

Clarkson, random Clarkson inequalities and
Rademacher type for Banach spaces

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Introduction

In connection with generalized Clarkson's inequalities (Kato [4]; see also [11]), a high-dimensional version of Clarkson-Boas-Koskela-type inequalities (cf. [2], [1], [8]), Tonge [13] presented random Clarkson inequalities for L_p . On the other hand, the authors [6] characterized those Banach spaces satisfying (p, p') -Clarkson's inequality as of (Rademacher) type p with type p constant one, and using this, they [12] proved that if (p, p') -Clarkson's inequality holds in a Banach space X , then the random Clarkson inequality also holds in X , where the unknown absolute constant included in Tonge's original inequalities was replaced by one. We point out here that there are fairly many Banach spaces in which (p, p') -Clarkson's inequality holds (cf. [1], [3], [9]; see also [11]).

In this note, we present a more general high-dimensional version of Clarkson-Boas-koskela-type inequalities for Banach spaces X of type p , and using this, it is shown that if X is of type p , then Tonge-type random Clarkson inequality including an absolute constant holds in X ; in this case we can take type p constant as such an absolute constant.

1. Preliminaries

In the following, let $p', q', r', \dots$ denote the conjugate numbers of $p, q, r, \dots$

1.1. Clarkson's inequalities (Clarkson [2]): $\forall x, y \in L_p$

$$(CI-1) \quad (\|x + y\|^{p'} + \|x - y\|^{p'})^{1/p'} \leq 2^{1/p'} (\|x\|^p + \|y\|^p)^{1/p},$$

if $1 \leq p \leq 2$.

$$(CI-2) \quad (\|x + y\|^p + \|x - y\|^p)^{1/p} \leq 2^{1/p} (\|x\|^{p'} + \|y\|^{p'})^{1/p'},$$

if $2 \leq p \leq \infty$.

Let $1 \leq p \leq 2$. We say that a Banach space X satisfies (p, p') -Clarkson's inequality if (CI-1) holds in X . It is known (cf. [6]) that X satisfies (p, p') -Clarkson's inequality if and only if its dual X' does. This shows that one of the inequalities (CI-1) and (CI-2) implies the others. We note here that Clarkson [2] proved the other several inequalities for L_p , but those follow from (CI-1) or (CI-2).

Boas [1] and Koskela [8] considered some generalizations of Clarkson inequalities for a much wider class of parameters, which we call Clarkson-Boas-Koskela-type inequalities:

1.2. Clarkson-Boas-Koskela's inequalities (Boas[1], Koskela [8] for L_p ; see also [5]): Let $1 \leq p, r, s \leq \infty$. Then, $\forall x, y \in L_p$

$$(CBKI) \quad (\|x + y\|^s + \|x - y\|^s)^{1/s} \leq 2^{c(r,s;p)} (\|x\|^r + \|y\|^r)^{1/r},$$

where $c(r,s;p) = \max\{1/r', 1/s, 1/r' + 1/s - \min(1/p, 1/p')\}$.

In the inequality (CBKI), let $r = \min\{p, p'\}$ and $s = \max\{p, p'\}$. Then we have the Clarkson inequalities (CI-1) and (CI-2). We note that if $p = 1$ or ∞ , then (CBKI) holds in any Banach space X ; and if (CI-1) or (CI-2) holds in X , then (CBKI) also holds in X . (In the next section, this facts are claimed in more general forms.)

Now we introduce some generalizations of (CBKI) to higher dimensions.

1.3. Littlewood matrices $A_n = (\varepsilon_{ij}) (2^n \times 2^n)$:

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix} \quad (n = 1, 2, \dots).$$

1.4. Generalized Clarkson's inequalities (Kato [4] for L_p):

Let $1 \leq p, r, s \leq \infty$. Then, $\forall x_1, \dots, x_{2^n} \in L_p$

$$(GCI) \quad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} x_j \right\|^s \right\}^{1/s} \leq 2^{nc(r,s;p)} \left\{ \sum_{j=1}^{2^n} \|x_j\|^r \right\}^{1/r}.$$

where $c(r,s;p) = \max\{1/r', 1/s, 1/r' + 1/s - \min(1/p, 1/p')\}$.

For $n = 1$, (GCI) coincides with (CBKI), and so (GCI) may be regarded as a generalization of (CBKI) to higher dimensions. We note that (GCI) can be slightly generalized for a much wider class of parameters r and s . (For the details, see Maligranda and Persson [10].)

In the context of type and cotype for Banach spaces, the authors [6] considered some generalizations of the Clarkson inequality (CI-1) to higher dimensions, and proved that a Banach space X satisfies (p, p') -Clarkson's inequality, $1 \leq p \leq 2$, if and only if X is of (Rademacher) type p and its type p constant is one.

1.5. Type inequalities: Let $1 \leq p \leq 2$. A Banach space X is said to be of (Rademacher) type p if there is a constant $K > 0$ such that

$$(TpI) \quad \left(\frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^{p'} \right)^{1/p'} \leq K \left(\sum_{j=1}^n \|x_j\|^p \right)^{1/p}.$$

for all $x_1, x_2, \dots, x_n \in X$.

The smallest constant K satisfying (TpI) is said to be type p constant and denoted by $T_p(X)$. (The notion of type p is also defined by usual

Rademacher functions, but it is the same as the above definition.)

2. Clarkson-Boas-Koskela-type inequalities and Rademacher type for Banach spaces

In this section, we present a high-dimensional version of Clarkson-Boas-Koskela-type inequalities for a Banach space X .

2.1. Clarkson-Boas-Koskela-type inequalities with n elements:

Let $1 \leq p, r, s \leq \infty$. Then, $\forall x_1, x_2, \dots, x_n \in X$

$$(nCBKI) \quad \left(\sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^s \right)^{1/s} \leq 2^{n/s} c(r, s; p)^{-1/s} \left(\sum_{j=1}^n \|x_j\|^r \right)^{1/r},$$

where $c(r, s; p) = \max\{1/r', 1/s, 1/r' + 1/s - \min(1/p, 1/p')\}$.

2.2. Theorem Let $1 \leq p \leq 2$. Then for a Banach space X , the following statements are equivalent.

- (i) (p, p') -Clarkson's inequality holds in X .
- (ii) X is of type p and its type p constant is one.
- (iii) For each r and s with $1 \leq r, s \leq \infty$, (nCBKI) holds in X .

2.3. Corollary (Maligranda and Persson [10]). Let $1 \leq p \leq \infty$.

Then for each r and s with $1 \leq r, s \leq \infty$, (nCBKI) holds in L_p .

2.4. Remark. Corollary 2.3 is also true in the case $r < 1$ ($r \neq 0$) or $0 < s < 1$. In this case, $c(r, s; p)$ is replaced by $c(u, v; p)$, where $u = \max\{r, 1\}$ and $v = \max\{s, 1\}$, see [10]. We can show that Theorem 2.2 is also true for such cases.

Now we introduce (nCBKI) including an absolute constant, and give a generalization of Theorem 2.2.

2.5. Theorem. Let $1 \leq p \leq 2$. If a Banach space X is of type p , and its type p constant $T_p(X) = K$, then (nCBKI) with the constant K holds in X , that is, for each r and s with $1 \leq r, s \leq \infty$, and for all $x_1, x_2, \dots, x_n \in X$,

$$(nCBKI) \quad \left(\sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^s \right)^{1/s} \leq K 2^{n/s} c(r, s; p)^{-1/s} \left(\sum_{j=1}^n \|x_j\|^r \right)^{1/r}$$

holds, where $c(r, s; p) = \max\{1/r', 1/s, 1/r' + 1/s - 1/p'\}$.

2.6. Remark. In Theorem 2.5, it is easy to see that if (nCBKI) with an absolute constant K holds in X , then $T_p(X) \leq K$ (put $r = p$ and $s = p'$). This means that $T_p(X)$ coincides with the smallest constant K satisfying (nCBKI); the constant K depends on X and p , but it is independent of r, s and n . We note that Theorem 2.5 is also true in the case $r < 1$ ($r \neq 0$) or $0 < s < 1$. In this case, $c(r, s; p)$ is replaced by $c(u, v; p)$, where $u = \max\{r, 1\}$ and $v = \max\{s, 1\}$.

3. Random Clarkson inequalities and Rademacher type for Banach spaces

In [13], Tonge presented random Clarkson inequalities for L_p .

For a Banach space X , these inequalities are stated as follows:

3.1. Random Clarkson inequalities: Let $1 \leq p \leq 2$ and $1 \leq r, s \leq \infty$. Let $A = (a_{ij})$ be an $n \times n$ matrix, where a_{ij} are independent identically distributed random variables taking the values ± 1 with equal probability $1/2$. Then, E denoting mathematical expectation, for any $x_1, x_2, \dots, x_n$ in X .

$$(RCI) \quad E \left(\sum_{i=1}^n \left\| \sum_{j=1}^n a_{ij} x_j \right\|^s \right)^{1/s} \leq K n^{c(r,s;p)} \left(\sum_{j=1}^n \|x_j\|^r \right)^{1/r},$$

where $c(r,s;p) = \max\{1/r', 1/s, 1/r' + 1/s - 1/p'\}$, and K is a positive absolute constant depending only on p and X .

3.2. Remark. In [13], Tonge proved that (RCI) holds in L_p . The authors [12] recently generalized this result for more general Banach spaces X , and proved that if X satisfies (p, p') -Clarkson inequality, then (RCI) holds in X , where the unknown absolute constant K was replaced by one.

Using Theorem 2.5, we can easily prove the following Tonge-type random Clarkson inequality for Banach spaces of type p .

3.3. Theorem. Let $1 \leq p \leq 2$. If a Banach space X is of type p , and its type p constant $T_p(X) = K$, then (RCI) with the constant K holds in X ; and conversely, if (RCI) holds in X , then X is of type p .

3. 4. **Remark.** If X satisfies (p, p') -Clarkson inequality, then it is of type p with type p constant one (see Theorem 2. 2). Hence, the authors' result [12] mentioned above follows from Theorem 3. 3. (For the details, see Kato and Takahashi [7]).

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