

Modulus of continuity, a Hardy-Littlewood theorem and its application

By

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§ 1. Introduction

Let D be a simply connected proper domain in $\mathbb{C}$ and $\varphi : \Delta \rightarrow D$ a Riemann mapping from the unit disk $\Delta = \{|z| < 1\}$ onto D . The geometric function theory gives us various informations of the mapping φ . For example, if D is a quasi-disk, then we have

$$(1.1) \quad |\varphi'(z)| = O((1 - |z|)^{-\kappa})$$

for some $\kappa \in [0, 1)$ as $|z| \rightarrow 1$ (cf. [8]). On the other hand, if a simply connected domain D is an invariant component of a finitely generated Kleinian group G , we can say much more on the Riemann mapping φ . In fact, if D is a Jordan domain, then G must be a quasi-Fuchsian group by a theorem of Maskit ([3]). Hence, D is a quasi-disk, and the inequality (1.1) holds. Recently ([9]), we have shown that the converse is also true. Namely, we have shown the following;

Theorem 1.1. *Let $D \ni \infty$ be a simply connected invariant component of a finitely generated non-elementary Kleinian group G and φ a Riemann mapping from the unit disk onto D . Then the following are equivalent.*

1. *G is a quasi-Fuchsian group and D is a quasi-disk.*
2. *(1.1) holds for some $\kappa \in [0, 1)$ as $|z| \rightarrow 1$.*

In other words, the growth rate of the derivatives of the Riemann mappings characterizes quasi-Fuchsian groups.

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Remark. The condition (2) implies that D is a Hölder domain. It is known that every quasi-disk is a Hölder domain. Thus (1) implies (2). But the converse is not true in general.

It is a natural question what happens for φ if D is a simply connected invariant component of G other than a quasi-Fuchsian group. In fact, we have obtained the growth rate of $|\varphi'(z)|$ of Riemann mappings φ for regular b -groups and Kleinian groups with bounded geometry. Particularly, when G is a regular b -group, we have estimated the modulus of continuity of φ on the unit circle and we have shown the local connectivity of the limit set of G .

In this note, we will show a Hardy-Littlewood theorem to estimate the growth rate of $|\varphi'(z)|$ from the modulus of continuity and as a corollary, the growth rate of $|\varphi'(z)|$ for Kleinian groups with bounded geometry. It is an alternative proof of a result obtained in our previous paper [9].

§ 2. A Hardy-Littlewood theorem

Let f be a continuous function on the unit circle. The modulus of continuity of f is the function

$$\omega(t) = \sup_{|\theta_1 - \theta_2| \leq t} |f(e^{i\theta_1}) - f(e^{i\theta_2})|.$$

In 1932, Hardy and Littlewood [2] shows the following theorem called a Hardy-Littlewood theorem.

Theorem 2.1 (cf. [1] p. 74). *Let f be a holomorphic function on the unit disk Δ and continuous on $\overline{\Delta} = \Delta \cup \partial\Delta$. Suppose that there exists $\alpha \in (0, 1]$ such that*

$$|f(e^{i\theta_1}) - f(e^{i\theta_2})| = O(|\theta_1 - \theta_2|^\alpha).$$

Then

$$|f'(z)| = O((1 - |z|)^{\alpha-1})$$

holds as $|z| \rightarrow 1$.

In this section, we shall show the following theorem of Hardy-Littlewood type for holomorphic functions whose modulus of continuity is $|\log |\theta||^{-\alpha}$.

Theorem 2.2. *Let f be a holomorphic function on the unit disk Δ and continuous on $\overline{\Delta} = \Delta \cup \partial\Delta$. Suppose that there exists $\alpha > 0$ such that*

$$(2.1) \quad |f(e^{i\theta_1}) - f(e^{i\theta_2})| = O(|\log |\theta_1 - \theta_2||^{-\alpha}),$$

if $|\theta_1 - \theta_2| < \delta$ for some $\delta \in (0, 1)$. Then,

$$(2.2) \quad |f'(z)| = O((1 - |z|)^{-1} |\log(1 - |z|)|^{-\alpha})$$

holds as $|z| \rightarrow 1$.

Proof. By Cauchy's integral formula,

$$f'(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(f(e^{it}) - f(e^{i\varphi}))e^{it}}{(e^{it} - z)^2} dt. \quad (z = re^{i\varphi})$$

Thus, we have

$$|f'(z)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f(e^{i(t+\varphi)}) - f(e^{i\varphi})|}{1 - 2r \cos t + r^2} dt.$$

Since

$$1 - 2r \cos t + r^2 \geq (1 - r)^2 + \frac{4rt^2}{\pi^2},$$

it follows from (2.1) that

$$(2.3) \quad |f'(z)| \leq \frac{A}{2\pi} \int_{-\delta}^{\delta} \frac{|\log |t||^{-\alpha}}{(1 - r)^2 + 4r(t/\pi)^2} dt + B.$$

Setting $C_r = \pi^2(1 - r)^2/4r$ and $t = \sqrt{C_r} \tan \theta$, we have

$$\begin{aligned} I(z) &:= \frac{A}{2\pi} \int_{-\delta}^{\delta} \frac{|\log |t||^{-\alpha}}{(1 - r)^2 + 4r(t/\pi)^2} dt = \frac{A\pi}{8r} \int_{-\delta}^{\delta} \frac{|\log |t||^{-\alpha}}{t^2 + C_r} dt \\ &= \frac{A\pi}{8r\sqrt{C_r}} \int_{-\beta_r}^{\beta_r} \frac{d\theta}{|\log \sqrt{C_r} + \log |\tan \theta||^{\alpha}} \\ &= \frac{A\pi}{4r\sqrt{C_r}} \int_0^{\beta_r} \frac{d\theta}{|\log \sqrt{C_r} + \log |\tan \theta||^{\alpha}}, \end{aligned}$$

where $\beta_r = \arctan \frac{\delta}{\sqrt{C_r}} \in (0, \frac{\pi}{2})$. As $r = |z| \rightarrow 1$, $C_r \rightarrow 0$ and $\beta_r \rightarrow \frac{\pi}{2}$. We take $r > 0$ sufficiently close to 1 so that $C_r < 1$.

When $\theta \in (0, \frac{\pi}{4}]$, $\tan \theta \in (0, 1]$. Hence

$$\log \sqrt{C_r} + \log |\tan \theta| \leq \log \sqrt{C_r} < 0,$$

and

$$\left| \log \sqrt{C_r} + \log |\tan \theta| \right|^{-\alpha} \leq \left| \log \sqrt{C_r} \right|^{-\alpha}.$$

Thus, we have

$$(2.4) \quad \int_0^{\pi/4} \frac{d\theta}{\left| \log \sqrt{C_r} + \log |\tan \theta| \right|^{\alpha}} = O(|\log(1 - |z|)|^{-\alpha}),$$

because $C_r = O((1 - |z|)^2)$.

Next, we take a constant $\lambda \in (\frac{1}{2}, 1)$ and put $\gamma_r := \arctan\left(\frac{1}{\sqrt{C_r}}\right)^\lambda$. We may assume that $\gamma_r < \beta_r$. When $\theta \in (\frac{\pi}{4}, \gamma_r]$, $\tan \theta \in (1, C_r^{-\lambda/2})$ and we have

$$\log \sqrt{C_r} + \log |\tan \theta| \leq (1 - \lambda) \log \sqrt{C_r} < 0.$$

This implies

$$\left| \log \sqrt{C_r} + \log |\tan \theta| \right|^{-\alpha} \leq (1 - \lambda)^{-\alpha} \left| \log \sqrt{C_r} \right|^{-\alpha}$$

and we have

$$(2.5) \quad \int_{\pi/4}^{\gamma_r} \frac{d\theta}{\left| \log \sqrt{C_r} + \log |\tan \theta| \right|^\alpha} = O\left(|\log(1 - |z|)|^{-\alpha}\right).$$

Finally, we consider the case where $\theta \in (\gamma_r, \beta_r]$. Since $\arctan x = \int_0^x \frac{1}{t^2+1} dt$, we have

$$\begin{aligned} \beta_r - \gamma_r &= \int_{(\sqrt{C_r})^{-\lambda}}^{\delta/\sqrt{C_r}} \frac{dx}{x^2 + 1} \\ &\leq \left(\frac{\delta}{\sqrt{C_r}} - \frac{1}{\sqrt{C_r^\lambda}} \right) \frac{\sqrt{C_r^{2\lambda}}}{\sqrt{C_r^{2\lambda}} + 1} = O\left(C_r^{\lambda-1/2}\right). \end{aligned}$$

On the other hand,

$$\log \sqrt{C_r} + \log |\tan \theta| \leq \log \delta < 0,$$

because $\tan \theta \leq \frac{\delta}{\sqrt{C_r}}$. Therefore, we conclude

$$(2.6) \quad \begin{aligned} \int_{\gamma_r}^{\beta_r} \frac{d\theta}{\left| \log \sqrt{C_r} + \log |\tan \theta| \right|^\alpha} &\leq (\beta_r - \gamma_r) |\log \delta|^{-\alpha} \\ &= O((1 - |z|)^{2\lambda-1}). \end{aligned}$$

Combining (2.4), (2.5) and (2.6), we have

$$I(z) = O\left((1 - |z|)^{-1} |\log(1 - |z|)|^{-\alpha}\right).$$

Thus, we complete the proof of the theorem. $\square$

§ 3. Conformal mappings on invariant components of Kleinian groups

Let G be a finitely generated non-elementary Kleinian group. The group G is said to have *bounded geometry* if there exists a constant $\varepsilon > 0$ such that the injectivity radius with respect to the hyperbolic metric at any point in $\mathbb{H}^3/G$ is greater than ε .

We also assume that G has a simply connected invariant component D and denote by φ a Riemann mapping from the unit disk Δ onto D as before. Many things are known for Kleinian groups with bounded geometry (cf. [5]). For example, the limit set of G is locally connected whenever it is connected. Particularly, H. Miyachi ([6]) shows the following;

Proposition 3.1. *Let G be a Kleinian group with bounded geometry having a simply connected invariant component D and $\varphi : \Delta \rightarrow D$ a Riemann mapping. Then, φ has a continuous extension to $\partial\Delta$ and*

$$|\varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})| = O(|\log |\theta_1 - \theta_2||^{-\alpha})$$

holds as $|\theta_1 - \theta_2| \rightarrow 0$.

From this proposition and Theorem 2.1, we immediately obtain a theorem which is shown in [9] by a different method;

Theorem 3.2. *Let G, D and φ be the same ones as in Proposition 3.1. Then,*

$$(3.1) \quad |\varphi'(z)| = O((1 - |z|)^{-1} |\log(1 - |z|)|^{-\alpha})$$

holds as $|z| \rightarrow 1$.

Remark. In [9], we have also shown that if G is a regular b -group, then

$$(3.2) \quad |\varphi'(z)| = O((1 - |z|)^{-1} |\log(1 - |z|)|^{-2})$$

and we obtain the modulus of continuity on $\partial\Delta$,

$$(3.3) \quad |\varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})| = O(|\log |\theta_1 - \theta_2||^{-1}).$$

by using (3.2). From Theorem 2.1 it seems to be difficult to show (3.2) from (3.3). Actually, Nolder and Oberlin [7] show the following;

Proposition 3.3. *Let $\omega(t)$ be a differentiable non-negative increasing function on $[0, \infty)$ having the decreasing derivative $\omega'(t)$. The following are equivalent:*

1. *If f is a holomorphic function with the modulus of continuity θ , then*

$$|f'(z)| = O(\omega'(1 - |z|)).$$

- 2.

$$\limsup_{t \rightarrow 0+} \frac{\omega(t)}{t\omega'(t)} < \infty.$$

In our case, (3.3) implies $\omega(t) = (-\log t)^{-1}$ for small $t > 0$ and $\omega'(t) = t^{-1}(\log t)^{-2}$. However,

$$\limsup_{t \rightarrow 0+} \frac{\omega(t)}{t\omega'(t)} = \lim_{t \rightarrow 0+} (-\log t) = +\infty.$$

Hence, the second condition is not satisfied and we can not apply the above proposition to get (3.3) from (3.2).

We have also characterized quasi-Fuchsian groups in terms of the growth of derivatives of Riemann mappings of invariant components.

Proposition 3.4 ([9]). *Let G be a Kleinian group having a simply connected invariant component D with $\partial D \subset \mathbb{C}$ and φ a conformal mapping of the unit disk Δ onto D . Suppose that D/G has no punctures. Then, the following conditions are equivalent.*

1. *There exist constants $\alpha > 0$, $A > 0$ and a point $\zeta_0 \in D$ such that for any $z \in \varphi^{-1}(G\zeta_0) \setminus \varphi^{-1}(\infty)$,*

$$(3.4) \quad |\varphi'(z)| \leq \frac{A}{(1 - |z|)|\log(1 - |z|)|^{2+\alpha}}$$

holds.

2. *G is a quasi-Fuchsian group.*

From Theorem 2.1 and the above one, immediately we have;

Theorem 3.5. *Let G , D and φ be the same ones as above. Suppose that $\varphi : \Delta \rightarrow D$ has the continuous extension to $\partial\Delta$. If the extension φ on $\partial\Delta$ satisfies*

$$|\varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})| = O(|\log |\theta_1 - \theta_2||^{-2-\alpha})$$

for some $\alpha > 0$. Then G is a quasi-Fuchsian group.

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