

RIMS-1960

**On Intrinsic Hodge-Tate-ness of
Galois Representations of Dimension Two**

By

Yuichiro HOSHI

April 2022



京都大学 数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

ON INTRINSIC HODGE-TATE-NESS OF GALOIS REPRESENTATIONS OF DIMENSION TWO

YUICHIRO HOSHI

APRIL 2022

ABSTRACT. — In the present paper, we first prove that, for an arbitrary reducible Hodge-Tate p -adic representation of dimension two of the absolute Galois group of a p -adic local field and an arbitrary continuous automorphism of the absolute Galois group, the p -adic Galois representation obtained by pulling back the given p -adic Galois representation by the given continuous automorphism is Hodge-Tate. Next, we also prove the existence of an irreducible Hodge-Tate p -adic representation of dimension two of the absolute Galois group of a p -adic local field and a continuous automorphism of the absolute Galois group such that the p -adic Galois representation obtained by pulling back the given p -adic Galois representation by the given continuous automorphism is not Hodge-Tate.

CONTENTS

INTRODUCTION	1
§1. AUT-INTRINSIC HODGE-TATE-NESS OF REPRESENTATIONS	2
§2. THE CASE OF REDUCIBLE REPRESENTATIONS OF DIMENSION TWO	6
§3. THE CASE OF IRREDUCIBLE REPRESENTATIONS OF DIMENSION TWO	9
REFERENCES	12

INTRODUCTION

In the present paper, we study the *intrinsic Hodge-Tate-ness* of p -adic representations of the absolute Galois group of a p -adic local field. In the present Introduction, let p be a prime number, k a finite extension of $\mathbb{Q}_p$, and $\bar{k}$ an algebraic closure of k . Write $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$ for the absolute Galois group of k determined by the algebraic closure $\bar{k}$. For a given $\mathbb{Q}_p$ -vector space V of finite dimension and a given continuous representation $\rho: G_k \rightarrow \text{Aut}_{\mathbb{Q}_p}(V)$ of G_k , we shall say that ρ is *Aut-intrinsically Hodge-Tate* if, for an arbitrary continuous automorphism α of G_k , the composite $\rho \circ \alpha: G_k \xrightarrow{\sim} G_k \rightarrow \text{Aut}_{\mathbb{Q}_p}(V)$ is Hodge-Tate [cf. Definition 1.3].

Let us first recall that the author of the present paper proved that

if p is *odd*, and $k = \mathbb{Q}_p$, then there exists a p -adic representation of G_k that is *Hodge-Tate* but *not Aut-intrinsically Hodge-Tate* [cf. [1, Remark 3.3.1]].

Moreover, in the present paper, we establish a refinement of this result. That is to say, we verify that

2020 MATHEMATICS SUBJECT CLASSIFICATION. — 11S20.

KEY WORDS AND PHRASES. — p -adic local field, p -adic Galois representation, Hodge-Tate, intrinsically Hodge-Tate, Aut-intrinsically Hodge-Tate, group of MLF-type.

there exists a p -adic representation of G_k that is *Hodge-Tate* but *not Aut-intrinsically Hodge-Tate* whenever p is *odd*, i.e., without the assumption that $k = \mathbb{Q}_p$ [cf. Corollary 1.5].

On the other hand, let us also observe that it is likely to be well-known that

an arbitrary *Hodge-Tate* p -adic representation of *dimension* 1 of G_k is *Aut-intrinsically Hodge-Tate* [cf. Theorem 2.7].

In this state of affairs, one may have the following question:

Is there a p -adic representation of *dimension* 2 of G_k that is *Hodge-Tate* but *not Aut-intrinsically Hodge-Tate*?

In the present paper, we give an answer to this question.

First, we consider the case where a given continuous representation is *reducible*. The first main result of the present paper is as follows [cf. Theorem 2.10]:

THEOREM A. — *Let V be a $\mathbb{Q}_p$ -vector space of dimension 2 and $\rho: G_k \rightarrow \text{Aut}_{\mathbb{Q}_p}(V)$ a continuous representation. Suppose that the continuous representation ρ is **reducible**. Then ρ is **Hodge-Tate** if and only if ρ is **Aut-intrinsically Hodge-Tate**.*

Next, we consider the case where a given continuous representation is *irreducible*. The second main result of the present paper is as follows [cf. Corollary 3.4]:

THEOREM B. — *Let p be an **odd** prime number. Then there exist a finite extension K of $\mathbb{Q}_p$, an algebraic closure $\overline{K}$ of K , a $\mathbb{Q}_p$ -vector space V of dimension 2, and a continuous representation $\rho: \text{Gal}(\overline{K}/K) \rightarrow \text{Aut}_{\mathbb{Q}_p}(V)$ that is **irreducible**, **abelian**, **crystalline** [hence also **Hodge-Tate**], but **not Aut-intrinsically Hodge-Tate**.*

ACKNOWLEDGMENTS

This research was supported by JSPS KAKENHI Grant Number 21K03162 and by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

1. AUT-INTRINSIC HODGE-TATE-NESS OF REPRESENTATIONS

In the present §1, we introduce the notion of *Aut-intrinsic Hodge-Tate-ness* of p -adic representations [cf. Definition 1.3 below]. Moreover, we prove the existence of a p -adic representation that is *potentially crystalline* [hence also *Hodge-Tate*] but *not Aut-intrinsically Hodge-Tate* [cf. Corollary 1.5 below]. Finally, we also recall some basic facts concerning *abelian Hodge-Tate* p -adic representations [cf. Lemma 1.8 below and Lemma 1.9 below].

DEFINITION 1.1. — We shall refer to a field isomorphic to a finite extension of $\mathbb{Q}_p$, for some prime number p , as an *MLF*. Here, “MLF” is to be understood as an abbreviation for “mixed-characteristic local field”.

In the remainder of the present §1, let k be an MLF and $\bar{k}$ an algebraic closure of k . Write $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$.

DEFINITION 1.2. — We shall write

- $k^{(d=1)} \subseteq k$ for the [unique] minimal MLF contained in k [i.e., the unique subfield of k isomorphic to $\mathbb{Q}_p$, for some prime number p],
- $\mathcal{O}_k \subseteq k$ for the ring of integers of k ,
- p_k for the characteristic of the residue field of $\mathcal{O}_k$,
- d_k for the extension degree of the [necessarily finite] extension $k/k^{(d=1)}$,
- $(k^\times)^\wedge$ for the profinite completion of the multiplicative module $k^\times$ of k ,
- G_k^{ab} for the topological abelianization of G_k , i.e., the quotient of G_k by the closure of the commutator subgroup of G_k , and
- $\text{rec}_k: (k^\times)^\wedge \xrightarrow{\sim} G_k^{\text{ab}}$ for the isomorphism induced by the reciprocity homomorphism $k^\times \hookrightarrow G_k^{\text{ab}}$ in local class field theory.

DEFINITION 1.3. — Let V be a $\mathbb{Q}_{p_k}$ -vector space of finite dimension and $\rho: G_k \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Then we shall say that ρ is *Aut-intrinsically Hodge-Tate* if, for an arbitrary continuous automorphism α of G_k , the composite $\rho \circ \alpha: G_k \xrightarrow{\sim} G_k \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ is Hodge-Tate.

The following result is a formal consequence of the main result of [1].

THEOREM 1.4. — *For each $\square \in \{\circ, \bullet\}$, let $k_\square$ be an MLF and $\bar{k}_\square$ an algebraic closure of $k_\square$. Let $\alpha: \text{Gal}(\bar{k}_\circ/k_\circ) \rightarrow \text{Gal}(\bar{k}_\bullet/k_\bullet)$ be an **open** continuous homomorphism [which thus implies that $p_{k_\circ} = p_{k_\bullet}$ — cf., e.g., [3, Proposition 3.4, (iii)] and [4, Proposition 3.6]]. Then the following two conditions are equivalent:*

(1) *There exists an isomorphism $\bar{k}_\bullet \xrightarrow{\sim} \bar{k}_\circ$ of fields that is **compatible** with the respective natural actions of $\text{Gal}(\bar{k}_\bullet/k_\bullet)$, $\text{Gal}(\bar{k}_\circ/k_\circ)$ on $\bar{k}_\bullet$, $\bar{k}_\circ$ relative to the given open continuous homomorphism $\alpha: \text{Gal}(\bar{k}_\circ/k_\circ) \rightarrow \text{Gal}(\bar{k}_\bullet/k_\bullet)$.*

(2) *For an arbitrary $\mathbb{Q}_{p_{k_\bullet}}$ -vector space $V_\bullet$ of finite dimension and an arbitrary continuous representation $\rho_\bullet: \text{Gal}(\bar{k}_\bullet/k_\bullet) \rightarrow \text{Aut}_{\mathbb{Q}_{p_{k_\bullet}}}(V_\bullet)$, if $\rho_\bullet$ is **potentially crystalline**, then the composite $\rho_\bullet \circ \alpha: \text{Gal}(\bar{k}_\circ/k_\circ) \xrightarrow{\sim} \text{Gal}(\bar{k}_\bullet/k_\bullet) \rightarrow \text{Aut}_{\mathbb{Q}_{p_{k_\bullet}}}(V_\bullet) = \text{Aut}_{\mathbb{Q}_{p_{k_\circ}}}(V_\bullet)$ is **Hodge-Tate**.*

PROOF. — The implication (1) $\Rightarrow$ (2) is immediate. To verify the implication (2) $\Rightarrow$ (1), suppose that condition (2) is satisfied. Then it follows immediately from [9, Chapter III, §A.4, Proposition 5], together with a similar argument to the argument applied in the proof of [1, Lemma 1.4], that the open continuous homomorphism α is of *HT-qLT-type* [cf. [1, Definition 1.3, (ii)]]. Thus, it follows from [1, Theorem 3.3] [cf. also Remark 1.4.1 below] that condition (1) is satisfied, as desired. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Theorem 1.4. $\square$

REMARK 1.4.1. — Unfortunately, the proof of [1, Theorem 3.3], which was applied in the proof of Theorem 1.4 of the present paper, contains an *inessential inaccuracy* [cf. (i) below]. In light of the importance of [1, Theorem 3.3] in the present paper, we thus pause to discuss how this inaccuracy may be amended.

(i) In the final portion of the proof of [1, Claim 3.3.A], the author of the present paper has claimed that $\beta_{k_\bullet, k_\circ}$ is *inertially compatible* with α . However, it is not clear that $\beta_{k_\bullet, k_\circ}$ is inertially compatible with α .

(ii) Thus, the statement of [1, Claim 3.3.A] should be replaced by the following text:

(*) Suppose that $k_\circ$ is *Galois* over $\mathbb{Q}_p$. Then the field $k_\circ$ is *isomorphic* to the field $k_\bullet$.

Here, let us observe that the argument given in the proof of [1, Claim 3.3.A] proves this assertion.

(iii) Next, let us observe that one verifies immediately from the various definitions involved that if the MLF k is *Galois* over $k^{(d=1)}$, then, for an arbitrary positive real number ν , the algebraic extension of k in $\bar{k}$ that corresponds to the higher ramification subgroup of G_k associated to ν in the “upper numbering” is *Galois* over $k^{(d=1)}$. In particular, in the situation of [1, Theorem 3.3], one may conclude — by applying the assertion (*) of (ii) to the various restrictions of the given continuous isomorphism [cf. the first paragraph of the proof of [1, Theorem 3.3]] α (respectively, of the inverse of the given continuous isomorphism α) to the open subgroups of $G_{k_\circ}$ (respectively, of $G_{k_\bullet}$) that correspond to the finite extensions of $k_\circ$ in $\bar{k}_\circ$ (respectively, of $k_\bullet$ in $\bar{k}_\bullet$) *Galois* over $\mathbb{Q}_p$ — that if $k_\circ$ is *Galois* over $\mathbb{Q}_p$, then the continuous isomorphism [cf. the first paragraph of the proof of [1, Theorem 3.3]] α is *compatible* with the respective *higher ramification subgroups* of $G_{k_\circ}, G_{k_\bullet}$ associated to the positive real numbers in the “upper numbering”. Thus, the conclusion of [1, Theorem 3.3] in the case where $k_\circ$ is *Galois* over $\mathbb{Q}_p$, hence also the conclusion of [1, Theorem 3.3] for an *arbitrary* $k_\circ$, follows immediately from [6, Theorem].

COROLLARY 1.5. — *Let k be an MLF and $\bar{k}$ an algebraic closure of k . Suppose that p_k is **odd**. Then there exist a $\mathbb{Q}_{p_k}$ -vector space V of finite dimension and a continuous representation $\rho: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ that is **potentially crystalline** [hence also **Hodge-Tate**] but **not Aut-intrinsically Hodge-Tate**.*

PROOF. — Let us first recall that if $d_k = 1$ (respectively, $d_k \neq 1$), then it follows from, for instance, the discussion given at the final portion of [7, Chapter VII, §5] (respectively, [4, Proposition 3.6] and [5, Corollary 1.6, (iv)]) that we have a continuous automorphism of $\text{Gal}(\bar{k}/k)$ such that an arbitrary automorphism of the field $\bar{k}$ is *not compatible* with the natural action of $\text{Gal}(\bar{k}/k)$ on $\bar{k}$ relative to the continuous automorphism of $\text{Gal}(\bar{k}/k)$. Thus, Corollary 1.5 follows from Theorem 1.4. This completes the proof of Corollary 1.5. $\square$

REMARK 1.5.1. — The content of Corollary 1.5 in the case where $d_k = 1$ is essentially contained in [1, Remark 3.3.1].

In the remainder of the present §1, let us recall some basic facts concerning *abelian Hodge-Tate p -adic representations*. Let E be either k or $k^{(d=1)}$. Suppose that E is

absolutely Galois, i.e., that the finite extension $E/k^{(d=1)}$ is Galois [cf. [3, Definition 4.2, (i)]]].

DEFINITION 1.6. — We shall write E_+ for the $\mathbb{Q}_{p_k}$ -vector space [necessarily of finite dimension] obtained by forming the underlying additive module of the MLF E . Thus, we have a natural injective continuous homomorphism $\mathcal{O}_E^\times \hookrightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(E_+)$, i.e., by multiplication, by means of which we regard $\mathcal{O}_E^\times$ as a [necessarily closed] subgroup of the topological group $\text{Aut}_{\mathbb{Q}_{p_k}}(E_+)$:

$$\mathcal{O}_E^\times \subseteq \text{Aut}_{\mathbb{Q}_{p_k}}(E_+).$$

DEFINITION 1.7. — Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$ and σ an element of $\text{Gal}(E/k^{(d=1)})$. If $E = k$ (respectively, $E = k^{(d=1)}$), then we shall write

$$\Phi_\sigma: \mathcal{O}_k^\times \longrightarrow \mathcal{O}_E^\times$$

for the continuous automorphism of $\mathcal{O}_k^\times$ determined by σ (respectively, the continuous homomorphism $\mathcal{O}_k^\times \rightarrow \mathcal{O}_{k^{(d=1)}}^\times$ determined by the norm map with respect to the finite extension $k/k^{(d=1)}$). Moreover, we shall write

$$\chi_{\pi, \sigma}: G_k^{\text{ab}} \xrightarrow[\sim]{\text{rec}_k^{-1}} (k^\times)^\wedge \longrightarrow \mathcal{O}_k^\times \xrightarrow{\Phi_\sigma} \mathcal{O}_E^\times$$

— where the second arrow is the surjective continuous homomorphism obtained by considering the quotient by the closed submodule of the topological module $(k^\times)^\wedge$ topologically generated by $\pi \in k^\times$.

LEMMA 1.8. — Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$ and $\phi: G_k^{\text{ab}} \rightarrow \mathcal{O}_E^\times$ a continuous homomorphism. Then the following two conditions are equivalent:

- (1) The continuous representation obtained by forming the composite

$$G_k \longrightarrow G_k^{\text{ab}} \xrightarrow{\phi} \mathcal{O}_E^\times \hookrightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(E_+)$$

— where the first arrow is the natural surjective continuous homomorphism, and the third arrow is the natural inclusion — is **Hodge-Tate**.

- (2) There exist an integer i_σ for each $\sigma \in \text{Gal}(E/k^{(d=1)})$ and an open subgroup J of the inertia subgroup of G_k such that

- the restriction to J of the composite of the natural surjective continuous homomorphism $G_k \twoheadrightarrow G_k^{\text{ab}}$ and the given homomorphism $\phi: G_k^{\text{ab}} \rightarrow \mathcal{O}_E^\times$

coincides with

- the restriction to J of the composite of the natural surjective continuous homomorphism $G_k \twoheadrightarrow G_k^{\text{ab}}$ and the homomorphism

$$\prod_{\sigma \in \text{Gal}(E/k^{(d=1)})} \chi_{\pi, \sigma}^{i_\sigma}: G_k^{\text{ab}} \longrightarrow \mathcal{O}_E^\times.$$

PROOF. — This assertion follows from [9, Chapter III, §A.5, Corollary]. □

LEMMA 1.9. — Let $\phi: G_k^{\text{ab}} \rightarrow \mathcal{O}_k^\times$ be a continuous homomorphism. Suppose that the continuous representation obtained by forming the composite

$$G_k \twoheadrightarrow G_k^{\text{ab}} \xrightarrow{\phi} \mathcal{O}_k^\times \hookrightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(k_+)$$

— where the first arrow is the natural surjective continuous homomorphism, and the third arrow is the natural inclusion — is **Hodge-Tate**. Then the image of some open submodule of $\mathcal{O}_{k(d=1)}^\times$ by the composite

$$\mathcal{O}_{k(d=1)}^\times \hookrightarrow \mathcal{O}_k^\times \xrightarrow{\text{rec}_k} G_k^{\text{ab}} \xrightarrow{\phi} \mathcal{O}_k^\times$$

— where the first arrow is the natural inclusion — is **contained** in the submodule $\mathcal{O}_{k(d=1)}^\times \subseteq \mathcal{O}_k^\times$.

PROOF. — This assertion follows immediately from Lemma 1.8. $\square$

2. THE CASE OF REDUCIBLE REPRESENTATIONS OF DIMENSION TWO

In the present §2, we introduce the notion of *intrinsic Hodge-Tate-ness* of p -adic representations [cf. Definition 2.2 below]. Moreover, we prove that an arbitrary *reducible Hodge-Tate* p -adic representation of *dimension 2* is *Aut-intrinsically Hodge-Tate* [cf. Theorem 2.10 below].

DEFINITION 2.1. — We shall refer to a group isomorphic to the absolute Galois group of an MLF as a group of *MLF-type* [cf. [2, Definition 1.1]]. Here, “MLF” is to be understood as an abbreviation for “mixed-characteristic local field”. Let us always regard a group of MLF-type as a profinite group by means of the profinite topology discussed in [2, Proposition 1.2, (i)].

In the remainder of the present §2, let G be a group of MLF-type. Thus, by applying various functorial group-theoretic reconstruction algorithms established in the study of mono-anabelian geometry to the group G of MLF-type, we obtain

- a prime number $p(G)$ [cf. [4, Definition 3.5, (i)]],
- a positive integer $d(G)$ [cf. [4, Definition 3.5, (ii)]],
- a normal closed subgroup $I(G) \subseteq G$ of G [cf. [4, Definition 3.5, (iii)]],
- topological modules $\mathcal{O}^\times(G) \subseteq k^\times(G)$ [cf. [4, Definition 3.10, (i), (iv)]], and
- a topological field $\mathbb{Q}_p(G)$ [cf. [3, Definition 4.5, (iii)] and [3, Lemma 4.6, (i)]].

Moreover, in the remainder of the present §2, let V be a $\mathbb{Q}_p(G)$ -vector space of finite dimension and $\rho: G \rightarrow \text{Aut}_{\mathbb{Q}_p(G)}(V)$ a continuous representation.

DEFINITION 2.2. — We shall say that the given continuous representation ρ is *intrinsically Hodge-Tate* if, for an arbitrary MLF-envelope $(k, \bar{k}, \alpha: \text{Gal}(\bar{k}/k) \xrightarrow{\sim} G)$ of G [cf. [2, Definition 1.1]], the continuous [cf. [2, Proposition 1.2, (ii)]] representation obtained by forming the composite $\rho \circ \alpha: \text{Gal}(\bar{k}/k) \xrightarrow{\sim} G \rightarrow \text{Aut}_{\mathbb{Q}_p(G)}(V) = \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ [cf. [3, Lemma 4.6, (i)] and [4, Proposition 3.6]] is Hodge-Tate.

REMARK 2.2.1. — In the situation of Definition 1.3, it is immediate that the implications

$$\begin{aligned} \rho \text{ is intrinsically Hodge-Tate} &\implies \rho \text{ is Aut-intrinsically Hodge-Tate} \\ &\implies \rho \text{ is Hodge-Tate} \end{aligned}$$

hold.

DEFINITION 2.3. — Let V' be a $\mathbb{Q}_p(G)$ -vector space of finite dimension and $\rho': G \rightarrow \text{Aut}_{\mathbb{Q}_p(G)}(V')$ a continuous representation. Then we shall say that ρ is *inertially isomorphic* to ρ' if there exists an open subgroup $J \subseteq I(G)$ of $I(G)$ such that the restriction of ρ to $J \subseteq (I(G) \subseteq) G$ is isomorphic to the restriction of ρ' to $J \subseteq (I(G) \subseteq) G$.

DEFINITION 2.4. — Let w be an integer. Then we shall say that the continuous representation ρ is *w-cyclotomic* if ρ is isomorphic to the continuous representation of dimension 1 obtained by considering the w -th power of the character $G \rightarrow \mathbb{Q}_p(G)^\times$ determined by the maximal pro- $p(G)$ quotient of the cyclotome $\Lambda(G)$ associated to G [cf. [4, Definition 4.1, (iii)]].

REMARK 2.4.1. — Let k be an MLF and $\bar{k}$ an algebraic closure of k . Write $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$.

(i) Let us recall from [4, Proposition 3.6] that the normal closed subgroup $I(G_k) \subseteq G_k$ of G_k coincides with the *inertia subgroup* of G_k .

(ii) Let us recall from [4, Proposition 4.2, (iv)] that the character $G_k \rightarrow \mathbb{Q}_p(G_k)^\times \xleftarrow{\sim} \mathbb{Q}_{p_k}^\times$ [cf. [3, Lemma 4.6, (i)] and [4, Proposition 3.6]] determined by the maximal pro- $p(G_k)$, i.e., pro- p_k [cf. [4, Proposition 3.6]], quotient of the cyclotome $\Lambda(G_k)$ associated to G_k coincides with the *p_k -adic cyclotomic character* of G_k .

LEMMA 2.5. — Let k be an MLF, $\bar{k}$ an algebraic closure of k , V a $\mathbb{Q}_{p_k}$ -vector space **of dimension 1**, and $\rho: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Then the following two conditions are equivalent:

- (1) The continuous representation ρ is **Hodge-Tate**.
- (2) The continuous representation ρ is **inertially isomorphic** to the **w-cyclotomic** representation of $\text{Gal}(\bar{k}/k)$ for some integer w .

PROOF. — This assertion follows — in light of Remark 2.4.1, (i), (ii) — from Lemma 1.8, together with [9, Chapter III, §A.4, Corollary]. $\square$

THEOREM 2.6. — Let G be a group of MLF-type, V a $\mathbb{Q}_p(G)$ -vector space **of dimension 1**, and $\rho: G \rightarrow \text{Aut}_{\mathbb{Q}_p(G)}(V)$ a continuous representation. Then the following two conditions are equivalent:

- (1) The continuous representation ρ is **intrinsically Hodge-Tate**.
- (2) The continuous representation ρ is **inertially isomorphic** to the **w-cyclotomic** representation of G for some integer w .

PROOF. — This assertion follows from Lemma 2.5. $\square$

THEOREM 2.7. — *Let k be an MLF, $\bar{k}$ an algebraic closure of k , V a $\mathbb{Q}_{p_k}$ -vector space of dimension 1, and $\rho: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Then the following three conditions are equivalent:*

- (1) *The continuous representation ρ is **Hodge-Tate**.*
- (2) *The continuous representation ρ is **intrinsically Hodge-Tate**.*
- (3) *The continuous representation ρ is **Aut-intrinsically Hodge-Tate**.*

PROOF. — It follows from Remark 2.2.1 that, to verify Theorem 2.7, it suffices to verify the implication (1) $\Rightarrow$ (2). On the other hand, the implication (1) $\Rightarrow$ (2) follows from Lemma 2.5 and Theorem 2.6. This completes the proof of Theorem 2.7. $\square$

LEMMA 2.8. — *Let k be an MLF, $\bar{k}$ an algebraic closure of k , V a $\mathbb{Q}_{p_k}$ -vector space of dimension 2, and $\rho: G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Suppose that the continuous representation ρ is **reducible**. Then the continuous representation ρ is **Hodge-Tate** if and only if there exist integers w, w' and a G_k -stable $\mathbb{Q}_{p_k}$ -subspace $W \subseteq V$ of V of dimension 1 such that the continuous representations $G_k \rightarrow \text{Aut}(W)$, $G_k \rightarrow \text{Aut}(V/W)$ determined by ρ are, respectively, **inertially isomorphic** to the **w -cyclotomic**, **w' -cyclotomic** representations of G_k , and, moreover, one of the following two conditions is satisfied:*

- (1) *There exists an open subgroup J of the inertia subgroup of G_k such that the natural surjective homomorphism $V \twoheadrightarrow V/W$ has a **J-equivariant splitting**.*
- (2) *The equality $w = w'$ does **not hold**.*

PROOF. — First, we verify sufficiency. Suppose that there exist w, w', W as in the statement of Lemma 2.8. If condition (1) is satisfied, then it follows immediately — in light of Remark 2.4.1, (i), (ii), and [9, Chapter III, §A.1, Corollary 2] — from Lemma 2.5 that the continuous representation ρ is **Hodge-Tate**. If condition (2) is satisfied, then it follows immediately — in light of Remark 2.4.1, (i), (ii), and [9, Chapter III, §A.1, Corollary 2] — from [10, Proposition 8, (b)] that the continuous representation ρ is **Hodge-Tate**. This completes the proof of sufficiency.

Next, we verify necessity. Suppose that the continuous representation ρ is **Hodge-Tate**. Then since [we have assumed that] the continuous representation ρ is **reducible** and of dimension 2, there exists a G_k -stable $\mathbb{Q}_{p_k}$ -subspace $W \subseteq V$ of V of dimension 1. Now since ρ is **Hodge-Tate**, and both W and V/W are of dimension 1, it follows from Lemma 2.5 that there exist integers w, w' such that the continuous representations $G_k \rightarrow \text{Aut}(W)$, $G_k \rightarrow \text{Aut}(V/W)$ determined by ρ are, respectively, **inertially isomorphic** to the **w -cyclotomic**, **w' -cyclotomic** representations of G_k . Now suppose that condition (2) is **not satisfied**. Then it follows immediately from [8, Corollary 1] that condition (1) is satisfied, as desired. This completes the proof of necessity, hence also of Lemma 2.8. $\square$

THEOREM 2.9. — *Let G be a group of MLF-type, V a $\mathbb{Q}_p(G)$ -vector space of dimension 2, and $\rho: G \rightarrow \text{Aut}_{\mathbb{Q}_p(G)}(V)$ a continuous representation. Suppose that ρ is **reducible**. Then the continuous representation ρ is **intrinsically Hodge-Tate** if and only if there*

exist integers w, w' and a G -stable $\mathbb{Q}_p(G)$ -subspace $W \subseteq V$ of V **of dimension 1** such that the continuous representations $G \rightarrow \text{Aut}(W)$, $G \rightarrow \text{Aut}(V/W)$ determined by ρ are, respectively, **inertially isomorphic** to the **w -cyclotomic**, **w' -cyclotomic** representations of G , and, moreover, one of the following two conditions is satisfied:

- (1) There exists an open subgroup $J \subseteq I(G)$ of $I(G)$ ($\subseteq G$) such that the natural surjective homomorphism $V \twoheadrightarrow V/W$ has a **J -equivariant splitting**.
- (2) The equality $w = w'$ does **not hold**.

PROOF. — This assertion follows — in light of Remark 2.4.1, (i) — from Lemma 2.8. $\square$

THEOREM 2.10. — Let k be an MLF, $\bar{k}$ an algebraic closure of k , V a $\mathbb{Q}_{p_k}$ -vector space **of dimension 2**, and $\rho: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Suppose that the continuous representation ρ is **reducible**. Then the following three conditions are equivalent:

- (1) The continuous representation ρ is **Hodge-Tate**.
- (2) The continuous representation ρ is **intrinsically Hodge-Tate**.
- (3) The continuous representation ρ is **Aut-intrinsically Hodge-Tate**.

PROOF. — It follows from Remark 2.2.1 that, to verify Theorem 2.10, it suffices to verify the implication (1) $\Rightarrow$ (2). On the other hand, the implication (1) $\Rightarrow$ (2) follows — in light of Remark 2.4.1, (i) — from Lemma 2.8 and Theorem 2.9. This completes the proof of Theorem 2.10. $\square$

3. THE CASE OF IRREDUCIBLE REPRESENTATIONS OF DIMENSION TWO

In the present §3, we prove the existence of an *irreducible crystalline* [hence also *Hodge-Tate*] p -adic representation of dimension 2 that is *not Aut-intrinsically Hodge-Tate* [cf. Corollary 3.4 below].

In the present §3, let k be an MLF and $\bar{k}$ an algebraic closure of k . Write $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$. We shall also apply the notational conventions introduced in Definition 1.2.

LEMMA 3.1. — Suppose that p_k is **odd**, and that $d_k = 2$. Write $\text{Nm}: k^\times \rightarrow (k^{(d=1)})^\times$ for the norm map with respect to the finite extension $k/k^{(d=1)}$. Then the following assertions hold:

- (i) There exists an **open** submodule $U \subseteq \mathcal{O}_k^\times$ of $\mathcal{O}_k^\times$ such that
 - (1) the topological module U has a natural structure of **free $\mathbb{Z}_{p_k}$ -module of rank 2**, and, moreover,
 - (2) the submodule $U \subseteq \mathcal{O}_k^\times$ is **preserved** by an arbitrary continuous automorphism of $\mathcal{O}_k^\times$.
- (ii) Let $U \subseteq \mathcal{O}_k^\times$ be as in (i). Then the topological modules $U \cap \mathcal{O}_{k^{(d=1)}}^\times$, $U \cap \text{Ker}(\text{Nm})$ have natural structures of **free $\mathbb{Z}_{p_k}$ -modules of rank 1**, respectively.
- (iii) Let $U \subseteq \mathcal{O}_k^\times$ be as in (i). Then the **equality** $U \cap \mathcal{O}_{k^{(d=1)}}^\times \cap \text{Ker}(\text{Nm}) = \{1\}$ holds.

(iv) Let $U \subseteq \mathcal{O}_k^\times$ be as in (i). Then the closed submodule of U topologically generated by the closed submodules $U \cap \mathcal{O}_{k(d=1)}^\times$ and $U \cap \text{Ker}(\text{Nm})$ is **open**.

(v) There exists a continuous automorphism α of G_k such that, for an arbitrary nonzero integer n , if one writes $\alpha_\times^n$ for the continuous automorphism of $\mathcal{O}_k^\times$ induced by α^n [cf. [4, Proposition 3.11, (iv)]], then the intersection $\alpha_\times^n(\mathcal{O}_{k(d=1)}^\times) \cap \mathcal{O}_{k(d=1)}^\times$ is **not open** in $\mathcal{O}_{k(d=1)}^\times$. In particular, the continuous automorphism $\alpha_\times^n$ of $\mathcal{O}_k^\times$ does **not preserve** the submodule $\mathcal{O}_{k(d=1)}^\times \subseteq \mathcal{O}_k^\times$.

PROOF. — Assertions (i), (ii) follow from [4, Lemma 1.2, (i)] [cf. also our assumption that $d_k = 2$]. Assertion (iii) is immediate [cf. the fact that U is *torsion-free* — cf. condition (1) of assertion (i)]. Assertion (iv) follows from assertions (ii), (iii), together with condition (1) of assertion (i).

Finally, we verify assertion (v). Let α be a continuous automorphism of G_k as in the discussion preceding [5, Theorem 1.5] [cf. also [4, Proposition 3.6]]. Write β for the continuous automorphism of the submodule $U \subseteq \mathcal{O}_k^\times$ obtained by forming the restriction of $\alpha_\times^n$ [cf. condition (2) of assertion (i)]. Thus, it follows immediately from [5, Theorem 1.5] [cf. also [4, Definition 3.10, (vi)]] that

(a) the continuous automorphism β is *not the identity automorphism* of U , but

(b) the image of the *square* of the endomorphism of U given by “ $a \mapsto \beta(a) \cdot a^{-1}$ ” consists of the identity element of U .

Moreover, it follows immediately from [5, Lemma 2.3, (i)] that

(c) the continuous automorphism β *preserves* the submodule $U \cap \text{Ker}(\text{Nm})$ of U .

Thus, it follows immediately from assertion (iv) [cf. also condition (1) of assertion (i)], together with (b) and (c), that if the continuous automorphism β *preserves* some open submodule of the submodule $U \cap \mathcal{O}_{k(d=1)}^\times$, then β is the *identity automorphism* of U — in contradiction to (a). In particular, the continuous automorphism β does *not preserve* any open submodule of the submodule $U \cap \mathcal{O}_{k(d=1)}^\times$, which thus implies [cf. assertion (ii)] that $\beta(U \cap \mathcal{O}_{k(d=1)}^\times) \cap U \cap \mathcal{O}_{k(d=1)}^\times = \{1\}$. Thus, it follows immediately from [4, Lemma 1.2, (i)] that $\alpha_\times^n(\mathcal{O}_{k(d=1)}^\times) \cap \mathcal{O}_{k(d=1)}^\times$ is *not open* in $\mathcal{O}_{k(d=1)}^\times$, as desired. This completes the proof of assertion (v), hence also of Lemma 3.1. $\square$

REMARK 3.1.1. — One may conclude from the final portion of Lemma 3.1, (v), that it is *impossible* to establish a functorial group-theoretic reconstruction algorithm for constructing, from an *arbitrary* group H of MLF-type, a closed submodule of the topological module $\mathcal{O}^\times(H)$ which “*corresponds*” to the closed submodule $\mathcal{O}_{k(d=1)}^\times \subseteq \mathcal{O}_k^\times$ of the topological module $\mathcal{O}_k^\times$. Put another way, one may conclude from the final portion of Lemma 3.1, (v), that the closed submodule $\mathcal{O}_{k(d=1)}^\times \subseteq \mathcal{O}_k^\times$ should be considered to be “*not group-theoretic*”.

PROPOSITION 3.2. — Suppose that p_k is **odd**, and that d_k is **even**. Suppose, moreover, that k is **absolutely abelian**, i.e., that k is absolutely Galois, and the Galois group $\text{Gal}(k/k^{(d=1)})$ is abelian [cf. [3, Definition 4.2, (ii)]]. Then there exists a continuous automorphism α of G_k such that, for an arbitrary nonzero integer n , if one writes $\alpha_\times^n$ for the continuous automorphism of $\mathcal{O}_k^\times$ induced by α^n [cf. [4, Proposition 3.11, (iv)]], then the intersection $\alpha_\times^n(\mathcal{O}_{k(d=1)}^\times) \cap \mathcal{O}_{k(d=1)}^\times$ is **not open** in $\mathcal{O}_{k(d=1)}^\times$.

PROOF. — Let us first observe that since d_k is *even*, and k is *absolutely abelian*, one verifies easily that there exists a *quadratic* extension of $k^{(d=1)}$ contained in k . Moreover, since k is *absolutely abelian*, it follows immediately from the implication (1) $\Rightarrow$ (2) of [3, Theorem F, (i)] that G_k is a *characteristic* subgroup of the absolute Galois group of the quadratic extension of $k^{(d=1)}$ determined by the algebraic closure $\bar{k}$. Thus, one may conclude that we may assume without loss of generality, by applying a similar argument to the argument applied in the proof of [5, Lemma 2.6, (ii)] and replacing k by the quadratic extension of $k^{(d=1)}$, that $d_k = 2$. On the other hand, if $d_k = 2$, then the desired conclusion follows from Lemma 3.1, (v). This completes the proof of Proposition 3.2. $\square$

THEOREM 3.3. — *Let k be an MLF and $\bar{k}$ an algebraic closure of k . Suppose that p_k is **odd**, that d_k is **even**, and that k is **absolutely abelian**. Then there exist a $\mathbb{Q}_{p_k}$ -vector space V **of dimension d_k** and a continuous representation $\rho: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ that is **irreducible, abelian, crystalline** [hence also **Hodge-Tate**], but **not Aut-intrinsically Hodge-Tate**.*

PROOF. — Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$. Write ρ for the continuous representation of $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$ [necessarily of dimension d_k] obtained by forming the composite

$$G_k \twoheadrightarrow G_k^{\text{ab}} \xrightarrow{\chi_{\pi, \text{id}_k}} \mathcal{O}_k^{\times} \hookrightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(k_+)$$

— where the first arrow is the natural surjective continuous homomorphism, and the third arrow is the natural inclusion. Then one verifies easily that this continuous representation ρ is *irreducible* and *abelian*. Moreover, it follows immediately from [9, Chapter III, §A.4, Proposition 5] that this continuous representation ρ is *crystalline*.

Next, to verify that the continuous representation ρ is *not Aut-intrinsically Hodge-Tate*, let us recall that it follows immediately from the various definitions involved that the composite

$$\mathcal{O}_k^{\times} \xrightarrow{\text{rec}_k} G_k^{\text{ab}} \xrightarrow{\chi_{\pi, \text{id}_k}} \mathcal{O}_k^{\times}$$

is an *automorphism* that restricts to an *automorphism* of the submodule $\mathcal{O}_{k^{(d=1)}}^{\times} \subseteq \mathcal{O}_k^{\times}$. In particular, if α is a continuous automorphism of G_k as in Proposition 3.2, then it follows immediately from Lemma 1.9, together with the various definitions involved, that the composite $\rho \circ \alpha: G_k \xrightarrow{\sim} G_k \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(k_+)$ is *not Hodge-Tate*, which thus implies that the continuous representation ρ is *not Aut-intrinsically Hodge-Tate*, as desired. This completes the proof of Theorem 3.3. $\square$

COROLLARY 3.4. — *Let p be an **odd** prime number. Then there exist an MLF K such that $p_K = p$, an algebraic closure $\bar{K}$ of K , a $\mathbb{Q}_{p_K}$ -vector space V **of dimension 2**, and a continuous representation $\rho: \text{Gal}(\bar{K}/K) \rightarrow \text{Aut}_{\mathbb{Q}_{p_K}}(V)$ that is **irreducible, abelian, crystalline** [hence also **Hodge-Tate**], but **not Aut-intrinsically Hodge-Tate**.*

PROOF. — This assertion is a formal consequence of Theorem 3.3. $\square$

REFERENCES

- [1] Y. Hoshi: A note on the geometricity of open homomorphisms between the absolute Galois groups of p -adic local fields. *Kodai Math. J.* **36** (2013), no. **2**, 284–298.
- [2] Y. Hoshi: Mono-anabelian reconstruction of number fields. *On the examination and further development of inter-universal Teichmüller theory*, 1–77, *RIMS Kôkyûroku Bessatsu*, **B76**, Res. Inst. Math. Sci. (RIMS), Kyoto, 2019.
- [3] Y. Hoshi: Topics in the anabelian geometry of mixed-characteristic local fields. *Hiroshima Math. J.* **49** (2019), no. **3**, 323–398.
- [4] Y. Hoshi: Introduction to mono-anabelian geometry. *Publications mathématiques de Besançon. Algèbre et théorie des nombres. 2021*, 5–44, *Publ. Math. Besançon Algèbre Théorie Nr.*, 2021, Presses Univ. Franche-Comté, Besançon, 2021.
- [5] Y. Hoshi and Y. Nishio: *On the outer automorphism groups of the absolute Galois groups of mixed-characteristic local fields*. RIMS Preprint **1931** (November 2020).
- [6] S. Mochizuki: A version of the Grothendieck conjecture for p -adic local fields. *Internat. J. Math.* **8** (1997), no. **4**, 499–506.
- [7] J. Neukirch, A. Schmidt, and K. Wingberg: *Cohomology of number fields*. Second edition. Grundlehren der Mathematischen Wissenschaften, **323**. Springer-Verlag, Berlin, 2008.
- [8] S. Sen: Lie algebras of Galois groups arising from Hodge-Tate modules. *Ann. of Math. (2)* **97** (1973), 160–170.
- [9] J.-P. Serre: *Abelian l -adic representations and elliptic curves*. With the collaboration of Willem Kuyk and John Labute. Revised reprint of the 1968 original. Research Notes in Mathematics, **7**. A K Peters, Ltd., Wellesley, MA, 1998.
- [10] J.-T. Tate: p -divisible groups. 1967 *Proc. Conf. Local Fields (Driebergen, 1966)* pp. 158–183 Springer, Berlin.

(Yuichiro Hoshi) RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

Email address: `yuichiro@kurims.kyoto-u.ac.jp`