5 \\ \phi(\zeta^{\mathfrak{m}}(5, 5)) &= f_5 f_5 - \frac{16}{385} f_2^5 \\ \phi(\zeta^{\mathfrak{m}}(7, 3)) &= 15 f_3 f_7 + 6 f_5 f_5 + f_7 f_3 - \frac{32}{385} f_2^5\end{aligned}$$

For details on how to compute these formulas, see [11]. The part of highest length in the symbols f_{odd} is canonically defined, i.e., the rational coefficient of the power of f_2 depends on the choice of the homomorphism ϕ , but not the rest. For example,

$$\text{gr}_2^C(\phi)(\zeta^{\mathfrak{m}}(7, 3)) = 15 f_3 f_7 + 6 f_5 f_5 + f_7 f_3$$

is independent of the choice of map ϕ . The interpretation of this formula is that $\zeta(7, 3)$ is a period of a rank 5 iterated extension of the following Tate motives:

$$\mathbb{Q}, \mathbb{Q}(-3), \mathbb{Q}(-5), \mathbb{Q}(-7), \mathbb{Q}(-10),$$

i.e., $\zeta(7, 3)$ is an entry in a corresponding 5×5 period matrix. The main point of motivic periods is that they enable us to manipulate such complicated objects with great ease, avoiding lengthy computations in homological algebra.

3.2. The proof of the Deligne-Ihara conjecture and recent developments.

Theorem 3.3. *Any map (3.1) is surjective. In other words, there is an isomorphism*

$$\phi : \mathcal{H} \xrightarrow{\sim} \mathbb{Q}\langle f_3, f_5, \dots \rangle \otimes_{\mathbb{Q}} \mathbb{Q}[f_2].$$

It follows that every real, effective motivic period of a mixed Tate motive over $\mathbb{Z}$ arises as a motivic multiple zeta value. We deduce that every such motive occurs in the fundamental groupoid of X . Theorem 2.1 follows from Tannakian arguments.

Some comments:

- The proof of surjectivity in [6] goes by showing that the Hoffman elements

$$(3.2) \quad \zeta^{\mathfrak{m}}(n_1, \dots, n_r) \quad \text{where} \quad n_i \in \{2, 3\}$$

have independent images under any ϕ . They therefore form a basis for $\mathcal{H}$, which, by applying the period map per , implies a conjecture due to Hoffman [33] stating that the $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$ span the space of multiple zetas over $\mathbb{Q}$. The proof uses in an essential way a theorem due to Zagier [51] who evaluated $\zeta(2, \dots, 2, 3, 2, \dots, 2)$ as an explicit linear form in $\zeta(2n+1)$ whose coefficients are rational multiples of powers of π . The coefficients in this linear form play an important role in the proof. Zagier's original proof was a highly ingenious combination of combinatorial and analytic methods. There have subsequently been several new proofs of this theorem (e.g., [41], [32], [49]).

In her thesis, Claire Glanois [25] constructed a new basis for $\mathcal{H}$ using a clever variant of motivic Euler sums (iterated integrals on $\mathbb{P}^1 \setminus \{0, -1, 1, \infty\}$) which actually turn out to be motivic multiple zeta values.

- The proof is by induction on a filtration by the ‘number of 3’s’ on the elements (3.2). This filtration, and the Hoffman basis itself, seems mysterious at first sight, but is in fact naturally induced by a ‘block filtration’ on motivic multiple zeta values [5] which is the filtration associated to the block decomposition introduced by S. Charlton [14]. Adam

Keilthy has shown in his forthcoming Ph.D thesis that the block filtration extends to all motivic iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and proved many beautiful properties of it.

- It is better to rephrase the previous theorem after taking the associated graded for the coradical filtration. Thus theorem 3.3 is equivalent to proving that the canonical homomorphism $\mathrm{gr}^C \phi$ is an isomorphism:

$$(3.3) \quad \mathrm{gr}^C(\phi) : \mathrm{gr}^C \mathcal{H} \xrightarrow{\sim} \mathrm{gr}^C \mathbb{Q}(f_3, f_5, \dots) \otimes_{\mathbb{Q}} \mathbb{Q}[f_2] .$$

Stated in this form, the proof can be simplified, since several of the arguments actually hold for general motivic periods, as explained in [9] .

- The proof used a mysterious 2-adic argument which can be explained using the 2-adic Frobenius on the $\mathbb{Z}_2$ -module generated by the motivic Hoffman basis. One needs to apply the infinitesimal coaction to elements (3.2) followed by $\mathrm{per}_2 \otimes \mathrm{id} \pmod{2}$, where per_2 is the 2-adic period homomorphism on de Rham zeta values. The key observation is that

$$v_2(\zeta_2(3, 2, \dots, 2)) = 1$$

where $\zeta_2(3, 2, \dots, 2) = \mathrm{per}_2 \zeta^{\mathrm{dr}}(3, 2, \dots, 2) \in \mathbb{Z}_2$, and v_2 denotes the 2-adic valuation. This statement follows from the calculation $v_2(\zeta_2(2k+1)) = v_2(\frac{4^k}{2k})$ which can be proved using a formula due to Ünver [50] §5.11. This will be discussed elsewhere.

- Deligne has analogous results for the projective line minus certain roots of unity [17]. See also Glanois [25] for further progress in this direction. These are situations in which there is no Ihara-Takao phenomenon. As a result, the motivic structure of the fundamental groupoid is much simpler than for the projective line minus three points.

3.3. Inverse problem. Given a mixed Tate motive over $\mathbb{Z}$ with prescribed weights, what are its possible periods? To answer this question, one must compute the *inverse* of (3.3). Equivalently, given a word w in the f_{2n+1} , find a linear combination of motivic multiple zeta values whose image under ϕ is w to leading order.

Definition 3.4. Let $a_1, \dots, a_r \geq 1$. Let us denote by

$$\zeta_{2a_1+1, \dots, 2a_r+1}^{\mathfrak{m}} \in C_r \mathcal{H}$$

any motivic multiple zeta value of weight $2a_1 + \dots + 2a_r + r$ whose leading term in any f -alphabet decomposition (3.1) is $f_{2a_1+1} \dots f_{2a_r+1}$. In other words,

$$\mathrm{gr}_r^C \phi(\zeta_{2a_1+1, \dots, 2a_r+1}^{\mathfrak{m}}) = f_{2a_1+1} \dots f_{2a_r+1} .$$

By theorem 3.3 it exists. It is only well-defined up to motivic multiple zeta values of coradical filtration $\leq r-1$ (in fact, one can do better: $\leq r-2$). We shall write

$$\zeta_{2a_1+1, \dots, 2a_r+1} = \mathrm{per}(\zeta_{2a_1+1, \dots, 2a_r+1}^{\mathfrak{m}})$$

for some choice of element $\zeta_{2a_1+1, \dots, 2a_r+1}^{\mathfrak{m}}$. It is only well-defined modulo multiple zeta values of coradical filtration $\leq r-2$. We have $\zeta_{2n+1} = \zeta(2n+1)$.

Examples 3.5. The usual indexation for multiple zeta values (and especially the Hoffmann multiple zeta values) are very ill-suited to this problem, as a glance at the following examples shows. Here are some choices of representatives for $\zeta_{a,b}$:

$$(3.4) \quad \begin{aligned} \zeta_{3,3} &= \zeta(3,3) \\ \zeta_{5,3} &= -\frac{1}{5}\zeta(3,5) \\ \zeta_{3,7} &= \zeta(3)\zeta(7) + \frac{1}{14}(\zeta(3,7) + 3\zeta(5)^2) \end{aligned}$$

They are well-defined up to addition of a power of π . All hell breaks loose in weight twelve, since we are forced to include multiple zeta values of depth four:

$$(3.5) \quad \zeta_{3,9} = \frac{1}{19.691} \left(2^4 3^2 \zeta(5, 3, 2, 2) - \frac{3^3 5.179}{2.7} \zeta(5, 7) - 2 \cdot 3^3 29 \zeta(7, 5) \right. \\ \left. - 3 \cdot 7^2 \zeta(3) \zeta(9) + 2^4 3 \zeta(3)^4 + 2^5 3^3 11 \zeta(3, 7) \zeta(2) + 2^5 3^2 31 \zeta(7, 3) \zeta(2) \right. \\ \left. - 2^4 3^4 \zeta(3, 5) \zeta(4) - 2^5 3^2 \zeta(5, 3) \zeta(4) - 2^3 3 \cdot 5^2 \zeta(3)^2 \zeta(6) + \frac{3.128583229}{247.691} \zeta(12) \right)$$

This number is only well-defined modulo $\pi^{12} \mathbb{Q}$ (but this particular representative will appear as a multiple modular value later). The problem ‘write down the periods of all mixed Tate motives over $\mathbb{Z}$ with three weight-graded pieces $\mathbb{Q}, \mathbb{Q}(-3), \mathbb{Q}(-12)$ ’ is therefore surprisingly complicated. A period matrix of such an object can be written in the form

$$\begin{pmatrix} 1 & \zeta_3 & \zeta_{3,9} \\ 0 & (2\pi i)^3 & (2\pi i)^3 \zeta_9 \\ 0 & 0 & (2\pi i)^{12} \end{pmatrix}$$

with respect to suitable bases. This illustrates the combinatorial complexity which implicitly hides behind theorem 2.1, and is due to the Ihara-Takao phenomenon.

3.4. Ihara-Takao and the depth-defect. The depth filtration on motivic multiple zeta values is compatible with the coradical filtration:

$$\zeta^{\mathfrak{m}}(n_1, \dots, n_r) \in C_r \mathcal{H}.$$

Thus, an f -alphabet decomposition (3.1) of a motivic multiple zeta value of depth r involves words in the letters f_{2n+1} of length at most r . Alas, the converse is false: the depth filtration is strictly smaller than the coradical filtration. This phenomenon first becomes visible in weight 12, where it is related to the fact that δ_{12}^f in (2.1) vanishes in depth 2. For example, one can show that $\zeta_{3,9}$ and $\zeta_{5,7}$ cannot individually be expressed as multiple zeta values of depth 2. Only the following linear combination lies in the subspace generated by double zeta values:

$$(3.6) \quad \zeta_{5,7} + 3 \zeta_{3,9} = \frac{1}{9} \zeta(3, 9) + 3 \zeta(3) \zeta(9) + \frac{5}{3} \zeta(5) \zeta(7) - \frac{31.139}{2.691} \zeta(12).$$

The expression on the left-hand side is dual to the left-hand side of (2.1).

4. A SKETCH OF THE THEORY IN GENUS ONE

The element δ_{12}^f in the Ihara-Takao relation (2.1) occurs in the same weight as the first cusp form for $\mathrm{SL}_2(\mathbb{Z})$. This suggests studying the fundamental group of the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves, which, together with $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$, comprise the two ‘generators’ in Grothendieck’s tower of moduli spaces $\mathcal{M}_{g,n}$ of curves of genus g with n marked points [28].

The analytic space associated to $\mathcal{M}_{1,1}$ is the orbifold quotient

$$\mathcal{M}_{1,1}(\mathbb{C}) = \Gamma \backslash \mathfrak{H}$$

where $\mathfrak{H}$ is the upper half plane $\mathfrak{H} = \{\tau \in \mathbb{C} : \mathrm{Im} \tau > 0\}$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. A natural choice for a basepoint is the unit tangent vector at the cusp, denoted by

$$(4.1) \quad \partial/\partial q \quad \text{or} \quad \vec{1}_\infty$$

where $q = \exp(2\pi i \tau)$. The former denotes the unit tangent vector on the punctured q -disc and induces a tangential basepoint on $\mathcal{M}_{1,1}$. We tend to use the latter notation to denote a tangential basepoint on the universal covering, i.e., a tangent vector of length 1 on the tangent space at $\tau = i\infty$ on the compactified upper half plane. They coincide on $\mathcal{M}_{1,1}(\mathbb{C})$. This basepoint has an

infinite Γ -stabiliser. Other natural choices of basepoint include the images of the points $\tau = i$ or $\tau = e^{2\pi i/3}$ whose Γ -stabilisers have order 4 and 6 respectively. Better still, one can consider the groupoid of all three basepoints simultaneously and the paths between them.

There is a canonical isomorphism

$$\pi_1(\mathcal{M}_{1,1}(\mathbb{C}); \vec{1}_\infty) = \Gamma.$$

The first problem one encounters is that, although the profinite completion of $\mathrm{SL}_2(\mathbb{Z})$ is huge (e.g., by Belyi's theorem [2]), its unipotent completion is trivial:

$$\pi_1^{\mathrm{un}}(\mathcal{M}_{1,1}(\mathbb{C}); \vec{1}_\infty) = 1$$

There are many ways to see this: perhaps the simplest is that $H^1(\mathcal{M}_{1,1}; \mathbb{Q})$ vanishes as there are no modular forms of full level in weight 2. The remedy is to consider not the unipotent completion but a generalisation called the *relative completion* [29] of Γ with respect to the inclusion $\Gamma \subset \mathrm{SL}_2(\mathbb{Q})$. This is a pro-algebraic group over $\mathbb{Q}$

$$\mathcal{G}_{1,1}^B = \pi_1^{\mathrm{rel}}(\mathcal{M}_{1,1}; \vec{1}_\infty)$$

which is an extension of the algebraic group SL_2

$$1 \longrightarrow \mathcal{U}_{1,1}^B \longrightarrow \mathcal{G}_{1,1}^B \longrightarrow \mathrm{SL}_2 \longrightarrow 1$$

by a pro-unipotent algebraic group $\mathcal{U}_{1,1}^B$. It admits a Zariski-dense homomorphism $\Gamma \rightarrow \mathcal{G}_{1,1}^B(\mathbb{Q})$. It can be thought of as an algebraic envelope of Γ that is neither too small (as in the case of the unipotent completion) nor too large (as in the case of the pro-algebraic fundamental group), but lies in a Goldilocks zone in-between.

- The relative completion has various realisations $\mathcal{G}_{1,1}^\bullet, \mathcal{U}_{1,1}^\bullet$ which have a natural Tannakian description. In each case, one considers a certain Tannakian category of sheaves on $\mathcal{M}_{1,1}$ (e.g., local systems, or algebraic vector bundles with integrable connection) which have an increasing filtration with graded pieces of a specified type. One defines $\mathcal{G}_{1,1}^\bullet$ to be the group of tensor automorphisms of the functor ‘fiber at (4.1)’. Most of the properties of $\mathcal{G}_{1,1}^\bullet$ follow automatically. Since unipotent completion is a special case of relative completion, this construction also applies in the case of the projective line minus three points.
- The de Rham description is particularly accessible. One shows [30] that $\mathfrak{u}_{1,1}^{dR} = \mathrm{Lie} \mathcal{U}_{1,1}^{dR}$ is isomorphic to the completed free Lie algebra generated by

$$\prod_{n \geq 0} H_{dR}^1(\mathcal{M}_{1,1}; \mathbb{V}_n^{dR})^\vee \otimes V_n^{dR}$$

where $\mathbb{V}_n^{dR} = \mathrm{Sym}^n \mathcal{H}$, and $\mathcal{H} = R^1 \pi_* \Omega_{\mathcal{E}/\mathcal{M}_{1,1}}^\bullet$ where $\pi : \mathcal{E} \rightarrow \mathcal{M}_{1,1}$ is the universal elliptic curve. It is an algebraic vector bundle with the Gauss-Manin connection. Here V_n^{dR} denotes the fiber of $\mathbb{V}_n^{dR}$ at the basepoint (4.1). In the theory of modular forms, one often identifies

$$V_n^{dR} = \bigoplus_{i+j=n} X^i Y^j \mathbb{Q}$$

with the space of homogeneous polynomials of degree n in two variables X and Y , with its natural right SL_2 -action. By the Eichler-Shimura theorem,

$$\dim_{\mathbb{Q}} H_{dR}^1(\mathcal{M}_{1,1}; \mathbb{V}_n^{dR}) = 1 + 2 \dim_{\mathbb{C}} S_{n+2}(\Gamma)$$

where $S_{n+2}(\Gamma)$ denotes the space of cusp forms for Γ . In fact, there is a canonical injection $M_{n+2}(\Gamma; \mathbb{Q}) \rightarrow H_{dR}^1(\mathcal{M}_{1,1}; \mathbb{V}_n^{dR})$ from the space of modular forms of level 1 and weight $n+2$ with rational Fourier coefficients into this space. The remaining classes can be generated, for example, using weakly holomorphic modular forms (i.e., with a pole at the

cuspidal). It follows that the Lie algebra $\mathfrak{u}_{1,1}^{dR}$ is isomorphic to the completed free Lie algebra on symbols

$$\mathbf{e}_{n+2} \otimes V_n^{dR}, \quad \mathbf{e}'_f \otimes V_n^{dR}, \quad \mathbf{e}''_f \otimes V_n^{dR}$$

where $\mathbf{e}_{n+2}$ corresponds to an Eisenstein series, and $\mathbf{e}'_f, \mathbf{e}''_f$ correspond to a cusp form (resp. weakly holomorphic cusp form) of weight $n+2$. The reader is warned that unlike the case of the projective line minus three points, the de Rham relative completion is not canonically graded and the choices of generators are non-canonical. The periods of relative completion (called ‘multiple modular values’ in [7]) include the regularised iterated integrals of vector valued modular forms along the imaginary axis. In the simplest case, these reduce to classical Eichler integrals of cusp forms (see below).

- A key ingredient in the study of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is a bound in the extension groups in the category of mixed Tate motives over $\mathbb{Z}$, which follows from Borel’s deep results in algebraic K theory. Such results are lacking in the modular situation, but we can work instead in a category of realisations (as in [18]) over $\mathbb{Q}$ with coefficients in $\overline{\mathbb{Q}}$. Hereafter we use the word ‘motive’ loosely to mean an object in the subcategory generated by $\mathcal{O}(\mathcal{G}_{1,1})$. This formalism is sufficient to exhibit elements of a ‘motivic’ Lie algebra generalising the algebra generated by the σ_{2n+1} which play a role in the Deligne-Ihara conjecture. Such elements (which are only defined up to commutators) correspond to non-trivial simple extensions of $\mathbb{Q}$ inside $\mathcal{O}(\mathcal{G}_{1,1})$. In contrast to the case of genus zero, we now expect infinitely many different types of such extensions.

We briefly describe some of the ‘motivic’ elements which act non-trivially on the relative completion $\mathcal{G}_{1,1}$, and in particular the Lie algebra $\mathfrak{u}_{1,1}$.

4.0.1. Zeta elements. Analogues of the familiar zeta elements σ_{2n+1} indeed appear, and correspond to extensions of Tate objects occurring inside $\mathcal{O}(\mathcal{G}_{1,1})$:

$$0 \longrightarrow \mathbb{Q}(2n+1) \longrightarrow \mathcal{E} \longrightarrow \mathbb{Q} \longrightarrow 0.$$

One can prove ([7], §19) that their de Rham versions act on $\mathfrak{u}_{1,1}^{dR}$ via

$$(4.2) \quad \sigma_{2n+1}^{dR} = \frac{(2n)!}{2} \text{ad}(\mathbf{e}_{2n+2} Y^{2n}) + \dots$$

In fact, some of the higher order terms are also known. This already has applications to the theory of the projective line minus 3 points, as it enables us to construct zeta elements σ_{2n+1} which are canonical up to depth four [12].

4.0.2. Modular elements. These have no analogue in the case of the projective line minus three points. For any eigencusp form f of weight w and any integer $d \geq w$, there exist *modular elements* $\sigma_f(d)$ which correspond to a non-trivial extension

$$0 \longrightarrow M_f(d) \longrightarrow \mathcal{E} \longrightarrow \mathbb{Q} \longrightarrow 0,$$

where M_f is the simple object of rank 2 (‘motive’ in the category of realisations) associated to the cusp form f . One proves [7] that they act on $\mathfrak{u}_{1,1}^{dR}$ via

$$\sigma_{f(d)}^{dR} = \text{ad}(\mathbf{b}_{f(d)}) + \delta_{f(d)}$$

where the ‘geometric part’ $\mathbf{b}_{f(d)} \in [\mathfrak{u}_{1,1}^{dR}, \mathfrak{u}_{1,1}^{dR}]$ and the ‘arithmetic part’ $\delta_{f(d)}$ is an outer derivation of $\mathfrak{u}_{1,1}^{dR}$. Their leading terms are known. In the case $d = w$, we have

$$(4.3) \quad \delta_{f(w)}(\mathbf{e}'_f Y^{w-2}) = \sum_{2a+2b=w-2} c_f^{a,b} [\mathbf{e}_{2a+2} Y^{2a}, \mathbf{e}_{2b+2} Y^{2b}] + \dots$$

where $c_f^{a,b}$ are algebraic numbers in the field K_f generated by the Fourier coefficients of f , which are proportional to the coefficients in the even period polynomial of f . More precisely, one has the equality of points in projective space $\mathbb{P}^{n-1}(K_f)$

$$(c_f^{1,n-1} : \dots : c_f^{n-1,1}) = (i^3 \Lambda(f, 3) : i^5 \Lambda(f, 5) : \dots : i^{2n-1} \Lambda(f, 2n-1))$$

where $w = 2n + 2$ and Λ is the completed L -function of f .

4.0.3. Higher elements. We expect $\sigma_{2n+1}^{dR}, \sigma_{f(d)}^{dR}, \dots$ to be the beginning of an infinite sequence of families of elements. The next family is $\sigma_{f \otimes g(d)}^{dR}$ corresponding to an extension of $\mathbb{Q}$ by a Rankin-Selberg motive $M_f \otimes M_g(d)$, for d first in the ‘semi-critical’ range, and later for all values of d . After this come extensions of $\mathbb{Q}$ by $M_f \otimes M_g \otimes M_h(d)$, and so on.

4.1. A genus one Deligne-Ihara conjecture? Beilinson’s conjecture predicts exactly how many motivic extensions of $\mathbb{Q}$ by

$$(4.4) \quad (\mathrm{Sym}^{m_1} M_{f_1} \otimes \dots \otimes \mathrm{Sym}^{m_r} M_{f_r})(d)$$

one should expect to find in nature. Such extensions correspond to derivations

$$\sigma_{2n+1}^{dR}, \sigma_{f(d)}^{dR}, \sigma_{f \otimes g(d)}^{dR}, \sigma_{(\mathrm{Sym}^2 f)(d)}^{dR}, \dots$$

which will be non-trivial if the extension actually occurs in $\mathcal{O}(\mathcal{G}_{1,1})$. One hopes this is true, and furthermore, that these derivations generate a free Lie algebra, by analogy with conjecture 1. This would imply a classification theorem for mixed modular motives, assuming an analogue of Borel’s theorem which would bound the extension groups of mixed modular motives. As a first approximation, we can prove [7]:

Theorem 4.1. *Any representative for the zeta and modular elements $\sigma_{2n+1}^{dR}, \sigma_{f(d)}^{dR}$ act freely upon $\mathfrak{u}_{1,1}^{dR}$, i.e., generate a free Lie algebra.*

In fact, one can prove much more, namely that any non-zero ‘motivic’ derivations satisfying quite a weak condition necessarily generate a free Lie algebra [7], §21.

Here, however, comes a surprise. One can show that not every predicted extension of $\mathbb{Q}$ by (4.4) can actually arise in $\mathcal{O}(\mathcal{G}_{1,1})$: there are certain exceptional cases for large $m_1 + \dots + m_r$ and small d which *cannot* occur due to a Hodge-theoretic obstruction. This is very mysterious.

4.2. The Ihara-Takao relation revisited. Let Δ denote the Ramanujan cusp form of weight 12. The critical values of its completed L -function satisfy

$$(i^3 \Lambda(\Delta, 3) : i^5 \Lambda(\Delta, 5) : i^7 \Lambda(\Delta, 7) : i^9 \Lambda(\Delta, 9)) = (14 : -9 : -9 : 14) .$$

It follows that some normalisation of the associated modular element (which is only defined up to scalar multiple) $\sigma_{\Delta(12)}^{dR}$ acts via the derivation

$$\mathbf{e}'_{\Delta} Y^{10} \mapsto 14 [\mathbf{e}_4 Y^2, \mathbf{e}_{10} Y^8] - 9 [\mathbf{e}_6 Y^4, \mathbf{e}_8 Y^6] + (\text{higher order terms})$$

Therefore if we rescale this particular choice of generator and define

$$d_{12}^{dR} = 1440 \sigma_{\Delta(12)}^{dR} (\mathrm{ad} \mathbf{e}'_{\Delta} Y^{10})$$

and substitute in equation (4.2), we obtain the congruence

$$(4.5) \quad d_{12}^{dR} \equiv [\sigma_3^{dR}, \sigma_9^{dR}] - 3 [\sigma_5^{dR}, \sigma_7^{dR}]$$

modulo higher order Lie brackets. Compare with (2.1). Note, however, that the previous equation takes place in a different Lie algebra, namely $\mathrm{Der} \mathfrak{u}_{1,1}^{dR}$. To make the connection with the projective line minus three points, one can use a ‘monodromy’ homomorphism from $\mathrm{Der} \mathfrak{u}_{1,1}^{dR}$ to the derivations on the de Rham π_1 of the punctured infinitesimal Tate curve, which sends d_{12}^{dR} to

Object	$\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$	$\mathcal{M}_{1,1}$
$\pi_1^?$	Unipotent completion $\pi_1^{\text{un}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$	Relative completion $\mathcal{G}_{1,1} = \pi_1^{\text{rel}}(\mathcal{M}_{1,1})$
basepoints	$\vec{1}_0, -\vec{1}_1$	$\frac{\partial}{\partial q} \quad \vec{1}_\infty$
Lie algebra of π_1^{dR}	Free Lie algebra generated by e_0, e_1 dual to $\frac{dz}{z}, \frac{dz}{1-z}$	Free Lie algebra generated by $\{\mathbf{e}_w, \mathbf{e}'_f, \mathbf{e}''_f\} \otimes V_{w-2}^{dR}$ for $f \in S_w(\mathbb{Q})$ cusp form
Paths	The straight line path dch from $\vec{1}_0$ to $-\vec{1}_1$	The images of the paths $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ i.e., $i\mathbb{R}_{>0}$ and a loop in q -disk
Periods	Regularised iterated integrals of one forms $\frac{dz}{z}$ and $\frac{dz}{1-z}$ a.k.a. multiple zeta values	Regularised iterated integrals of vector-valued modular forms of first and second kinds a.k.a. multiple modular values
Pure periods	Polynomials in $2\pi i$	Polynomials in $\omega_f^+, i\omega_f^-, \eta_f^+, i\eta_f^-$
‘Motives’	Mixed Tate Motives over $\mathbb{Z}$ $\mathcal{MT}(\mathbb{Z})$	Mixed Modular ‘Motives’ $\mathcal{MMM}_{\text{SL}_2(\mathbb{Z})}$
Simple objects	$\mathbb{Q}(n)$	$(\text{Sym}^{m_1} M_{f_1} \otimes \dots \otimes \text{Sym}^{m_r} M_{f_r})(d)$
Motivic Lie algebra	<u>Generators</u> ζ -elements: $\sigma_3, \sigma_5, \dots$	<u>Generators</u> ζ -elements: $\sigma_3, \sigma_5, \dots$ Modular elements: $\sigma_f(d)$ Rankin-Selberg elements, $\dots$

zero. Hain has shown how to relate the latter to the projective line minus three points. We can deduce, for example, that

$$0 \equiv [\sigma_3^{dR}, \sigma_9^{dR}] - 3[\sigma_5^{dR}, \sigma_7^{dR}]$$

in the associated depth-graded Lie algebra [7].

A version of the above should hold ℓ -adically as well. One expects a ‘Galois’ Lie algebra constructed out of a filtration on $G_{\mathbb{Q}}$ induced by its action on the lower central series of the ℓ -adic realisation $\mathcal{G}_{1,1}^{\ell}$. One expects that ℓ -adic versions of the modular elements $\sigma_{f(d)}^{\ell}$ should be non-zero. Repeating the above argument in the ℓ -adic setting should explain the origin of δ_{12}^{ℓ} in the Ihara-Takao relation (2.1). The fact that it vanishes modulo 691 should be related to the congruence $\Delta \equiv \mathbb{G}_{12} \pmod{691}$ and its ℓ -adic interpretation via the action of $G_{\mathbb{Q}}$ on $(M_{\Delta})_{\ell}$. This seems to be related to the fascinating talk of Sharifi in this volume.

5. MULTIPLE MODULAR VALUES OF LEVEL 1: AN OVERVIEW

The rest of this talk concerns multiple modular values, which are the periods of the relative completion of the fundamental group of $\mathcal{M}_{1,1}$ and the genus one analogues of multiple zeta values. The totally holomorphic multiple modular values are very concrete and provide an elementary perspective on the underlying motivic structure. After giving a brief overview of general principles, the remaining sections consist of examples which may help to build intuition for the theory.

Let $f_1, \dots, f_r$ be modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ which have rational Fourier coefficients. Let $w_1, \dots, w_r$ denote their weights.

Definition 5.1. A totally holomorphic multiple modular value of length r is $(2\pi i)^{w_1+\dots+w_r-r}$ times a regularised iterated integral ([7], §5)

$$(5.1) \quad \Lambda(f_1, \dots, f_r; n_1, \dots, n_r) = \int_0^1 f_1(t) t^{n_1-1} dt \dots f_r(t) t^{n_r-1} dt$$

where each n_i , for $1 \leq i \leq r$, satisfies $0 < n_i < w_i$.

The case where all f_i are cuspidal was considered by Manin, and the integral is an ordinary integral from 0 to ∞ . In general, the tangential basepoint at the upper range of integration implies a certain regularisation procedure described in the sequel to this talk. The numbers (5.1) are periods of the relative completion of the fundamental group of $\mathcal{M}_{1,1}$, but not the only periods of the latter (see §6.4).

5.1. Properties. It follows from the general properties of iterated integrals that the numbers (5.1) satisfy a reflection symmetry and shuffle product relations. In particular, they generate an algebra. In fact, much more is known:

- The values $\Lambda(f_1, \dots, f_r; n_1, \dots, n_r)$ for fixed f_i and varying $0 < n_i < w_i$ satisfy non-abelian cocycle relations ([44], [7] §5).
- Surprisingly, there is a mechanism of transference of periods between integrals of *different* sets of modular forms. It is a higher analogue of the Peterson inner product. It generates relations between multiple modular values

$$\Lambda(f_1, \dots, f_i; -) \Lambda(f_{i+1}, \dots, f_n; -) \quad \text{for } 1 \leq i < n$$

where $f_1, \dots, f_n$ are any fixed set of modular forms.

- Certain linear combinations of iterated Eisenstein integrals are multiple zeta values. This is a consequence of the fact that the relative completion of the fundamental group of $\mathcal{M}_{1,1}$ has a monodromy representation on the pro-unipotent fundamental group of the punctured Tate curve, whose periods are multiple zeta values. Enriquez has written down a combinatorial relation between the Drinfeld associator and the periods of the latter [22]. This is closely related to the elliptic MZV's developed by him and Matthes [23, 46].

This list is not exhaustive: Beilinson's conjectures on extensions of motives should imply relations which presently have no elementary or direct proof.

Remark 5.2. The transference principle can be viewed as a manifestation of some kind of convolution on modular forms. For example, it implies a relation between multiple modular values of the form $\Lambda(f_1, f_2) \Lambda(f_3)$ and those of the form $\Lambda(f_1) \Lambda(f_2, f_3)$. It follows that the multiple modular values $\Lambda(f_1, f_2; n_1, n_2)$ for any two modular forms f_1, f_2 pick up information from *a priori* unrelated modular forms f_3 . This phenomenon is related in simple cases to the Rankin-Selberg method, but is more general.

5.2. Two relations to multiple zeta values. We describe two different connections with multiple zeta values. One is conjectural, the other is a theorem.

Recall that the Eisenstein series of weight $2k \geq 4$ is the modular form of level one and weight $2k$ with Fourier expansion normalised as follows:

$$\mathbb{G}_{2k} = -\frac{b_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n,$$

where σ denotes the divisor function. Recall definition 3.4.

Conjecture 2. Let $a_1, \dots, a_r \geq 1$ and $a = a_1 + \dots + a_r$. Then

$$(2\pi)^{2a+r} \Lambda(\mathbb{G}_{2a_1+2}, \dots, \mathbb{G}_{2a_r+2}; 1, \dots, 1) \equiv (-1)^a \frac{(2a_1+1)! \dots (2a_r+1)!}{2^r} \zeta_{2a_1+1, \dots, 2a_r+1}$$

modulo multiple modular values of unipotency (coradical) filtration $\leq r-1$.

We expect the following stronger variant to be true: the difference between the left and right-hand sides of the equation are linear combinations of multiple modular values of the form $\Lambda(\mathbb{G}_{2a'_1+2}, \dots, \mathbb{G}_{2a'_k+2}; n_1, \dots, n_k)$ where $k \leq r$. In all the examples we can actually take $k = r$ and $a'_1 = a_1, \dots, a'_k = a_k$. In other words, we find that a representative for the multiple zeta value $\zeta_{2a_1+1, \dots, 2a_r+1}$ can be expressed as a $\mathbb{Q}$ -linear combination of the multiple modular values $\Lambda(\mathbb{G}_{2a_1+2}, \dots, \mathbb{G}_{2a_r+2}; n_1, \dots, n_r)$.

The motivation for this conjecture is theorem 22.2 in [7], where a similar statement is proved on the level of mixed Hodge structures. Note that iterated integrals of the Eisenstein series $\mathbb{G}_{2k}$ are *not* periods of mixed Tate motives, and are not expected to be multiple zeta values in general (see the examples below). So if conjecture 2 is true, then we have a solution to the inverse problem §3.3. The price to pay is the possible introduction of non multiple-zeta value periods.

Now compare this with the following theorem, which is a corollary of a much more general and precise theorem about motivic periods which will be part of A. Saad's forthcoming doctoral thesis at the University of Oxford.

Theorem 5.3. (Saad) *Every multiple zeta value of depth r and weight $2a + r$ can be expressed as a rational linear combination of iterated integrals of Eisenstein series*

$$(2\pi)^{2a+r} \Lambda(\mathbb{G}_{2a_1+2}, \dots, \mathbb{G}_{2a_r+2}; n_1, \dots, n_r) \quad \text{where} \quad 1 \leq n_i \leq 2a_i + 1.$$

The relationship between this theorem and conjecture 2 is subtle: already in length $r = 2$, conjecture 2 relates $\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 1, 1)$ to the number $\zeta_{3,9}$, which cannot be expressed as multiple zeta values of depth ≤ 2 (we expect). Indeed, it is not captured by the previous theorem for $r = 2$ except in the linear combination (3.6), but will reappear in depth $r = 4$.

6. EXAMPLES OF MULTIPLE MODULAR VALUES IN LENGTH ONE.

Multiple modular values in length one are generated by: powers of $2\pi i$, odd zeta values, periods of cusp forms (all of which are totally holomorphic) and quasi-periods of cusp forms (which are not totally holomorphic).

For f a modular form of weight $2k$, the totally holomorphic values (5.1) are classical. As the notation suggests, they are critical values of the completed L -function

$$\Lambda(f; s) = (2\pi)^{-s} \Gamma(s) L(f, s)$$

where $L(f, s) = \sum_{n \geq 1} a_n(f) n^{-s}$, and $a_n(f)$ are the Fourier coefficients of f . The functional equation $\Lambda(f; s) = (-1)^k \Lambda(f; 2k - s)$ holds.

6.1. **Eisenstein series.** Since the L -function of $\mathbb{G}_{2k}$ factorises as a product

$$L(\mathbb{G}_{2k}, s) = \zeta(s)\zeta(s - 2k + 1)$$

of two Riemann zeta functions, we deduce that the even values satisfy

$$\Lambda(\mathbb{G}_{2k}; 2i) = \frac{(-1)^i}{2} \frac{b_{2i}}{2i} \frac{b_{2k-2i}}{2k-2i}.$$

The values $\Lambda(\mathbb{G}_{2k}; n)$ vanish for odd values of $3 \leq n \leq 2k - 3$ except for

$$(2\pi)^{2k-1} \Lambda(\mathbb{G}_{2k}; 2k - 1) = -\frac{(2k - 2)!}{2} \zeta(2k - 1)$$

and $\Lambda(\mathbb{G}_{2k}; 2k - 1) = (-1)^k \Lambda(\mathbb{G}_{2k}; 1)$. This confirms conjecture 2 (and theorem 5.3) in the case of length one. The numbers $\zeta(2k - 1)$ are periods of a simple extension

$$0 \longrightarrow \mathbb{Q} \longrightarrow M_{\mathbb{G}_{2k}} \longrightarrow \mathbb{Q}(1 - 2k) \longrightarrow 0$$

of Tate motives. This extension (or a Tate twist of its dual), can be realised as a subquotient of the affine ring of the relative completion $\mathcal{O}(\mathcal{G}_{1,1})$ and corresponds to the Eisenstein series (it is dual to $\mathbf{e}_{2k} \otimes V_{2k-2}^{dR} \in \mathbf{u}_{1,1}^{dR}$). This extension is the source of the ‘zeta elements’ σ_{2k-1}^{dR} .

6.2. **General case.** For any modular form f of weight w set:

$$(6.1) \quad P_f(y) = \sum_{k=1}^{w-1} i^{w-k-1} \binom{w-2}{k-1} \Lambda(f; k) y^{k-1}.$$

The function P_f can be interpreted as the value of a canonical cocycle C_f in a certain cochain complex. The cocycle relations imply functional relations for P_f . When f is cuspidal, these are equivalent to the period polynomial equations:

$$\begin{aligned} P_f(y) + y^{w-2} P_f(-y^{-1}) &= 0 \\ P_f(y) + (1-y)^{w-2} P_f\left(\frac{1}{1-y}\right) + y^{w-2} P_f\left(\frac{y-1}{y}\right) &= 0. \end{aligned}$$

The first equation follows from the functional equation of Λ . A variant of these equations is satisfied for an Eisenstein series $f = \mathbb{G}_w$ ([7], §7). Kohnen and Zagier’s ‘extra relation’ [39], which expresses orthogonality of cusp forms to Eisenstein series, is a consequence of transference equations ([7], §8). Manin showed [43] that when f is a Hecke eigenform, the function P_f is an eigenfunction for a certain action of Hecke operators. From this he deduced that

$$(6.2) \quad P_f = \omega_f^+ P_{f,+} + i \omega_f^- P_{f,-}$$

where $\omega_f^+, \omega_f^- \in \mathbb{R}$ and $P_{f,+}, P_{f,-} \in K_f[y]$ where K_f is the field generated by the Fourier coefficients of f . These polynomials play a key role in the Ihara-Takao relation and its generalisations.

Example 6.1. Let Δ denote the unique normalised cusp form of weight 12:

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \frac{a_n}{n^s}$$

where $a_1 = 1, a_2 = -24, a_3 = 252, \dots$. Equation (6.2) holds with

$$(6.3) \quad \begin{aligned} P_{\Delta,+} &= \frac{36}{691} (y^{10} - 1) + y^2 - 3y^4 + 3y^6 - y^8 \\ P_{\Delta,-} &= 4y - 25y^3 + 42y^5 - 25y^7 + 4y^9 \end{aligned}$$

where $\omega_{\Delta,+}, \omega_{\Delta,-}$ take the numerical values

$$\omega_{\Delta,+} = 0.114379022438848 \dots, \quad \omega_{\Delta,-} = 0.009269276162370 \dots.$$

The numbers $(2\pi i)^{11} \omega_{\Delta,\pm}$ are examples of totally holomorphic multiple modular values.

6.3. Motivic interpretation. The Manin relation (6.2) can be interpreted as an instance of Deligne’s conjecture on critical values of L -functions. Associated to a Hecke eigen cusp form f is a rank two motive M_f with coefficients in the field K_f generated by the Fourier coefficients of f [47]. It has Hodge numbers of type $(0, w_f - 1)$ and $(w_f - 1, 0)$. The periods of $M_f(w_f - 1)$ are encapsulated by a two-by-two matrix:

$$\begin{pmatrix} P_{f,+} & P_{f,-} \end{pmatrix} \begin{pmatrix} \eta_f & \omega_f \\ \eta_{f,+} & \omega_{f,+} \\ i\eta_{f,-} & i\omega_{f,-} \end{pmatrix}$$

The periods of M_f are given by the same matrix with all entries multiplied by $(2\pi i)^{w_f-1}$. Here, $(2\pi i)^{w_f-1}\omega_f, (2\pi i)^{w_f-1}\eta_f$ are a K_f -basis of the de Rham realisation $(M_f)_{dR}$ and $P_{f,\pm}$ correspond to a basis for the Betti realisation $(M_f)_B$. Poincaré duality states that

$$M_f^\vee = M_f(w_f - 1)$$

and hence $\bigwedge^2 M_f = K_f(1 - w_f)$. This implies a Legendre-style relation:

$$i\eta_{f,+}\omega_{f,-} - i\omega_{f,+}\eta_{f,-} \in (2\pi i)^{1-w_f} K_f^\times.$$

The holomorphic class $\omega_f \in F^{w_f-1}$ corresponds to the modular form f itself; the class η_f can be generated by a modular form of the ‘second kind’, and is only well-defined up to adding a K_f -rational multiple of ω_f .

Deligne’s conjecture predicts that the values of $\Lambda(f, s)$ at a critical point s (i.e., $s = 1, \dots, w_f - 1$) are proportional, depending on the parity of s , to either of the two holomorphic periods ω_f^+, ω_f^- . This is indeed confirmed by Manin’s equation (6.2).

Let us rephrase this in a different manner. The values $\Lambda(f; n)$ are periods of an extension of mixed ‘motives’ of the following form:

$$0 \longrightarrow K_f(n) \longrightarrow \mathcal{E} \longrightarrow M_f(w_f - 1) \longrightarrow 0.$$

It (or rather, it tensored with the representation $V_{w_f-2}^{dR}$ of SL_2 of dimension $w_f - 2$) arises as a subquotient of $\mathcal{O}(\mathcal{G}_{1,1})$ and corresponds to the form f (dual to the class $\mathbf{e}'_f \otimes V_{w_f-2}^{dR}$). However, unlike the case of the Eisenstein series, this extension necessarily splits when $1 \leq n \leq w_f - 1$. This is why $\Lambda(f; n)$, for $1 \leq n \leq w_f - 1$, is in fact a period of the summand $M_f(w_f - 1)$.

6.4. Quasi-periods and non-totally holomorphic MMV’s. In order to obtain the two other non-classical periods $\eta_{f,+}, i\eta_{f,-}$ (which could be called ‘quasi-periods’ following the terminology for algebraic curves), one needs to consider modular forms of the second kind with poles, for example, at the cusp.

Consider the Ramanujan cusp form Δ . There exists a unique weakly holomorphic modular form Δ' of weight 12 which has a simple pole at the cusp, and whose Fourier coefficients a_0, a_1 vanish. It is a weak Hecke eigenform with the same eigenvalues as Δ and has integer coefficients. Explicitly,

$$\Delta' = q^{-1} + 47709536q^2 + 39862705122q^3 + 7552626810624q^4 + \dots$$

Its periods are [13]

$$\eta_{\Delta,+} = 211.113366616704346\dots, \quad \eta_{\Delta,-} = 17.055972753974248\dots,$$

and one checks that $i(\eta_{\Delta,+}\omega_{\Delta,-} - \omega_{\Delta,+}\eta_{\Delta,-}) = 10! \times (2\pi i)^{-11}$.

In general, the periods $(2\pi i)^{w_f-1}\eta_{f,\pm}$ are examples of non totally-holomorphic multiple modular values. Since they are not canonically defined, they fall outside the remit of the conjectures of special values of L -functions. We shall not say any more about non-totally holomorphic multiple modular values, as very few are known. In principle, one should be able to compute them using the methods of Luo [42].

7. EXAMPLES OF DOUBLE EISENSTEIN INTEGRALS

We first discuss concrete examples of multiple modular values in length two before explaining their general structure in the next section.

7.1. Examples in low weights. These only involve multiple zeta values.

Example 7.1. (Two Eisenstein series of weight 4). Write $\Lambda_{i,j} = \Lambda(\mathbb{G}_4, \mathbb{G}_4; i, j)$ for brevity, where $1 \leq i, j \leq 3$. One can prove that:

$$\begin{aligned}\Lambda_{1,1} &= \frac{\zeta(3)^2}{2^7 \pi^6} \quad , \quad \Lambda_{1,2} = \frac{\zeta(3)}{2^5 3^2 5 \pi^3} - \frac{5 \zeta(5)}{2^8 3 \pi^5} \quad , \quad \Lambda_{1,3} = \frac{\zeta(3)^2}{2^6 \pi^6} - \frac{11.19}{2^{10} 3^4 5^3} \\ \Lambda_{2,1} &= -\frac{\zeta(3)}{2^8 3.5 \pi^3} + \frac{5 \zeta(5)}{2^8 3 \pi^5} \quad , \quad \Lambda_{2,2} = \frac{1}{2^{11} 3^4} \quad , \quad \Lambda_{2,3} = \frac{\zeta(3)}{2^5 3^2 5 \pi^3} - \frac{5 \zeta(5)}{2^8 3 \pi^5} \\ \Lambda_{3,1} &= \frac{11.19}{2^{10} 3^4 5^3} \quad , \quad \Lambda_{3,2} = -\frac{\zeta(3)}{2^8 3.5 \pi^3} + \frac{5 \zeta(5)}{2^8 3 \pi^5} \quad , \quad \Lambda_{3,3} = \frac{\zeta(3)^2}{2^7 \pi^6}\end{aligned}$$

In particular, one can check the symmetry $\Lambda_{i,j} = \Lambda_{4-j,4-i}$ and the shuffle product relation $\Lambda_{i,j} + \Lambda_{j,i} = \Lambda(\mathbb{G}_4; i) \Lambda(\mathbb{G}_4; j)$.

Example 7.2. (Double integrals of two Eisenstein series of weights 6 and 4.) Consider the values of $\Lambda(\mathbb{G}_6, \mathbb{G}_4; n_1, n_2)$ for $1 \leq n_1 \leq 5$ and $1 \leq n_2 \leq 3$. There is a symmetry

$$\Lambda(\mathbb{G}_6, \mathbb{G}_4; n_1, n_2) = \Lambda(\mathbb{G}_4, \mathbb{G}_6; 4 - n_2, 6 - n_1)$$

and shuffle relation

$$\Lambda(\mathbb{G}_6, \mathbb{G}_4; n_1, n_2) + \Lambda(\mathbb{G}_4, \mathbb{G}_6; n_2, n_1) = \Lambda(\mathbb{G}_6; n_1) \Lambda(\mathbb{G}_4; n_2).$$

The corners

$$\Lambda(\mathbb{G}_6, \mathbb{G}_4; 1, 1) = \frac{3 \zeta(3, 5)}{2^6 5 \pi^8} - \frac{503}{2^{11} 3^4 5^3 7}$$

and

$$\Lambda(\mathbb{G}_6, \mathbb{G}_4; 5, 3) = \frac{3 \zeta(3, 5)}{2^6 5 \pi^8} + \frac{3 \zeta(3) \zeta(5)}{2^6 \pi^8} - \frac{503}{2^{11} 3^4 5^3 7}$$

give rise to the first non-trivial multiple zeta values in weight 8, namely $\zeta(3, 5)$, which is conjecturally algebraically independent from the values of the Riemann zeta function at integers $\zeta(n)$. The remaining values, denoted by $\Lambda_{i,j} = \Lambda(\mathbb{G}_6, \mathbb{G}_4; i, j)$, are:

$$\begin{aligned}\Lambda_{1,2} &= -\frac{\zeta(3)}{2^6 3^2 5.7 \pi^3} - \frac{\zeta(5)}{2^8 3 \pi^5} + \frac{7 \zeta(7)}{2^9 \pi^7} \quad , \quad \Lambda_{1,3} = \frac{\zeta(3)^2}{2^6 7 \pi^6} + \frac{19.23}{2^{10} 3^5 5^2 7^2} - \frac{3 \zeta(3) \zeta(5)}{2^6 \pi^8} \\ \Lambda_{2,1} &= \frac{\zeta(3)}{2^8 3^2 5.7 \pi^3} - \frac{7 \zeta(7)}{2^{10} \pi^7} \quad , \quad \Lambda_{2,2} = -\frac{\zeta(3)^2}{2^8 7 \pi^6} - \frac{107}{2^{10} 3^5 5^2 7^2} \quad , \quad \Lambda_{2,3} = \frac{\zeta(3)}{2^9 3.5.7 \pi^3} - \frac{\zeta(5)}{2^7 3.5 \pi^5} \\ \Lambda_{3,1} &= \frac{\zeta(3)^2}{2^7 3.7 \pi^6} - \frac{1187}{2^{10} 3^6 5^2 7^2} \quad , \quad \Lambda_{3,2} = -\frac{\zeta(3)}{2^7 3^3 7 \pi^3} + \frac{\zeta(5)}{2^8 3.5 \pi^5} \quad , \quad \Lambda_{3,3} = \frac{\zeta(3)^2}{2^7 3.7 \pi^6} - \frac{1187}{2^{10} 3^6 5^2 7^2} \\ \Lambda_{4,1} &= \frac{\zeta(3)}{2^8 3^2 7 \pi^3} - \frac{\zeta(5)}{2^7 3.5 \pi^5} \quad , \quad \Lambda_{4,2} = -\frac{\zeta(3)^2}{2^8 7 \pi^6} + \frac{521}{2^{11} 3^5 5^2 7^2} \quad , \quad \Lambda_{4,3} = \frac{\zeta(3)}{2^9 5.7 \pi^3} - \frac{7 \zeta(7)}{2^{10} \pi^7} \\ \Lambda_{5,1} &= \frac{\zeta(3)^2}{2^6 7 \pi^6} + \frac{19.23}{2^{10} 3^5 5^2 7^2} \quad , \quad \Lambda_{5,2} = -\frac{\zeta(3)}{2^6 3^2 5.7 \pi^3} + \frac{7 \zeta(7)}{2^9 \pi^7}\end{aligned}$$

Continuing in this manner, one can show that all $\Lambda(\mathbb{G}_a, \mathbb{G}_b; n_1, n_2)$ for $0 \leq n_1 < a, 0 < n_2 < b$, and $a + b \leq 10$ are multiple zeta values.

7.2. Examples in weight 12. The first modular periods appear.

Example 7.3. Consider the double Eisenstein integrals $\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; n_1, n_2)$. There are too many periods to write down in full, but we find exactly two new periods which are of modular type and not expected to be multiple zeta values, namely:

$$\pi^{-1}\Lambda(\Delta; 12) = 600\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 2, 5) + 480\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 3, 4)$$

which is a *non-critical* value of the completed L -function of Δ . This can be proved using the Rankin-Selberg method [7] §9. There is a new number:

$$c(\Delta; 12) = 70\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 3, 5) = 0.000225126548190262999168981015\dots$$

which is only well-defined modulo $\mathbb{Q}$. For example, we might equally well have taken the following quantity as our representative for it

$$\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 2, 6) = \frac{13}{2^{12}3^55^27.11} + \Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 3, 5)$$

which differs by a rational number. See §7.3 for an interpretation of these numbers. We now turn to the most interesting multiple zeta values which occur.

The multiple modular value in the corner satisfies

$$(7.1) \quad \Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 1, 1) \stackrel{?}{=} \frac{2^23^2}{691}c(\Delta; 12) - \frac{3^{25.7}}{2^6} \frac{\zeta_{3,9}}{\pi^{12}}$$

where $\zeta_{3,9}$ is the multiple zeta value of weight 12 and depth 4 given by (3.5). Since $c(\Delta; 12)$ is of coradical filtration one (see below), this is indeed consistent with conjecture 2, and the stronger versions of it discussed immediately after.

Remark 7.4. Another interesting value is

$$(7.2) \quad \Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 3, 1) \stackrel{?}{=} \frac{4027}{2^83^55^27.11^2} + \frac{c(\Delta; 12)}{3^25} - \frac{5^{27}}{2^511} \frac{\zeta_{3,7}}{\pi^{10}}$$

where $\zeta_{3,7}$ is the multiple zeta value (3.4). The appearance of this number is explained by ‘transference’ from a different double Eisenstein integral (see §8.3).

In the previous equations, $\stackrel{?}{=}$ denotes an equality which is true to hundreds of digits, but is presently unproved. A potential strategy of proof is sketched below.

Example 7.5. Consider the double Eisenstein integrals $\Lambda(\mathbb{G}_6, \mathbb{G}_8; n_1, n_2)$. The values for $0 < n_1 < 6, 0 < n_2 < 8$ satisfy similar properties to the previous example. They are conjecturally linear combinations of $c(\Delta; 12)$ and $\pi^{-1}\Lambda(\Delta; 12)$ and multiple zeta values. Conjecture 2 concerns

$$(7.3) \quad \Lambda(\mathbb{G}_6, \mathbb{G}_8; 1, 1) \stackrel{?}{=} -\frac{2.3^4}{7.691}c(\Delta; 12) - \frac{3^{35}}{2^7} \frac{\zeta_{5,7}}{\pi^{12}}$$

where $\zeta_{5,7}$ can be deduced from (3.5) and (3.6).

7.3. Interpretation of the modular periods arising in weight 12. Consider an extension of the following form, where M_Δ was considered in §6.3:

$$(7.4) \quad 0 \longrightarrow M_\Delta \longrightarrow \mathcal{E} \longrightarrow \mathbb{Q}(-12) \longrightarrow 0.$$

Beilinson’s conjecture suggests that such an extension is essentially unique: i.e., there is a one-dimensional space of such extension classes which can arise from the cohomology of algebraic varieties over $\mathbb{Q}$. We find several equivalent such extensions (tensored with V_{10}^{dR}) inside $\mathcal{O}(\mathcal{G}_{1,1})$ and the cocycle relations indeed confirm that they all have the same periods, in accordance with Beilinson’s conjecture. In detail, one finds that

$$\mathcal{E}_{dR} = (M_\Delta)_{dR} \oplus \mathbb{Q}(-12)_{dR}$$

is canonically split by the Hodge and weight filtrations: it has Hodge types $(0, 11), (11, 0)$ and $(12, 12)$. The iterated integrals of two Eisenstein series $\mathbb{G}_4, \mathbb{G}_{10}$ correspond to classes in $F^{12}\mathcal{E}_{dR}$. They are dual to $[\mathbf{e}_4 V_2^{dR}, \mathbf{e}_{10} V_8^{dR}] \subseteq \mathbf{u}_{1,1}^{dR}$. Computing the iterated integrals yields the following period matrix for $\mathcal{E}(11)$:

$$\begin{pmatrix} \eta_+ & \boxed{\omega_+} & \boxed{\Lambda(\Delta; 12)} \\ i\eta_- & \boxed{i\omega_-} & 90(2\pi i)c(\Delta; 12) \\ 0 & 0 & \boxed{2\pi i} \end{pmatrix}$$

The space of real Frobenius invariants in $\mathcal{E}_B(11)$ is one-dimensional (rows with real entries), but the space of real Frobenius anti-invariants (imaginary rows) is two-dimensional. Changing basis in $\mathcal{E}_B(11)^\vee$ in a way that respects Frobenius can add the third row to the second, which effectively modifies $c(\Delta; 12)$ by addition of a rational. The boxed elements are well-defined up to scalar multiple. They are all predicted by Deligne's conjecture on special values of L -functions, or Beilinson's in the case of the non-critical value $\Lambda(\Delta; 12)$. The three other non-zero entries are not canonically defined and are not predicted by the standard conjectures on L -functions.

Since $\Lambda(\Delta; 12) \neq 0$, the extension (7.4) is non-trivial, which proves the existence of the first modular element $\sigma_{\Delta(12)}^{dR}$. The subspace $(M_\Delta)_{dR}$ is dual to $\mathbf{e}_\Delta V_{10}^{dR} \in \mathbf{u}_{1,1}^{dR}$ (where $\mathbf{e}_\Delta$ denotes the pair $(\mathbf{e}'_\Delta, \mathbf{e}''_\Delta)$). Therefore $\sigma_{\Delta(12)}^{dR}$ is a derivation which sends

$$\mathbf{e}_\Delta V_{10}^{dR} \mapsto [\mathbf{e}_4, \mathbf{e}_{10}] V_{10}^{dR} + \dots$$

where the right-hand V_{10}^{dR} is a factor of $V_2^{dR} \otimes V_8^{dR} \cong V_{10}^{dR} \oplus V_8^{dR} \oplus V_6^{dR}$.

Equivalent extensions show up in many other places inside $\mathcal{O}(\mathcal{G}_{1,1})$.

7.4. Modular incarnation of the Ihara-Takao relation. The equation (4.5) can be viewed on the level of periods as the identity

$$9\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 1, 1) + 14\Lambda(\mathbb{G}_6, \mathbb{G}_8; 1, 1) = -\frac{3^{35.7}}{2^6}(\zeta_{5,7} + 3\zeta_{3,9}) ,$$

as deduced by taking 9×equation (7.1) and adding 14×equation (7.3). The modular period $c(\Delta; 12)$ cancels out and the expression in brackets is dual to the terms in the Ihara-Takao identity (2.1). Here we see in a very concrete manner the exchange of information between the depth filtration on multiple zeta values, the appearance of modular periods as double Eisenstein integrals, and the Ihara-Takao relation.

8. STRUCTURE OF DOUBLE INTEGRALS

In order to continue our exploration of multiple modular values of length two, we need to express the answers more compactly using generating functions.

8.1. Generating functions for length two. For f, g modular forms of level one and weights w_f, w_g , define a polynomial in two variables by:

$$P_{f,g}(y_1, y_2) = \sum_{k, \ell} i^{w_f + w_g - k - \ell - 2} \binom{w_f - 2}{k - 1} \binom{w_g - 2}{\ell - 1} \Lambda(f, g; k, \ell) y_1^{k-1} y_2^{\ell-1} .$$

It has bidegree $(w_f - 2, w_g - 2)$. The shuffle product implies that

$$(8.1) \quad P_{f,g}(y_1, y_2) + P_{g,f}(y_2, y_1) = P_f(y_1)P_g(y_2)$$

where the terms on the right-hand side are (6.1). In order to unpackage $P_{f,g}(y_1, y_2)$ into simpler pieces we must use a little representation theory of SL_2 . First define

$$\tilde{P}_{f,g}(X_1, X_2, Y_1, Y_2) = X_1^{w_f-2} X_2^{w_g-2} P_{f,g}\left(\frac{Y_1}{X_1}, \frac{Y_2}{X_2}\right)$$

to be the homogeneous version in four variables. One shows that this polynomial is a value of a canonical $\mathrm{SL}_2(\mathbb{Z})$ -cochain $C_{f,g}$ which satisfies the equation [7], §5

$$\delta C_{f,g} = C_f \cup C_g .$$

This, together with some known information about the value of cocycles at infinity implies many (but not all) relations between the coefficients $P_{f,g}$. We shall not say any more about this here and turn straight to examples. To this end, consider

$$\partial = \frac{\partial}{\partial X_1} \frac{\partial}{\partial Y_2} - \frac{\partial}{\partial Y_1} \frac{\partial}{\partial X_2}$$

which is an SL_2 -equivariant differential operator of degree -2 . The two-variable polynomial $P_{f,g}$ is uniquely determined by a finite sequence of simpler polynomials in a single variable of degree $w_f + w_g - 2k - 4$ defined for $k \geq 0$ by:

$$(8.2) \quad \delta^k P_{f,g} = \left(\partial^k \tilde{P}_{f,g} \right) (1, 1, y, y) .$$

The polynomial $\delta^0 P_{f,g}$ is simply the diagonal $P_{f,g}(y, y)$. The polynomials $\delta^k P_{f,g}$ vanish for all $k \geq \min\{w_f - 2, w_g - 2\}$.

Example 8.1. The discussion of example 7.3 can be summarised using the generating function $P_{\mathbb{G}_4, \mathbb{G}_{10}}$. It has the following shape

$$P_{\mathbb{G}_4, \mathbb{G}_{10}} = \frac{\zeta_{3,9}}{\pi^{12}} c_0 B^{(0)} + \frac{\zeta_{3,7}}{\pi^{10}} c_1 B^{(2)} + \frac{\Lambda(\Delta; 12)}{\pi} c_2 P_{\Delta, -}^{(0)} + c(\Delta; 12) c_3 P_{\Delta, +}^{(0)} + R$$

where c_0, c_1, c_2, c_3 are known rational numbers, $B^{(0)}, B^{(2)}$ are the unique solutions to

$$\begin{aligned} \delta^0 B^{(0)} &= y^{10} - 1 & \delta^0 B^{(2)} &= 0 \\ \delta^1 B^{(0)} &= 0 & \delta^1 B^{(2)} &= 0 \\ \delta^2 B^{(0)} &= 0 & \delta^2 B^{(2)} &= y^6 - 1 \end{aligned}$$

($B^{(0)} = y_1^2 y_2^8 - 1$ and $B^{(2)} = (y_1 - y_2)^2 (y_2^6 - 1)$), and $P_{\Delta, \pm}^{(0)}$ are the unique solutions to

$$\begin{aligned} \delta^0 P_{\Delta, -}^{(0)} &= P_{\Delta, -} & \delta^0 P_{\Delta, +}^{(0)} &= P_{\Delta, +} \\ \delta^1 P_{\Delta, -}^{(0)} &= 0 & \delta^1 P_{\Delta, +}^{(0)} &= 0 \\ \delta^2 P_{\Delta, -}^{(0)} &= 0 & \delta^2 P_{\Delta, +}^{(0)} &= 0 \end{aligned}$$

where $P_{\Delta, \pm}$ are the odd and even period polynomials (6.3) associated to Δ . The terms $B^{(-)}$ can be interpreted as coboundaries in a cochain complex. The remainder term R involves only single zeta values and products. It is mostly determined from $P_{\mathbb{G}_4}$ and $P_{\mathbb{G}_{10}}$ via (8.1), although some new single zeta values can occur, including $\frac{\zeta(11)}{\pi^{11}}$.

Example 8.2. Example 7.5 can be reformulated in a similar way. It is:

$$P_{\mathbb{G}_6, \mathbb{G}_8} = \frac{\zeta_{5,7}}{\pi^{12}} c_0 B^{(0)} + \frac{\zeta_{5,3}}{\pi^8} c_1 B^{(4)} + \frac{\Lambda(\Delta; 12)}{\pi} c_2 P_{\Delta, -}^{(0)} + c(\Delta; 12) c_3 P_{\Delta, +}^{(0)} + R$$

where $c_0, c_1, c_2, c_3 \in \mathbb{Q}$, the B are the unique polynomials of bidegree (4, 6) satisfying

$$\delta^i B^{(k)} = \begin{cases} y^{10-2i} - 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

(e.g, $B^{(0)} = y_1^4 y_2^6 - 1$), and $P_{\Delta, \pm}^{(0)}$ are the unique polynomials of bidegree (4, 6) with

$$\delta^i P_{\Delta, \pm}^{(0)} = \begin{cases} P_{\Delta, \pm} & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

and R involves only single zeta values and products.

8.2. General case in length two. Let f, g be as above. The generating function $P_{f,g} \in \mathbb{C}[y_1, y_2]$ has bidegree $(w_f - 2, w_g - 2)$ and takes the form

$$P_{f,g} = \sum_{k \geq 0} \left(c_{f,g;\mathbb{G}_{d_k}} B^{(k)} + \sum_h c_{f,g;h}^+ P_{h,+}^{(k)} + \sum_h i c_{f,g;h}^- P_{h,-}^{(k)} \right) + R$$

where $d_k = w_f + w_g - 2 - 2k$, the second and third sums range over a basis h of Hecke eigen cusp forms h of weight d_k , and $P_{h,\pm}$ are (a choice of) associated odd and even period polynomials (6.2) with coefficients in the field K_f generated by the eigenvalues of h . The coboundary polynomials B are the unique solutions to the equations:

$$\delta^j B^{(k)} = \begin{cases} 0 & \text{if } j \neq k \\ y^{d_k-2} - 1 & \text{if } j = k \end{cases}.$$

The polynomials $P_{h,\pm}^{(k)}$ are the unique solutions to the equations

$$\delta^j P_{h,\pm}^{(k)} = \begin{cases} 0 & \text{if } j \neq k \\ P_{h,\pm} & \text{if } j = k \end{cases}.$$

The remainder R is completely (but not uniquely!) determined from P_f and P_g . Its coefficients are linear combinations of products of the coefficients of the latter.

All the interesting information is contained in the numbers

$$c_{f,g;\mathbb{G}_{d_k}}, c_{f,g;h}^+, c_{f,g;h}^- \in \mathbb{R}$$

which are only well-defined modulo products and periods of lower unipotency filtration. The reason for the notation $c_{f,g;\mathbb{G}_{d_k}}$ is that the cocycle of the Eisenstein series $\mathbb{G}_{d_k}$ is Poincaré dual to the coboundary cocycle $\delta^k B^{(k)} = y^{d_k-2} - 1$.

Remark 8.3. The terms in the above formula for $P_{f,g}$ are a Hecke eigenbasis for the group cohomology $H^1(\mathrm{SL}_2(\mathbb{Z}), V_{w_f-2}^{dR} \otimes V_{w_g-2}^{dR})$.

Example 8.4. (Modular Ihara-Takao relation on generating functions). The polynomials $\delta^0 P_{\mathbb{G}_4, \mathbb{G}_{10}}$ and $\delta^0 P_{\mathbb{G}_6, \mathbb{G}_8}$ both have degree 10. All modular periods $\Lambda(\Delta; 12), c(\Delta; 12)$ cancel out in the linear combination $9 \delta^0 P_{\mathbb{G}_4, \mathbb{G}_{10}} + 14 \delta^0 P_{\mathbb{G}_6, \mathbb{G}_8}$ (compare §7.4).

8.3. Transference principle in length two. In length one, transference is equivalent to the well-known orthogonality of cocycles of Hecke eigenforms. In length two, transference is a relation between linear combinations of multiple modular values

$$\Lambda(f; a_1) \Lambda(g, h; a_2, a_3) \quad \text{and} \quad \Lambda(f, g; b_1, b_2) \Lambda(h; b_3)$$

where f, g, h are fixed Eisenstein series or cusp forms and $a_1, a_2, a_3, b_1, b_2, b_3$ can vary within the allowed range. Heuristically, we obtain relations of the form:

$$\begin{aligned} c_{\mathbb{G}_a, \mathbb{G}_b; \mathbb{G}_c} &\sim c_{\mathbb{G}_b, \mathbb{G}_c; \mathbb{G}_a} \\ c_{f, m; \mathbb{G}_a} &\sim \left(c_{m, \mathbb{G}_a; f}^+ \omega_{f,-} + c_{m, \mathbb{G}_a; f}^- \omega_{f,+} \right) \\ \left(c_{f, m; g}^+ \omega_g^- + c_{f, m; g}^- \omega_g^+ \right) &\sim \left(c_{g, m; f}^+ \omega_{f,-} + c_{g, m; f}^- \omega_{f,+} \right) \end{aligned}$$

where $\mathbb{G}_a, \mathbb{G}_b, \mathbb{G}_c$ are Eisenstein series, f, g are cuspidal eigenforms and m is a modular form and $\sim$ denotes a relation modulo products and periods of lower coradical filtration. These equations also hold for non-totally holomorphic multiple modular values.

Example 8.5. The first transference equation between the three Eisenstein series $\mathbb{G}_4, \mathbb{G}_8, \mathbb{G}_{10}$ is an explicit relation between the multiple modular values of $\Lambda(\mathbb{G}_4, \mathbb{G}_{10})$ and those of $\Lambda(\mathbb{G}_4, \mathbb{G}_8)$. For example, one can deduce the identity:

$$14\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 3, 5) - 9\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 3, 1) = \frac{2^{25.7}}{11}\Lambda(\mathbb{G}_4, \mathbb{G}_8; 1, 1) + \frac{157}{2^7 3^3 5^2 11^2}.$$

Now conjecture 2 predicts that

$$\Lambda(\mathbb{G}_4, \mathbb{G}_8; 1, 1) \stackrel{?}{=} \frac{3^{25}}{2^7} \frac{\zeta_{3,7}}{\pi^{10}} - \frac{83}{2^{10} 3^{25} 7^2 11}$$

and so we can understand the appearance (7.2) of the multiple zeta value $\zeta_{3,7}$ in $\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 3, 1)$ as having been transferred from $\Lambda(\mathbb{G}_4, \mathbb{G}_8; 1, 1)$. Conjecture 2 and transference are sufficient to explain all appearances of multiple zeta values (as opposed to products of simple zeta values) amongst double Eisenstein integrals.

Example 8.6. Transference between $\mathbb{G}_4, \mathbb{G}_{10}$ and Δ implies, amongst other things, that the quantity $c_{\mathbb{G}_4, \mathbb{G}_{10}; \Delta}^+ \omega_{\Delta, -} + c_{\mathbb{G}_4, \mathbb{G}_{10}; \Delta}^- \omega_{\Delta, +}$ which is a linear combination of multiple modular values of $\Lambda(\Delta)\Lambda(\mathbb{G}_4, \mathbb{G}_{10})$, is transferred to $c_{\Delta, \mathbb{G}_4; \mathbb{G}_{10}}$, which is a linear combination of multiple modular values of $\Lambda(\Delta, \mathbb{G}_4)\Lambda(\mathbb{G}_{10})$. For example:

$$(2^3 3^3 5^2 11)^{-1} (44\Lambda(\Delta, \mathbb{G}_4; 11, 1) - 110\Lambda(\Delta, \mathbb{G}_4; 10, 2) + 15\Lambda(\Delta, \mathbb{G}_4; 6, 2)) = \\ \Lambda(\Delta; 3) (30\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 2, 5) + 24\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 3, 4)) - 35\Lambda(\Delta; 2)\Lambda(\mathbb{G}_4, \mathbb{G}_{10}; 3, 5)$$

where the right hand side is

$$\left(\frac{\omega_{\Delta, +}}{900\pi} \Lambda(\Delta; 12) - \frac{\omega_{\Delta, -}}{5} c(\Delta; 12) \right)$$

Transference explains why periods of the cusp form Δ can occur as double Eisenstein integrals in weight 12, without appealing to the Rankin-Selberg method.

8.4. Summary of possible extensions in length two. Length two multiple modular values can give rise to the following ‘submotives’ (up to Tate twists) in the affine ring $\mathcal{O}(\mathcal{G}_{1,1})$ of the relative completion of the fundamental group of $\mathcal{M}_{1,1}$:

- (1) (Three Eisenstein series). The numbers $c_{\mathbb{G}_a, \mathbb{G}_b; \mathbb{G}_c}$ are periods of a biextension of Tate objects. We therefore expect them to be multiple zeta values of the form $\zeta_{2k+1, 2\ell+1}$ (definition 3.4). This is the content of conjecture 2.
- (2) (One cusp form f , and two Eisenstein series). The numbers $c_{f, \mathbb{G}_a; \mathbb{G}_b}$ or $c_{\mathbb{G}_a, \mathbb{G}_b; f}$ are periods of extensions of the form we have already discussed:

$$0 \longrightarrow M_f(d) \longrightarrow \mathcal{E} \longrightarrow \mathbb{Q} \longrightarrow 0$$

where d is non-critical. One period of this extension is a non-critical L -value $\Lambda(f; d)$, another is a ‘non-standard’ period of the form $c(f; d)$.

- (3) (Two cusp forms f, g , and one Eisenstein series). The numbers $c_{f, g; \mathbb{G}_a}$ or $c_{f, \mathbb{G}_a; g}$ are periods of Rankin-Selberg extensions of the form

$$0 \longrightarrow M_f \otimes M_g(d) \longrightarrow \mathcal{E} \longrightarrow \mathbb{Q} \longrightarrow 0$$

where d is ‘semi-critical’, i.e., where the rank of the extension group is 1. One of its periods is $\Lambda(f \otimes g; d)$, but there are 3 other non-standard periods.

- (4) (Three cusp forms f, g, h). The numbers $c_{f, g; h}$ are periods of triple Rankin-Selberg extensions of the form

$$0 \longrightarrow M_f \otimes M_g \otimes M_h(d) \longrightarrow \mathcal{E} \longrightarrow \mathbb{Q} \longrightarrow 0$$

where d corresponds to a central critical L -value, i.e., $M_f \otimes M_g \otimes M_h(d)$ has weight one, corresponding to the exceptional case in Beilinson’s conjecture.

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