

An opinion on non-Imaginary part of Gamma-Starlike Functions

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1 Introduction.

Let A denote the class of functions $f(z)$ analytic in $E = \{z : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$.

A function $f(z) \in A$ is called starlike with respect to the origin if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } E,$$

and a function $f(z) \in A$ is said to be convex if and only if

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0. \quad \text{in } E.$$

In [1], Lewandowski, Miller and Zlotkiewicz defined Gamma-Starlike Function as the following.

Definition. Let $f(z) \in A$ and suppose that $f(z) \neq 0$, $f'(z) \neq 0$, and $1 + \frac{zf''(z)}{f'(z)} \neq 0$ in $0 < |z| < 1$.

Suppose γ is a real number and

$$(1) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left(1 + \frac{zf''(z)}{f'(z)} \right)^\gamma > 0$$

for $z \in E$, where the power appearing in (1) are meant as principal values.

If $f(z) \in A$ satisfies the condition (1), then we say that $f(z)$ is a gamma-starlike function and we denote the class of such functions by L_γ .

Remarks. (i) Condition (1) is equivalent to the following condition

$$|(1-\gamma)\arg \frac{zf'(z)}{f(z)} + \gamma \arg \left(1 + \frac{zf''(z)}{f'(z)} \right)| < \frac{\pi}{2}.$$

(ii) If $\gamma = 0$, $L_0 \equiv S^*$, the class of starlike functions, while if $\gamma = 1$, $L_1 \equiv C$, the class of convex functions.

In [1], they obtained the following result.

Theorem A. $L(\gamma) \subset S^*$, for all real γ .

Let N be the class of functions $p(z)$ analytic in E and $p(0) = 1$. We call $p(z) \in N$ a Carathéodory function, if it satisfies the condition $\operatorname{Re} p(z) > 0$ in E .

2 Preliminary.

In this paper, we need the following lemma.

Lemma [2]. Let $p(z) \in N$ and suppose that there exists a point $z_0 \in E$ such that $\operatorname{Re} p(z) > 0$ for $|z| < |z_0|$, and $\operatorname{Re} p(z_0) = 0$ ($p(z_0) \neq 0$).

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is a real number and

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \quad \text{when} \quad p(z_0) = ia, \quad a > 0,$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \text{when} \quad p(z_0) = ia, \quad a < 0.$$

3 Main result.

Now, we prove the following Theorem.

Theorem. Let $f(z) \in A$ and let $f(z) \neq 0$, $f'(z) \neq 0$ and $1 + \frac{zf''(z)}{f'(z)} \neq 0$ in $0 < |z| < 1$.

Suppose that $\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma} \left(1 + \frac{zf''(z)}{f'(z)}\right)^\gamma \neq il$ in E , where $\gamma \geq \frac{1}{2}$, l is a real number and

$$|l| > \begin{cases} \sqrt{\left(\frac{2\gamma-1}{3}\right)} \left(\frac{3\gamma}{2\gamma-1}\right)^\gamma, & \gamma > \frac{1}{2}, \\ \frac{\sqrt{2}}{2}, & \gamma = \frac{1}{2}, \end{cases}$$

and for the case $\gamma < \frac{1}{2}$, suppose that

$$(2) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left(1 + \frac{zf''(z)}{f'(z)} \right)^\gamma > 0 \quad \text{in } E.$$

Then $f(z)$ is starlike in E .

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f(z)}.$$

If there exists a point $z_0 \in E$ such that $\text{Rep}(z) > 0$ for $|z| < |z_0|$, and $\text{Rep}(z_0) = 0$ ($p(z_0) \neq 0$), then from Lemma, we have

$$\left(\frac{z_0 f'(z_0)}{f(z_0)}\right)^{1-\gamma} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right)^\gamma = (p(z_0))^{1-\gamma} \left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right)^\gamma.$$

Therefore we obtain

$$\begin{aligned} \text{Re}\left(\frac{z_0 f'(z_0)}{f(z_0)}\right)^{1-\gamma} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right)^\gamma &= \text{Re} (ia)^{1-\gamma} (ia + ik)^\gamma \\ &= \text{Re} ia \left(1 + \frac{k}{a}\right)^\gamma \\ &= 0 \end{aligned}$$

where $p(z_0) = ia$ (a is a real number) and from Lemma, a and k are the same sign.

For the case $a > 0$, let us put

$$g(a) = a \left(1 + \frac{k}{a}\right)^\gamma.$$

From Lemma, we have

$$(3) \quad g(a) \geq a \left(\frac{3}{2} + \frac{1}{2a^2}\right)^\gamma, \quad (\gamma \geq 0).$$

Putting $q(a)$ the last term of (3), let us get the minimum value m of $q(a)$ for $a > 0$. Differentiation $q(a)$, we have

$$(4) \quad q'(a) = \left(\frac{3}{2} + \frac{1}{2a^2}\right)^{\gamma-1} \left(\frac{3}{2} + \frac{1}{2a^2} - \frac{\gamma}{a^2}\right).$$

Since we have

$$\left(\frac{3}{2} + \frac{1}{2a^2}\right)^{\gamma-1} > 0,$$

and so $q'(a)$ become 0 only at $a = \sqrt{\frac{2\gamma-1}{3}}$.

Therefore, for the case $\gamma > \frac{1}{2}$, $q(a)$ takes its minimum value m at $a = \sqrt{\frac{2\gamma-1}{3}}$, and

$$m = q\left(\sqrt{\frac{2\gamma-1}{3}}\right) = \sqrt{\frac{2\gamma-1}{3}} \left(\frac{3\gamma}{2\gamma-1}\right)^\gamma,$$

and for the case $\gamma = \frac{1}{2}$, $q(a)$ takes its minimum value m at $a = 0$, and

$$m = \lim_{a \rightarrow +0} q(a) = \lim_{a \rightarrow +0} a \left(\frac{3}{2} + \frac{1}{2a^2}\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{2}.$$

These contradict (2).

On the other hand, if there exists a point $z_0 \in E$ such that $\text{Rep}(z) > 0$ for $|z| < |z_0|$, $\text{Rep}(z_0) = 0$ ($p(z_0) \neq 0$) and $p(z_0) = ia$, $a < 0$.

Applying the same method as the above, we have

$$q(a) \leq \begin{cases} -\sqrt{\left(\frac{2\gamma-1}{3}\right)}\left(\frac{3\gamma}{2\gamma-1}\right)^\gamma, & \gamma > \frac{1}{2}, \\ -\frac{\sqrt{2}}{2}, & \gamma = \frac{1}{2}, \end{cases}$$

These also contradict (2).

For the case $\gamma < \frac{1}{2}$, if there exists a point $z_0 \in E$ such that $\operatorname{Re} p(z) > 0$ for $|z| < |z_0|$ and $\operatorname{Re} p(z_0) = 0$ ($p(z_0) \neq 0$).

Then from Lemma, we have

$$\begin{aligned} \operatorname{Re}\left(\frac{z_0 f'(z_0)}{f(z_0)}\right)^{1-\gamma} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right)^\gamma &= \operatorname{Re}(p(z_0))^{1-\gamma} \left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right)^\gamma \\ &= 0. \end{aligned}$$

This contradicts (2).

Therefore we have $\operatorname{Re} p(z) > 0$ in E , or $f(z)$ is starlike in E .

From Main theorem we easily have Theorem A, and so this theorem completely improved Theorem A [1].

Further, letting $\gamma = 1$ in Main theorem, we easily have

Corollary [3]. If $f(z) \in A(1)$ and

$$\left| \operatorname{Im} \frac{z f''(z)}{f'(z)} \right| < \sqrt{3} \quad \text{in } E,$$

then $f(z)$ is univalently starlike in E .

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References

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