

THE EXACT DEGREE OF PRECISION OF GENERALIZED GAUSS KRONROD

INTEGRATION RULES

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1. Introduction

In this paper we shall consider the Kronrod extensions (KE) to the Gauss-Gegenbauer integration rules (GGIR) and the Lobatto-Gegenbauer rules (LGIR). The Gegenbauer polynomials,  $C_n^\mu(x)$ ,  $\mu > -\frac{1}{2}$ , are those polynomials which are orthogonal with respect to the weight function  $w(x;\mu) \equiv (1-x^2)^{\mu-\frac{1}{2}}$  and have the following normalization [4, p. 174]

$$(1) \quad \int_{-1}^1 w(x;\mu) C_n^\mu(x) C_m^\mu(x) dx = \delta_{nm} h_{n\mu}$$

where

$$(2) \quad h_{n\mu} = \pi^{\frac{1}{2}} \Gamma(n+2\mu) \Gamma(\mu + \frac{1}{2}) / (n+\mu)n! \Gamma(\mu) \Gamma(2\mu)$$

which implies that  $C_n^\mu(x) = k_{n\mu} x^n + \dots$  where

$$(3) \quad k_{n\mu} = 2^n \Gamma(n+\mu) / n! \Gamma(\mu) .$$

$C_n^\mu(x)$  is even (odd) if  $n$  is even (odd). Special cases of  $C_n^\mu(x)$ ,

perhaps with a different normalization, are  $T_n(x)$ , the Chebyshev polynomials of the first kind ( $\mu = 0$ ),  $P_n(x)$ , the Legendre polynomials ( $\mu = 1/2$ ), and  $U_n(x)$ , the Chebyshev polynomials of the second kind ( $\mu = 1$ ).

The  $n$ -point GGIR is given by

$$(4) \quad If \equiv \int_{-1}^1 w(x; \mu) f(x) dx = \sum_{i=1}^n w_i f(x_i) + c_{n\mu} M_{2n}(f)$$

where we have omitted the dependence of  $w_i$  and  $x_i$  on  $\mu$  and  $n$ ,  $x_i$  are the zeros of  $C_n^\mu(x)$ ,

$$(5) \quad c_{n\mu} = 2^{2n} h_{n\mu} / k_{n\mu}^2$$

and  $M_j(f)$  is defined to be equal to  $f^{(j)}(\xi)/2^j j!$  for some  $\xi \in (-1, 1)$ . The corresponding LGIR has  $n+1$  points and is given by

$$(6) \quad If = \sum_{i=1}^{n+1} \bar{w}_i f(\bar{x}_i) + \bar{c}_{n\mu} M_{2n}(f)$$

where the  $\bar{x}_i$  are the zeros of  $(1-x^2) C_{n-1}^{\mu+1}(x)$  and

$$(7) \quad \bar{c}_{n\mu} = - \frac{2^{2n} h_{n-1, \mu+1}}{k_{n-1, \mu+1}^2} = -4 c_{n-1, \mu+1}$$

Since the weights of the integration rules considered do not play a part in the discussion, we shall not treat them here except to remark that Monegato [9, 10] has shown that the weights  $u_i$  in (8) below are positive for  $0 \leq \mu \leq 1$  and the  $v_i$ , for  $0 \leq \mu \leq 2$ .

The KEGGIR is given by

$$(8) \quad If = \sum_{i=1}^n u_i f(x_i) + \sum_{i=1}^{n+1} v_i f(y_i) + E_{p_n}(f)$$

where  $E_s(f) = 0$  if  $f$  is a polynomial of degree  $< s$  and  $p_n = 2[(3n+3)/2]$ . The  $y_i$  are the zeros of a certain polynomial  $E_{n+1,\mu}(x)$  which we shall study in the next section. For the moment we state a result of Szegő [16] that for  $0 \leq \mu \leq 2$ , the  $y_i$  are real, lie in  $[-1,1]$  and are separated by the  $x_i$ . (For  $\mu \neq 0$ , the  $y_i$  lie in  $(-1,1)$ .) The corresponding KELGIR is given by

$$(9) \quad If = \sum_{i=1}^{n+1} \bar{u}_i f(\bar{x}_i) + \sum_{i=1}^n \bar{v}_i f(\bar{y}_i) + E_{q_n}(f)$$

where  $q_n = 2[(3n+2)/2]$  and the  $\bar{y}_i$  are the zeros of  $E_{n,\mu+1}(x)$ . Thus, taking into account that  $\mu > -\frac{1}{2}$ , we see that practical KEGGIR's exist for  $0 \leq \mu \leq 2$  and KELGIR's, for  $-\frac{1}{2} < \mu \leq 1$ .

The first one to discover a KEGGIR was Kronrod [7] who dealt with the case  $\mu = 1/2$ , the Gauss-Legendre or standard Gauss rule. Subsequently, Patterson [13], Piessens and Branders [14] and Monegato [11] improved on Kronrod's original work and extended his results to the usual Lobatto case ( $\mu = 1/2$ ). Barrucand [2] was the first to point out the connection between the KE's and the Szegő polynomials  $E_{n+1,\mu}(x)$ . KE's to other integration rules are discussed by Baratella [1], Kahaner and Monegato [5], Monegato [9, 12] and Ramskii [15].

In the entire literature on this subject, it is stated that the KE's have error terms which vanish for polynomials of degree less than  $p_n$  (Gauss) or  $q_n$  (Lobatto), and in Kronrod's tables, he gives the error in the integration of  $x^{p_n}$  by the KEGIR with  $\mu = 1/2$ . However, nowhere is it proved that these KE's are of exact degree  $p_n - 1$  or  $q_n - 1$  as the case may be, that is, that there exists a polynomial of degree  $p_n$  or  $q_n$  for which the corresponding KE is not exact. Indeed, such a statement is not true for all  $\mu$ . Thus, as Monegato [9] points out, the KE of the  $n$ -point GGIR with  $\mu = 0$ , the first Gauss-Chebyshev rule, is exact for polynomials of degree  $\leq 4n-1$  and in fact is identical with the KE of the corresponding  $(n+1)$ -point LGIR, being the  $(2n+1)$ -point LGIR, the first Lobatto-Chebyshev rule. Furthermore, the KE of the  $n$ -point GGIR with  $\mu = 1$ , the second Gauss-Chebyshev rule, is exact for polynomials of degree  $\leq 4n+1$  and in fact, is identical with the  $(4n+1)$ -point GGIR. In the present work, we shall show that, except for  $\mu = 0, 1$  in the GGIR case and  $\mu = 0$  in the LGIR case, we have the result that the exact precision of the KEGIR is  $p_n - 1$  while that of the KELGIR is  $q_n - 1$ . Furthermore, if these rules are of simplex type, i.e. if we can express the error term in the form  $K_{n\mu} f^{(p_n)}(\xi)$  or  $K_{n\mu} f^{(q_n)}(\xi)$ , which we have not been able to prove, then we have the following result

$$(10) \quad If = \sum_{i=1}^n u_i f(x_i) + \sum_{i=1}^{n+1} v_i f(y_i) + d_{n\mu} c_{n\mu} M_{p_n}(f)$$

$$(11) \quad If = \sum_{i=1}^{n+1} \bar{u}_i f(\bar{x}_i) + \sum_{i=1}^n \bar{v}_i f(\bar{y}_i) + d_{n-1, \mu+1} \bar{c}_{n\mu} M_{q_n}(f)$$

where  $d_{n\mu}$  is easily computable and does not vanish for  $0 < \mu \leq 2$ ,  $\mu \neq 1$ , and all  $n \geq 2$ . For  $\mu = 2$  we have the explicit expression

$$(12) \quad d_{n\mu} = \begin{cases} -\frac{2}{n+3} \left( \frac{n+1}{n+3} \right)^m & n \text{ even} \\ -4(n+2)(n+1)^{m-1}/(n+3)^{m+1} & n \text{ odd} \end{cases}$$

where  $m = [(n+1)/2]$ .

## 2. The Szegő Polynomials $E_{n+1,\mu}$

We give here the main results of Szegő with some minor modification of his notation and refer to [16] for details. See also Davis and Rabinowitz [3, pp 82-89] and Monegato [11].

The Gegenbauer function of the second kind,  $Q_n^\mu(z)$ , defined by

$$(13) \quad Q_n^\mu(z) = \frac{\Gamma(2\mu)}{2\Gamma(\mu + \frac{1}{2})} \int_{-1}^1 w(t;\mu) \frac{C_n^\mu(t)}{z-t} dt$$

$$= \frac{\Gamma(2\mu)}{2\Gamma(\mu + \frac{1}{2})} z^{-n-1} \sum_{i=0}^{\infty} \beta_i z^{-2i}$$

where

$$(14) \quad \beta_i = \int_{-1}^1 w(t;\mu) C_n^\mu(t) t^{n+2i} dt, \quad i = 0, 1, \dots$$

is analytic in the entire complex plane with a slit on the closed interval  $[-1, 1]$ . Hence

$$(15) \quad \frac{1}{Q_n^\mu(z)} = z^{n+1} \sum_{i=0}^{\infty} \gamma_i z^{-2i} = E_{n+1,\mu}(z) + \delta_1 z^{-1} + \delta_2 z^{-2} + \dots$$

defining the polynomial  $E_{n+1,\mu}(z)$  which is even (odd) for  $n$  odd (even).

Thus,

$$(16) \quad Q_n^\mu(z) E_{n+1,\mu}(z) = 1 + b_1 z^{-n-2} + b_2 z^{-n-3} + \dots$$

and by the argument given in [16] or [3]

$$(17) \quad Q_n^\mu(z) E_{n+1,\mu}(z) = 1 + \sum_{i=0}^n c_i Q_{n+1+i}^\mu(z)$$

for certain constants  $c_0, \dots, c_n$  depending on  $\mu$  and  $n$ . Since  $Q_n^\mu(z)$

is an odd (even) function if  $n$  is even (odd), we have that

$Q_n^\mu(z) E_{n+1,\mu}(z)$  is always an odd function which implies that  $c_0 = 0$  if  $n$  is odd.

Now the functions of the second kind satisfy the following relations:

$$(18) \quad \lim_{\varepsilon \rightarrow 0} (Q_n^\mu(x+i\varepsilon) - Q_n^\mu(x-i\varepsilon)) = -i\pi \frac{\Gamma(2\mu)}{\Gamma(\mu + \frac{1}{2})} w(x;\mu) C_n^\mu(x)$$

$$(19) \quad \lim_{\varepsilon \rightarrow 0} (Q_n^\mu(x+i\varepsilon) + Q_n^\mu(x-i\varepsilon)) = 2 \tilde{Q}_n^\mu(x)$$

where  $\tilde{Q}_n^\mu(x)$  is defined on the segment  $[-1, 1]$ . Hence

$$(20) \quad C_n^\mu(x) E_{n+1,\mu}(x) = \sum_{i=0}^n c_i C_{n+1+i}^\mu(x)$$

and

$$(21) \quad \tilde{Q}_n^\mu(x) E_{n+1,\mu}(x) = 1 + \sum_{i=0}^n c_i \tilde{Q}_{n+1+i}^\mu(x)$$

From (20) it follows that

$$(22) \quad \int_{-1}^1 w(x;\mu) C_n^\mu(x) E_{n+1,\mu}(x) x^k dx = 0, \quad k = 0, 1, 2, \dots, n$$

so that by the theorem in [3, p. 77], an interpolatory integration rule based on the zeros of  $C_n^\mu(x)$  and  $E_{n+1,\mu}(x)$  is exact for all polynomials of degree  $\leq 3n+1$  which forms the basis for KEGGIR's.

Now, it can be shown that

$$(23) \quad Q_n^\mu(z) = \gamma_{n\mu} w^{-n-1} F(1-\mu, n+1; n+\mu+1; w^{-2})$$

$$= \gamma_{n\mu} \sum_{j=0}^{\infty} f_{j\mu} w^{-n-1-2j}$$

where  $z = \frac{1}{2}(w+w^{-1})$ ,  $\gamma_{n\mu} = \sqrt{\pi} \Gamma(n+2\mu)/\Gamma(n+\mu+1)$ ,  $F(a,b;c;z)$  is the usual hypergeometric function,  $f_{0\mu} = 1$ ,

$$(24) \quad f_{j\mu} = (1-\mu/j)(1-\mu/n+\mu+j)f_{j-1,\mu},$$

and we have not shown the dependence on  $n$  of the  $f_{j\mu}$ .

Setting  $w = e^{-i\theta}$  and  $x = \cos \theta$ , we get that

$$(25) \quad \tilde{Q}_n^\mu(x) = \gamma_{n\mu} \sum_{j=0}^{\infty} f_{j\mu} T_{n+1+2j}(x).$$

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Since  $E_{n+1,\mu}(x)$  contains only even or odd powers of  $x$ , we can write  $E_{n+1,\mu}(x)$  in the form

$$(26) \quad E_{n+1,\mu}(x) = \sum_{i=0}^{m-1} \lambda_{i\mu} T_{n+1-2i}(x) + \begin{cases} \lambda_{m\mu} T_1(x), & n \text{ even} \\ \frac{1}{2} \lambda_{m\mu}, & n \text{ odd} \end{cases}$$

To determine the coefficients  $\lambda_{i\mu}$ , we equate in view of (21) and (25) the coefficients of  $T_k(x)$ ,  $k = 1, \dots, n+1$  in the product

$$(27) \quad \tilde{Q}_n^\mu(x) E_{n+1,\mu}(x) = \gamma_{n\mu} \left( \sum_{j=0}^{\infty} f_{j\mu} T_{n+1+2j}(x) \right) \left( \sum_{i=0}^m \lambda_{i\mu} T_{n+1-2i}(x) \right)$$

to zero and the coefficient of  $T_0(x)$  to unity. Here the prime means that if  $n$  is odd, we replace  $\lambda_{m\mu}$  by  $\frac{1}{2} \lambda_{m\mu}$ . Since  $T_r(x)T_s(x) = \frac{1}{2} (T_{r+s}(x) + T_{|r-s|}(x))$ , we see that the  $\lambda_{i\mu}$  must satisfy the following equations

$$(28) \quad \begin{aligned} \lambda_{0\mu} &= 2\gamma_{n\mu}^{-1} \\ \sum_{i=0}^k f_{i\mu} \lambda_{k-i,\mu} &= 0 \quad k = 1, \dots, m \end{aligned}$$

Following Monegato [11], we define  $\alpha_{i\mu} = \lambda_{i\mu} / \lambda_{0\mu}$ , so that

$$\alpha_{0\mu} = 1, \quad \alpha_{1\mu} = -f_{1\mu} \text{ and}$$

$$(29) \quad \alpha_{k\mu} = -f_{k\mu} - \sum_{i=1}^{k-1} f_{i\mu} \alpha_{k-i,\mu} \quad k = 2, \dots, m$$

From this, we see that the  $\alpha_{i\mu}$  are the first  $m+1$  coefficients in the series

$$(30) \quad \sum_{i=0}^{\infty} \alpha_{i\mu} u^i = \left\{ \sum_{j=0}^{\infty} f_{j\mu} u^j \right\}^{-1}$$

so that we can also use (29) for indices  $k > m$ . Here also we have not indicated the dependence on  $n$  of the  $\lambda_{i\mu}$  and  $\alpha_{i\mu}$ .

### 3. The Exact Degree of Precision of KEGGIR's and KELGIR's

Let us define

$$(31) \quad f_k(x) = c_n^\mu(x) E_{n+1,\mu}(x) c_{n+1+k}^\mu(x), \quad k = 0, \dots, n.$$

Then from (20) it follows that  $lf_k = c_k h_{n+1+k,\mu}$ . Since the KEGGIR applied to  $f_k(x)$  vanishes, we have from (8) that  $E_{p_n}(f_k) = c_k h_{n+1+k,\mu}$  so that the exact precision of the KEGGIR is determined by the first index for which  $c_k \neq 0$ . We now show that for  $0 < \mu \leq 2$ ,  $\mu \neq 1$ ,  $c_0 \neq 0$  for  $n$  even and  $c_1 \neq 0$  for  $n$  odd.

Consider first the case  $n$  even. Substituting (25) and (27) into (21) and equating the coefficients of  $T_{n+2}(x)$ , we find that

$$(32) \quad c_0 \gamma_{n+1,\mu} = \frac{\gamma_{n\mu}}{2} \{ \lambda_{m\mu} f_{0\mu} + \lambda_{m\mu} f_{1\mu} + \lambda_{m-1,\mu} f_{2\mu} + \dots + \lambda_{0\mu} f_{m+1,\mu} \}$$

$$= \alpha_{m\mu} + \alpha_{m\mu} f_{1\mu} + \alpha_{m-1,\mu} f_{2\mu} + \dots + \alpha_{1\mu} f_{m\mu} + f_{m+1,\mu} = \alpha_{m\mu} - \alpha_{m+1,\mu} x^{\frac{n}{2}}.$$

Thus, it suffices to show that  $\alpha_{m\mu} - \alpha_{m+1,\mu}$  does not vanish. In fact, we shall show that the  $\alpha_{i\mu}$  are strictly monotonic. For  $0 < \mu < 1$ , the sequence  $\{f_{j\mu}\}$  is completely monotonic, i.e.  $(-1)^k \Delta^k f_{j\mu} > 0$  for all  $j$  and  $k$  [17, p. 137]. Hence, by a theorem of Kaluza [6], the sequence  $\{-\alpha_{i+1,\mu}\}$  is also completely monotonic and hence strictly monotonic. For  $1 < \mu < 2$ , the sequence  $\{-f_{j+1,\mu}\}$  is completely monotonic. From this it follows by some results in [6] that

$$\frac{\alpha_{i-1,\mu}}{\alpha_{i\mu}} > \frac{\alpha_{i\mu}}{\alpha_{i+1,\mu}}, \quad i = 1, 2, \dots$$

Since  $\sum_{i=0}^{\infty} \alpha_{i\mu}$  converges, and in fact equals  $\{F(1-\mu, n+1; n+\mu+1; 1)\}^{-1}$ , it follows that the sequence  $\{\alpha_{i\mu}\}$  is strictly monotonic. For  $\mu = 2$ , Szegő [16] gives an explicit expression for the  $\lambda_{i\mu}$ ,

$$(33) \quad \lambda_{i2} = \frac{2}{\sqrt{\pi}} \frac{1}{n+3} \left( \frac{n+1}{n+3} \right)^i, \quad i = 0, 1, \dots$$

which again shows that the  $\alpha_{i2}$  are strictly monotonic.

We now consider the case  $n$  odd. Proceeding as before, this time equating the coefficients of  $T_{n+3}(x)$ , we find that

$$(34) \quad c_1 \gamma_{n+2,\mu} = \frac{\gamma_{n\mu}}{2} \{ \lambda_{m\mu} f_{1\mu} + \lambda_{m-1,\mu} f_{0\mu} + \lambda_{m-1,\mu} f_{2\mu} + \lambda_{m-2,\mu} f_{3\mu} + \dots + \lambda_{0\mu} f_{m+1,\mu} \\ = \alpha_{m-1,\mu} + \alpha_{m\mu} f_{1\mu} + \alpha_{m-1,\mu} f_{2\mu} + \dots + \alpha_{1\mu} f_{m\mu} + f_{m+1,\mu} = \alpha_{m-1,\mu} - \alpha_{m+1,\mu}$$

Since the  $\alpha_{i\mu}$  are strictly monotonic, it follows that  $c_1 \neq 0$ .

For  $\mu = 0$ ,  $f_{j0} = 1, j = 0, 1, 2, \dots$  so that  $\lambda_{00} = -\lambda_{10} = 2n/\pi^{1/2}$ ,  $\lambda_{i0} = 0$ ,  $i > 1$  and  $E_{n+1,0} = \frac{2\pi}{\pi^{1/2}} \{ T_{n+1}(x) - T_{n-1}(x) \}$ ,  $n \geq 2$ .

Hence

$$(35) \quad c_n^0(x) E_{n+1,0}(x) = k_1 T_n \{T_{n+1} - T_{n-1}\} = \frac{k_1}{2} \{T_{2n+1} - T_{2n-1}\} = k_2 (1-x^2) U_{2n-1} = \\ = k_3 (1-x^2) c_{2n-1}^1(x)$$

and the zeros of  $c_n^0(x) E_{n+1,0}(x)$  are the abscissas of the  $(2n+1)$ -point LGIR for the weight  $w(x;0)$  which is of exact precision  $4n-1$ , as can also be seen from the fact that  $c_{n-2}$  is the first  $c_k$  which does not vanish.

$$\text{For } \mu = 1, \quad f_{01} = 1, \quad f_{j1} = 0, \quad j > 0 \quad \text{so that } \lambda_{01} = \frac{2}{\sqrt{\pi}}, \\ \lambda_{i1} = 0, \quad i > 0 \quad \text{and } E_{n+1,1}(x) = \frac{2}{\sqrt{\pi}} T_{n+1}(x). \quad \text{Hence}$$

$$(36) \quad c_n^1(x) E_{n+1,1}(x) = k_1' U_n(x) T_{n+1}(x) = k_2' c_{2n+1}^1(x)$$

and the zeros of  $c_n^1(x) E_{n+1,1}(x)$  are the abscissas of the  $(2n+1)$ -point GGIR for the weight  $w(x;1)$  which is of exact precision  $4n+1$  and which also follows from the fact that  $c_n$  is the first  $c_k$  which does not vanish.

In the case of the KELGIR, we define

$$(37) \quad \bar{f}_k(x) = (1-x^2) c_{n-1}^{\mu+1}(x) E_{n,\mu+1}^{\mu+1}(x) c_{n+\mu+k}^{\mu+1}(x), \quad k = 0, 1, \dots, n$$

so that  $\bar{f}_k = c_k h_{n+k,\mu+1}$ . Hence, since  $c_0 = c_0^{(n-1,\mu+1)} \neq 0$  for  $n-1$  even, i.e. for  $n$  odd, while  $c_1 \neq 0$  for  $n-1$  odd, we have that the  $(2n+1)$ -point KELGIR is of exact precision  $3n+1$  for  $n$  even and  $3n$  for  $n$  odd, provided that  $\mu \neq 0$ . For  $\mu = 0$ , we have as before that  $E_{n1}(x) = \frac{2}{\pi^{\frac{1}{2}}} T_n(x)$  so that

$$(38) \quad (1-x^2) c_{n-1}^1(x) E_{n1}(x) = \hat{k}_1 (1-x^2) c_{2n-1}^1(x)$$

whose zeros are again the abscissas of the  $(2n+1)$ -point LGIR for the weight  $w(x;0)$ .

If we now define

$$(39) \quad d_{n\mu} = \begin{cases} \alpha_{m\mu}^{-\alpha_{m+1,\mu}} & n \text{ even} \\ \alpha_{m-1,\mu}^{-\alpha_{m+1,\mu}} & n \text{ odd, } m = [(n+1)/2] \end{cases}$$

we have that for the Gauss case

$$(40) \quad d_{n\mu} = \begin{cases} c_0 \gamma_{n+1,\mu} & n \text{ even} \\ c_1 \gamma_{n+2,\mu} & n \text{ odd} \end{cases}$$

while for the Lobatto case

$$d_{n-1,\mu+1} = \begin{cases} c_0 \gamma_{n,\mu+1} & n \text{ even} \\ c_1 \gamma_{n+1,\mu+1} & n \text{ odd} \end{cases}$$

where we have suppressed the dependence of  $c_0$  and  $c_1$  on  $n$  and  $\mu$ .

This lead us immediately to formulas (10) and (11). For example,

applying (8) with  $n$  even to  $f_0(x)$ , we have that

$$(41) \quad c_0 h_{n+1,\mu} = K_{n\mu} k_{n\mu} 2\gamma_{n\mu}^{-1} 2^n k_{n+1,\mu} (3n+2)!$$

so that

$$(42) \quad K_{n\mu} = \frac{d_{n\mu}}{\gamma_{n+1,\mu}} \frac{h_{n+1,\mu} \gamma_{n\mu}}{2^{n+1} k_{n\mu} k_{n+1,\mu} (3n+2)!} = \frac{d_{n\mu} c_{n\mu}}{2^{p_n} p_n!}.$$

For  $n$  odd, we consider  $f_1(x)$  while in the Lobatto case we work with  $\bar{f}_0(x)$  and  $\bar{f}_1(x)$ .

#### 4. Remarks

a. Monegato [11] gives an error bound for KEGIR's with  $0 < \mu < 1$ . We shall show how to improve this bound slightly and extend it to the case  $1 < \mu < 2$  as well as to KELGIR's with  $-\frac{1}{2} < \mu \leq 1$ ,  $\mu \neq 0$ .

For  $n$  even, Monegato writes the error  $E_{p_n}(f)$  for  $f \in C^{3n+2}[-1,1]$  in the form

$$(43) \quad E_{p_n}(f) = \frac{2^{-2n}}{k_{n\mu}(3n+2)!} \int_{-1}^1 w(x;\mu) c_n^\mu(x) (\bar{E}_{n+1,\mu}^{(x)})^2 f^{(3n+2)}(\xi_x) dx$$

where

$$(44) \quad \bar{E}_{n+1,\mu}(x) = E_{n+1,\mu}(x)/\lambda_{0\mu} = \sum_{i=0}^m \alpha_{i\mu} T_{n+1-2i}(x).$$

Hence

$$(45) \quad |E_{p_n}(f)| \leq \frac{\pi \Gamma(n+2\mu) B_{n+1,\mu}^2}{2^{3n+2\mu-1} p_n! \Gamma(\mu+1) \Gamma(n+\mu)} M_{p_n}$$

where  $M_s = \max_{-1 \leq x \leq 1} |f^{(s)}(x)|$  and  $B_{n+1,\mu} = \max_{-1 \leq x \leq 1} |\bar{E}_{n+1,\mu}(x)|$ .

For  $0 < \mu < 1$ , Monegato states that  $B_{n+1,\mu} < 2$  and replaces

$B_{n+1,\mu}$  by 2 in (45). Now while this bound is the best available

for  $0 < \mu \leq \frac{1}{2}$ , we can improve on it for  $\frac{1}{2} < \mu < 1$ . In addition, a bound on  $B_{n+1,\mu}$  is also available for  $1 < \mu \leq 2$ . This follows from our observation above that

$$(46) \quad \sum_{i=0}^{\infty} \alpha_{i\mu} = \{F(1-\mu, n+1; n+\mu+1, 1)\}^{-1} \equiv T_{n\mu} = \frac{\Gamma(\mu)\Gamma(n+2\mu)}{\Gamma(n+\mu+1)\Gamma(2\mu-1)}, \quad \begin{matrix} \mu > \frac{1}{2} \\ \mu \neq 1, 2 \end{matrix}$$

Now for  $\frac{1}{2} < \mu < 1$ ,  $\alpha_{0\mu} = 1$ ,  $\alpha_{i\mu} < 0$ ,  $i > 0$ . Since

$$B_{n+1,\mu} \leq \sum_{i=0}^m |\alpha_{i\mu}| = 1 - \sum_{i=1}^m \alpha_{i\mu} < 1 - \sum_{i=1}^{\infty} \alpha_{i\mu}$$

it follows that  $B_{n+1,\mu} < 2 - T_{n\mu} < 2$ . For  $1 < \mu < 2$ , we have that  $\alpha_{i\mu} > 0$ , all  $i$ . Hence  $B_{n+1,\mu} \leq \sum_{i=0}^m \alpha_{i\mu} < T_{n\mu}$ . For  $\mu = 2$ ,

$$\sum_{i=0}^{\infty} \alpha_{i2} = \left(1 - \frac{n+1}{n+3}\right)^{-1} = \frac{n+3}{2} > B_{n+1,2}.$$

For  $n$  odd, using classical arguments, we have the same bound.

In the Lobatto case, we have similarly for  $n$  odd that

$$(47) \quad E_{q_n}(x) = \frac{2^{2-2n}}{k_{n-1,\mu+1}(3n)!} \int_{-1}^1 w(x; \mu+1) c_{n-1}^{\mu+1}(x) (\bar{E}_{n,\mu+1}(x))^{2f(3n)} (\bar{\xi}_x) dx$$

whence

$$(48) \quad |E_{q_n}(f)| \leq \frac{\pi \Gamma(n+2\mu+1) B_{n,\mu+1}^2}{2^{3n+2\mu-2} q_n! \Gamma(n+\mu) \Gamma(\mu+2)} M_{q_n}$$

where for  $-\frac{1}{2} < \mu < 0$ ,  $B_{n,\mu+1} < 2 - T_{n-1,\mu+1}$  and for  $0 < \mu < 1$ ,

$B_{n,\mu+1} < T_{n-1,\mu+1}$ . For  $\mu = 1$ ,  $B_{n2} < \frac{n+2}{2}$ . As before, the same bound holds for  $n$  even.

b. The Fourier-Gegenbauer coefficients of a function  $f(x)$  are defined by

$$(49) \quad FG_{n\mu}(f) = h_{n\mu}^{-1} \int_{-1}^1 w(x;\mu) C_n^\mu(x) f(x) dx, \quad n = 0, 1, \dots$$

As Barracund [2] points out, the integral is most efficiently evaluated by a  $(2n+1)$ -point KEGGIR applied to the function  $C_n^\mu(x)f(x)$  which reduces to the  $(n+1)$ -point formula

$$(50) \quad FG_{n\mu}(f) \approx h_{n\mu}^{-1} \sum_{i=1}^{n+1} v_i C_n^\mu(y_i) = \sum_{i=1}^{n+1} \tilde{v}_i f(y_i).$$

For  $\mu \neq 0, 1$ , we get a rule which is exact for polynomials of degree  $< p_n - n$ , which is the best possible. For assume that there existed an  $(n+1)$ -point rule, say

$$(51) \quad FG_n(f) \approx \sum_{i=1}^{n+1} \hat{v}_i f(\hat{y}_i)$$

exact for polynomials of degree  $p_n - n$ ,  $n$  even. This would imply that

$$(52) \quad \int_{-1}^1 w(x;\mu) C_n^\mu(x) E_{n+1,\mu}(x) \prod_{i=1}^{n+1} (x - \hat{y}_i) dx = 0$$

which contradicts our results above. Similarly for  $n$  odd.

For  $\mu = 0$ , the rule (50) is exact for polynomials of degree  $\leq 3n-1$ , a result which has already been reported in [8]. For  $\mu = 1$ , (50) is exact for polynomials of degree  $\leq 3n+1$  which is the best possible

result, so that the highest precision is achieved for Fourier-Chebyshev coefficients of the second kind. However, we should warn the user that the weights  $\tilde{v}_i$  in (50) alternate in sign inasmuch as the  $v_i$  are positive and the zeros of  $C_n^\mu(x)$  separate those of  $E_{n+1,\mu}(x)$  so that the  $C_n^\mu(y_i)$  alternate in sign.

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