

Algorithms related to Diophantine approximations
and their applications

by

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§1. On the simple continued fraction expansion

Any irrational number α in the interval $[0,1)$ has
a simple continued fraction expansion

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \dots}} \quad (1.1)$$

where the digits a_i 's are positive integers.

The expansion (1.1) is obtained through the following
map $S: [0,1) \rightarrow$ such that

$$S\alpha = \frac{1}{\alpha} - \left[\frac{1}{\alpha} \right] \quad (1.2)$$

Put

$$a(\alpha) = \left[\frac{1}{\alpha} \right]$$

and

$$a_n(\alpha) = a(S^{n-1}\alpha) \quad n=1,2,\dots,$$

then $a_1(\alpha), a_2(\alpha), \dots$ are just the digits in the
expansion (1.1). Accordingly, we call the pair $([0,1), S)$
the algorithm which induces a simple continued fraction expansion.

We define $(q_n, p_n) (= (q_n(\alpha), p_n(\alpha)))$ recursively by

$$\begin{pmatrix} q_0 & q_{-1} \\ p_0 & p_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = \begin{pmatrix} q_{n-1} & q_{n-2} \\ p_{n-1} & p_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}. \quad (1.3)$$

Then standard induction arguments show that

$$\alpha = \frac{p_n + s^n \alpha \cdot p_{n-1}}{q_n + s^n \alpha \cdot q_{n-1}} \quad (1.4)$$

and

$$\frac{p_n}{q_n} = \frac{1}{a_1 + \dots + \frac{1}{a_n}} \quad \text{and} \quad \frac{q_{n-1}}{q_n} = \frac{1}{a_n + \dots + \frac{1}{a_1}}$$

we call (q_n, p_n) n -th principal convergent of α .

We know several metrical results concerning the approximation of α with (q_n, p_n) .

Metrical theorems

$$(1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \alpha - \frac{p_n}{q_n} \right| = - \frac{\pi^2}{6 \log 2}$$

$$(2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log q_n = \frac{\pi^2}{12 \log 2}$$

$$(3) \quad \text{for } 0 \leq \lambda \leq 1$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{ n; |q_n \alpha - p_n| < \lambda, n \leq N \} = \begin{cases} \frac{\lambda}{\log 2} & 0 \leq \lambda \leq \frac{1}{2} \\ \frac{-\lambda + \log 2 \lambda + 1}{\log 2} & \frac{1}{2} \leq \lambda \leq 1 \end{cases}$$

(see [1])

$$(4) \quad \text{for } 0 \leq \lambda \leq \frac{1}{2}$$

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \# \{ (q, p); |q \alpha - p| < \lambda, (q, p) = 1, q \leq N \} = \frac{\pi^2}{12} \cdot \lambda$$

(see [5] and c.f. [2]

, where '=' means a.e. equality.

The main tool to obtain these theorems is the map $\bar{S}$ which will be called a natural extension of the algorithm $([0, 1), S)$, and the ergodicity of the map $\bar{S}$.

Let $\bar{X} = [0,1) \times [0,1)$ and define a map $\bar{S}$ of $\bar{X}$ onto itself by

$$\bar{S}(\alpha, \beta) = (S\alpha, \frac{1}{a_1 + \beta}) \quad \text{for } (\alpha, \beta) \in \bar{X},$$

then

Theorem 1 The map $\bar{S}$ is a one to one, onto map, and has an invariant measure $\bar{\mu}$ satisfying

$$d\bar{\mu} = \frac{1}{\log 2} \frac{d\alpha d\beta}{(1 + \alpha\beta)^2}.$$

Moreover, the dynamical system $(\bar{X}, \bar{S}, \bar{\mu})$ is ergodic.

Using the natural extension, we obtain the proofs of metrical theorems. In fact, in the case of (3), from the definition of $\bar{S}$ and (1.4), we have

$$\bar{S}^n(\alpha, 0) = (S^n\alpha, \frac{q_{n-1}}{q_n})$$

and

$$q_n |\alpha \cdot q_n - p_n| = \frac{S^n\alpha}{1 + \frac{q_{n-1}}{q_n} \cdot S^n\alpha}.$$

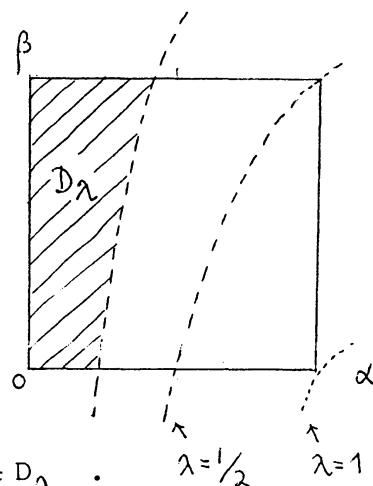
Let $D_\lambda = \{(\alpha, \beta) \in \bar{X} \mid \frac{\alpha}{1 + \alpha\beta} < \lambda\}$, then

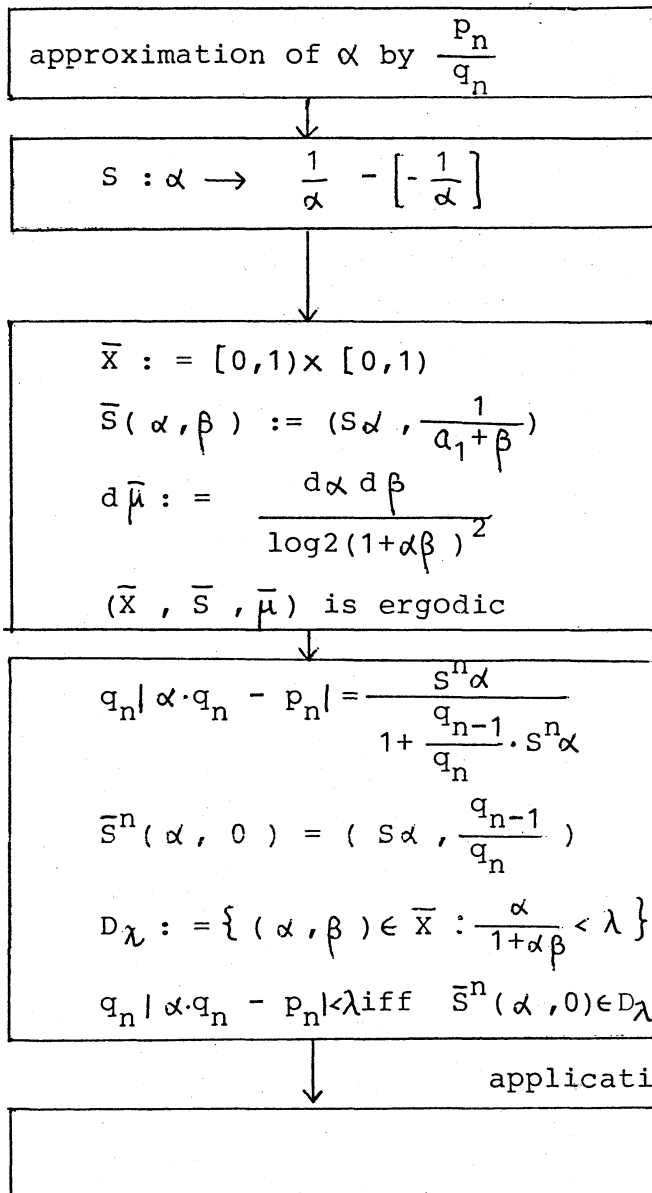
$$q_n |\alpha \cdot q_n - p_n| < \lambda \quad \text{iff} \quad \bar{S}^n(\alpha, 0) \in D_\lambda \quad \lambda = 1/2 \quad \lambda = 1$$

Therefore, from the individual ergodic theorem,

$$\begin{aligned} & \frac{1}{N} \# \{n; q_n |\alpha \cdot q_n - p_n| < \lambda, n \leq N\} \\ &= \frac{1}{N} \# \{n; \bar{S}^n(\alpha, 0) \in D_\lambda, n \leq N\} \xrightarrow{N \rightarrow \infty} \bar{\mu}(D_\lambda) \quad \text{a.e. } \alpha \end{aligned}$$

Putting in order the procedure of the proof, we obtain the following scheme.



Procedure

Diophantine approximation

Construction of the algorithm
which induces the approximationConstruction of the Natural
extension of the algorithm.(Construction of the
dynamical system)Interpretation of the terms
of diophantine approx. as of
ergodic theory.

§ 2 The presentation of problems

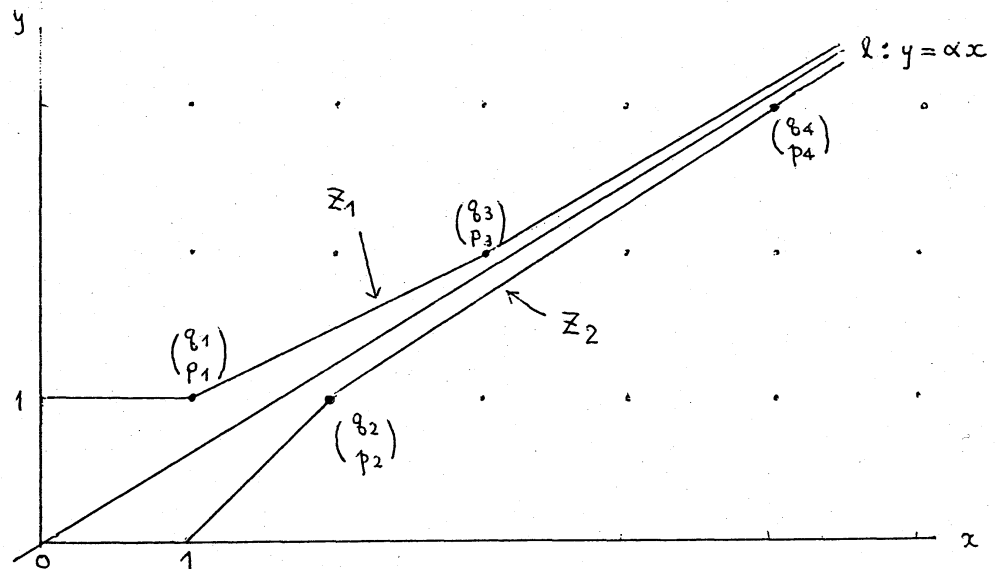
As a characterization of the principal convergents (q_n, p_n) $n=1, 2, \dots$, we know the convergents give a best approximation to α :

for any $A > 0$,

$$\min_{0 < q \leq A} |q\alpha - p| = |q_n\alpha - p_n|$$

where n is chosen as $q_n \leq A < q_{n+1}$.

Form the geometrical point of view, the convergents are also characterized as the vertices of approximating polygon Z_1 and Z_2 to the line $l: y = \alpha x$ (see figure).



The first presentation of problem is as follows.

Problem 1 : to construct the algorithm T_1 which induces a best approximation from below to α , that is, to construct the algorithm T_1 which induces the sequence (q'_n, p'_n) $n=1, 2, \dots$:

for any $A > 0$,

$$\min_{\substack{0 < q \leq A \\ q\alpha - p > 0}} (q\alpha - p) = q'_n\alpha - p'_n$$

where n is chosen as $q'_n \leq A < q'_{n+1}$.

Problem 2 : to construct the algorithm T_2 which induces the mediant convergents, that is, to construct the algorithm T_2 which induces the sequence (v_n, w_n) $n=1,2,\dots$ such that

$$\left\{ \frac{w_n}{v_n} : n = 1, 2, \dots \right\} = \bigcup_{m=1}^{\infty} \left\{ \frac{\lambda p_m + p_{m-1}}{\lambda q_m + q_{m-1}} ; \lambda = 1, 2, \dots, a_m \right\}.$$

The third problem is as follows .

Problem 3 : to construct the algorithm T_3 which induces a best approximation for inhomogeneous linear form :

$$\min_{0 < |x_k| < A} |\alpha x + \beta - y| = |x_n \cdot \alpha + \beta - y_n|.$$

And then we should answer what theorem can be obtained (in metrical point of view) by using the algorithm?

§ 3 Result

For problem 1 : The algorithm which induces the convergents

$$\begin{aligned} \text{from below to } \alpha : \min (q\alpha - p) &= q'_n \alpha - p'_n \\ 0 < q < A \\ q\alpha - p > 0 \end{aligned}$$

Algorithm : $X := [0, 1)$

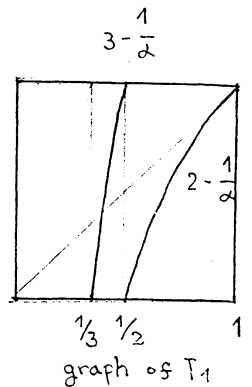
$$T_1 \alpha := - \left[-\frac{1}{\alpha} \right] - \frac{1}{\alpha}$$

$$a_n(\alpha) := a(T_1^{n-1} \alpha) \quad \text{where } a(\alpha) := - \left[-\frac{1}{\alpha} \right].$$

$$(a_n \in \{2, 3, 4, \dots\})$$

Expansion and approximation :

$$\alpha = \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_n - T_1^n \alpha}}}} = \frac{p_n - T_1^n \alpha \cdot p_{n-1} - 1}{q_n - T_1^n \alpha \cdot q_{n-1} - 1}$$



$$\text{where } \begin{pmatrix} q'_n & -q'_n & -1 \\ p'_n & -p'_n & -1 \end{pmatrix} := \begin{pmatrix} q'_n & -1 & -q'_n & -2 \\ p'_n & -1 & -p'_n & -2 \end{pmatrix} \begin{pmatrix} a_n & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} q'_0 & -q'_{-1} \\ p'_0 & -p'_{-1} \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

, then the sequence $(q'_n, p'_n)_{n=1,2,\dots}$ satisfies :

$$\begin{aligned} \min (q\alpha - p) &= q'_n \alpha - p'_n \\ 0 < q < A \\ q\alpha - p > 0 \end{aligned}$$

where n is chosen as $q'_n \leq A < q'_{n+1}$.

Natural extension : $\bar{X} := [0, 1) \times [0, 1)$

$$\bar{T}_1(\alpha, \beta) := (T_1 \alpha, \frac{1}{a_1 - \beta})$$

$$d\bar{\mu}_1(\alpha, \beta) := \frac{d\alpha d\beta}{(1 - \alpha\beta)^2},$$

then $(\bar{X}, \bar{T}_1, d\bar{\mu}_1)$ is ergodic.

interpretation : $q'_n (q'_n \alpha - p'_n) = \frac{T_1^n \alpha}{1 - T_1^n \alpha \cdot \frac{q'_n - 1}{q'_n}}$

$$\bar{T}_1^n(\alpha, 0) = (T_1^n \alpha, \frac{q'_n - 1}{q'_n}) ,$$

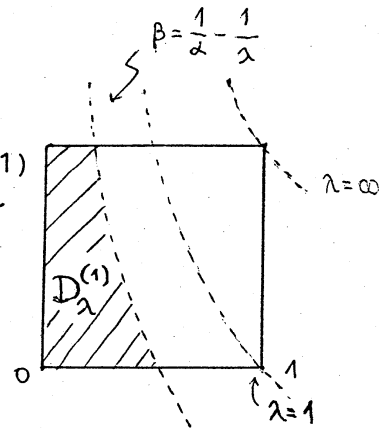
therefore

$$q'_n (q'_n \alpha - p'_n) < \lambda \quad \text{iff} \quad \bar{T}_1^n(\alpha, 0) \in D_\lambda^{(1)}$$

where

$$D_\lambda^{(1)} = \{(\alpha, \beta) \in \bar{X} : \frac{\alpha}{1 - \alpha \beta} < \lambda\}$$

metrical theorem : by ratio-ergodic theorem,



$$(1) \quad \frac{\#\{n : q'_n (q'_n \alpha - p'_n) < \lambda_1, n \leq N\}}{\#\{n : q'_n (q'_n \alpha - p'_n) < \lambda_2, n \leq N\}} \xrightarrow{(N \rightarrow \infty)} \frac{\bar{\mu}_1(D_{\lambda_1}^{(1)})}{\bar{\mu}_1(D_{\lambda_2}^{(1)})}$$

$$\text{where } \bar{\mu}_1(D_\lambda^{(1)}) = \begin{cases} \lambda & \text{if } 0 < \lambda \leq 1 \\ 1 + \log \lambda & \text{if } 1 < \lambda \end{cases}$$

(2) in particular, if $\lambda_1, \lambda_2 \leq 1$,

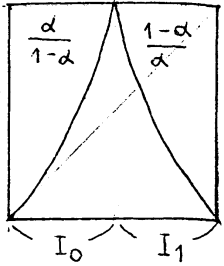
$$\frac{\#\{(q, p) : 0 < q(q\alpha - p) < \lambda_1, q \leq N\}}{\#\{(q, p) : 0 < q(q\alpha - p) < \lambda_2, q \leq N\}} \xrightarrow{(N \rightarrow \infty)} \frac{\lambda_1}{\lambda_2}$$

For problem 2 : The algorithm which induces the mediant convergents.

Algorithm : $X := [0, 1)$

$$T_2 \alpha = \begin{cases} \frac{\alpha}{1-\alpha} & \text{if } \alpha \in I_0 = [0, \frac{1}{2}) \\ \frac{1-\alpha}{\alpha} & \text{if } \alpha \in I_1 = [\frac{1}{2}, 1) \end{cases}$$

$$\varepsilon_n(\alpha) := \varepsilon \quad \text{if } T_2^{n-1} \alpha \in I_\varepsilon.$$



Expansion and approximation :

$$\begin{pmatrix} r_n & s_n \\ t_n & u_n \end{pmatrix} := A \varepsilon_1(\alpha) A \varepsilon_2(\alpha) \cdots A \varepsilon_n(\alpha)$$

where $A_0 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_1 := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

, then

$$\alpha = \frac{t_n + T_2^n \alpha \cdot u_n}{r_n + T_2^n \alpha \cdot s_n} \quad (\text{expansion})$$

$$\left\{ \frac{w_n}{v_n} : n=1, 2, \dots \right\} \quad (\text{sequence of mediant conv.})$$

where $w_n := t_n + u_n$ and $v_n := r_n + s_n$

Natural extension : $\bar{X} := [0, 1) \times [0, 1)$

$$\bar{T}_2(\alpha, \beta) := \begin{cases} (\frac{\alpha}{1-\alpha}, \frac{\beta}{1+\beta}) & \text{if } \alpha \in I_0 \\ (\frac{1-\alpha}{\alpha}, \frac{1}{1+\beta}) & \text{if } \alpha \in I_1 \end{cases}$$

$$d\bar{\mu}_2 = \frac{d\alpha d\beta}{(\alpha + \beta - \alpha\beta)^2}$$

then $(\bar{X}, \bar{T}_2, d\bar{\mu}_2)$ is ergodic.

Interpretation :

$$v_n | v_n \alpha - w_n | = \frac{1 - T_2^n \alpha}{\frac{r_n}{v_n} (1 - T_2^n \alpha) + T_2^n \alpha}$$

$$\bar{T}_2^n(\alpha, 1) = (T_2^n \alpha, \frac{r_n}{v_n})$$

therefore,

$$v_n | v_n \alpha - w_n | < \lambda \text{ iff } \bar{T}_2^n(\alpha, 1) \in D_\lambda^{(2)}$$

where $D_\lambda^{(2)} = \{ (\alpha, \beta) \in \bar{X} ; \frac{1-\alpha}{\beta(1-\alpha)+\alpha} < \lambda \}$

Metrical theorem [4] : for a.e. ,

$$(1) \frac{\#\{n ; v_n | v_n \alpha - w_n | < \lambda_1, n \leq N\}}{\#\{n ; v_n | v_n \alpha - w_n | < \lambda_2, n \leq N\}} \rightarrow \frac{\bar{\mu}_2(D_{\lambda_1}^{(2)})}{\bar{\mu}_2(D_{\lambda_2}^{(2)})}$$

where

$$\bar{\mu}_2(D_\lambda^{(2)}) = \begin{cases} \lambda & \text{if } \lambda \leq 1 \\ \log \lambda + 1 & \text{if } \lambda \geq 1 \end{cases}.$$

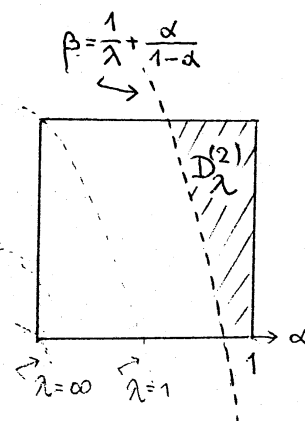
In particular, if $\lambda_1, \lambda_2 \leq 1$, then

$$(2) \frac{\#\{(q, p) ; |q(q\alpha - p)| < \lambda_1, q \leq N\}}{\#\{(q, p) ; |q(q\alpha - p)| < \lambda_2, q \leq N\}} \rightarrow \frac{\lambda_1}{\lambda_2} \quad (\text{a.e.})$$

By inducing modified algorithm of T_2 , we have

(3) for $\lambda \leq 1$,

$$\frac{\#\{(q, p) ; |q(q\alpha - p)| < \lambda, q \leq N\}}{\log N} \rightarrow \frac{12}{\pi^2} \cdot \lambda \quad (\text{a.e.})$$



For problem 3 : the algorithm which induces the approximation
for in-homogeneous linear class :

$$\min_{\substack{(x,y) \\ 0 < |x| < A}} |x(\alpha x + \beta - y)| = |\alpha x_n + \beta - y_n|.$$

Algorithm : $x := \{ (\alpha, \beta) : 0 \leq \beta < 1, -\beta \leq \alpha < -\beta + 1 \}$

$$T_3(\alpha, \beta) := \left(\frac{1}{\alpha} - a(\alpha, \beta), b(\alpha, \beta) - \frac{\beta}{\alpha} \right)$$

where $a(\alpha, \beta) := \left[\frac{1-\beta}{\alpha} \right] - \left[-\frac{\beta}{\alpha} \right]$ $b(\alpha, \beta) := - \left[-\frac{\beta}{\alpha} \right]$

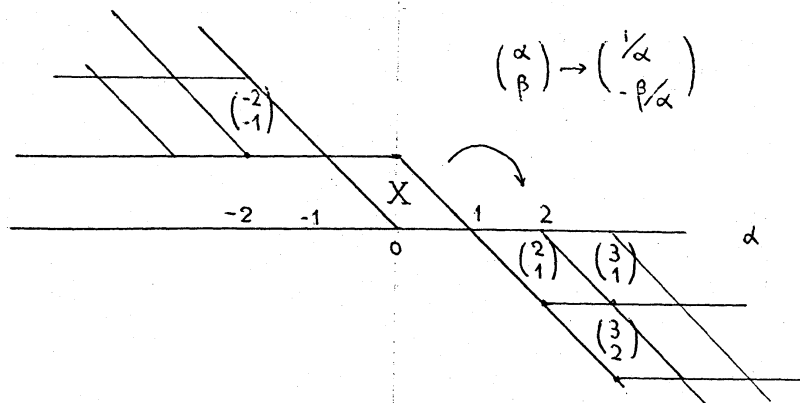
$$\text{digits} \begin{pmatrix} a_k \\ b_k \end{pmatrix} := \begin{pmatrix} a(T_3^{k-1}(\alpha, \beta)) \\ b(T_3^{k-1}(\alpha, \beta)) \end{pmatrix}$$

where digits a_k and b_k satisfies

$$(1) \quad |a_i| \geq 2, \quad |b_i| \geq 1$$

$$(2) \quad |a_i|^{-1} \geq |b_i|$$

and (3) $a_i b_i > 0$



Approximation :

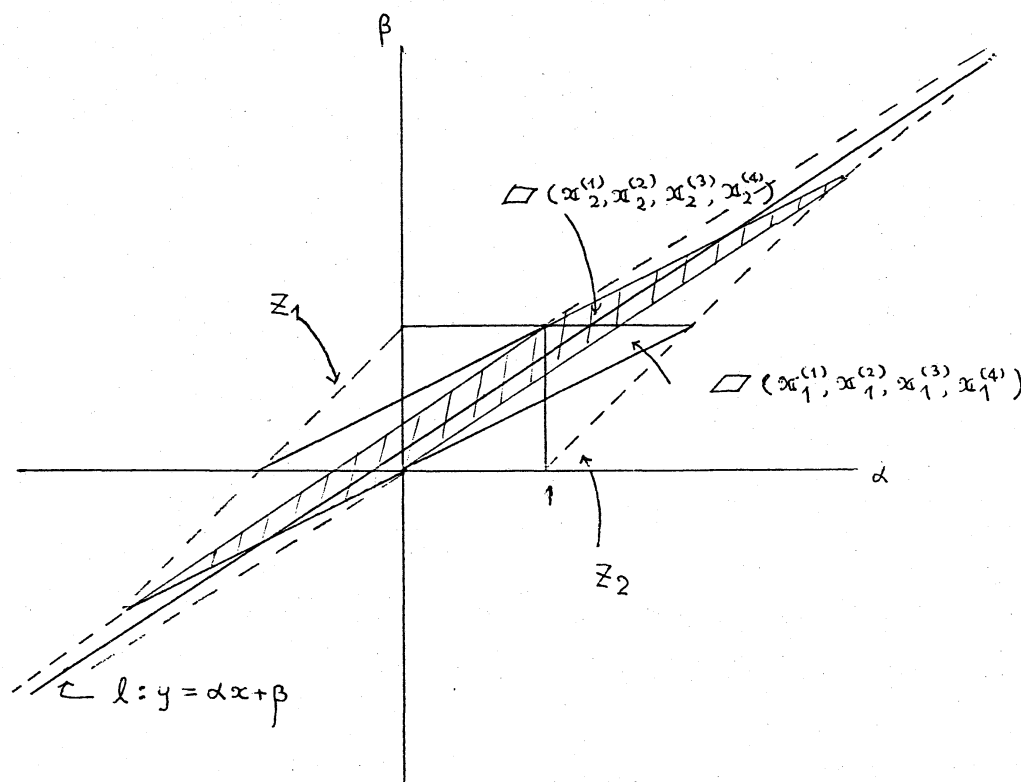
$$\begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} := \begin{pmatrix} q_{n-1} & q_{n-2} \\ p_{n-1} & p_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathcal{G}_n \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \sum_{k=0}^{n-1} q_k b_{k+1} \\ \sum_{k=0}^{n-1} p_k b_{k+1} \end{pmatrix},$$

then points

$$\mathbf{x}_n^{(i)} := \begin{pmatrix} x_n^{(i)} \\ y_n^{(i)} \end{pmatrix} := \varphi_n(a_i) \quad i=1,2,3,4$$

, where $a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $a_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $a_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $a_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, give an approximating parallelogram, and $\mathbf{x}_n^{(i)}: i=1,2,3,4 \quad n \in \mathbb{N}$ gives the approximating polygons Z_1 and Z_2 to the line $l: y = \alpha x + \beta$. (see figure)



Natural extension : $\bar{X} := X \times X^*$

where $X^* := \{ (\gamma, \delta) : 0 \leq \delta < 1 \text{ and } 0 < \delta - \gamma < 1 \}$

$$\bar{T}_3(\alpha, \beta, \gamma, \delta) := (T_3(\alpha, \beta), \frac{1}{a_1 + \gamma}, \frac{b_1 + \delta}{a_1 + \gamma})$$

$$d\bar{\mu}_3 := \frac{1}{\log 2} \frac{d\alpha d\beta d\gamma d\delta}{(1 + \alpha\gamma)^3},$$

$(\bar{X}, \bar{T}_3, \bar{\mu}_3)$ is ergodic.

Interpretation : Put $\bar{T}_3^n(\alpha, \beta, 0, 0) = (\alpha_n, \beta_n, \gamma_n, \delta_n)$, then

$$|x_n^{(i)}(\alpha x_n^{(i)} + \beta - y_n^{(i)})| = \begin{cases} \frac{1}{1+\alpha_n \gamma_n} \delta_n \beta_n & \text{if } i=1 \\ \frac{1}{1+\alpha_n \gamma_n} (\delta_n - \gamma_n)(1 - \beta_n) & \text{if } i=2 \\ \frac{1}{1+\alpha_n \gamma_n} (1+\gamma_n - \delta_n)(1+(\alpha_n + \beta_n)) & \text{if } i=3 \\ \frac{1}{1+\alpha_n \gamma_n} (1 - \delta_n)(\alpha_n + \beta_n) & \text{if } i=4 \end{cases},$$

therefore,

$$\min_{i=1,2,3,4} |x_n^{(i)}(\alpha x_n^{(i)} + \beta - y_n^{(i)})| < \lambda \text{ iff } \bar{T}_3^n(\alpha, \beta, 0, 0) \in \bigcup_{i=1,2,3,4} D_i(\lambda)$$

where

$$\begin{aligned} D_1(\lambda) &= \{(\alpha, \beta, \gamma, \delta) \in \bar{X} : \frac{\delta \beta}{1+\alpha \gamma} < \lambda\} \\ D_2(\lambda) &= \{(\alpha, \beta, \gamma, \delta) \in \bar{X} : \frac{(\delta - \gamma)(1 - \beta)}{1+\alpha \gamma} < \lambda\} \\ D_3(\lambda) &= \{(\alpha, \beta, \gamma, \delta) \in \bar{X} : \frac{(1 + \delta - \delta)(1 + \alpha + \beta)}{1 + \alpha \gamma} < \lambda\} \\ D_4(\lambda) &= \{(\alpha, \beta, \gamma, \delta) \in \bar{X} : \frac{(1 - \delta)(\alpha + \beta)}{1 + \alpha \gamma} < \lambda\} \end{aligned}$$

Metrical theorem [3] : for a.e. (α, β) ,

$$(1) \quad \lim_{n \rightarrow \infty} -\frac{1}{n} \log \min_{i=1,2,3,4} \{|\alpha x_n^{(i)} + \beta - y_n^{(i)}|\} = \frac{\pi^2}{12 \log 2}$$

$$(2) \quad \lim_{n \rightarrow \infty} \frac{1}{N} \# \{n ; \min_{i=1,2,3,4} |\alpha x_n^{(i)} + \beta - y_n^{(i)}| < \lambda, n \leq N\} = G(\lambda)$$

where
$$G(\lambda) = \frac{1}{\log 2} \int_{\bigcup_{i=1,2,3,4} D_i(\lambda)} \frac{d\alpha d\beta d\gamma d\delta}{(1 + \alpha \gamma)^3}$$

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