

Crystal Graph and Littlewood Richardson rule

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§0. Introduction

Recently, Professor Kashiwara constructed "Crystal base". We shall introduce the crystal base and the crystal graph associated with it, and their applications.

§1. Crystal base

1.1 Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with the Cartan subalgebra $\mathfrak{t}$ the set of simple roots $\{\alpha_i \in \mathfrak{t}^*\}$ and the set of simple coroots $\{h_i \in \mathfrak{t}\}$. We take an inner product $(\ , \)$ on $\mathfrak{t}^*$ such that $(\alpha_i, \alpha_i) \in \mathbb{Z}_{>0}$ and $(h_i, \lambda) = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for $\lambda \in \mathfrak{t}^*$. Then the q -analogue $U_q(\mathfrak{g})$ is the algebra over $\mathbb{Q}(q)$ generated by e_i, f_i and the invertible elements t_i satisfying the following relations;

$$(1.1.1) \quad t_i t_j = t_j t_i$$

$$(1.1.2) \quad \begin{aligned} t_i e_j t_i^{-1} &= q^{2(\alpha_j, \alpha_i)} e_j \\ t_i f_j t_i^{-1} &= q^{-2(\alpha_j, \alpha_i)} f_j \end{aligned}$$

$$(1.1.3) \quad [e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \quad \text{where} \quad q_i = q^{(\alpha_i, \alpha_i)}$$

For $i \neq j$, we have, setting $b = 1 - \langle h_i, \alpha_j \rangle$

$$(1.1.4) \quad \begin{aligned} \sum_{\mu=0}^b e_i^{(\mu)} e_j e_i^{(b-\mu)} &= 0 \\ \sum_{\mu=0}^b f_i^{(\mu)} f_j f_i^{(b-\mu)} &= 0 \end{aligned}$$

Here $e_i^{(k)} = \frac{e_i^k}{[k]_i!}$, $f_i^{(k)} = \frac{f_i^k}{[k]_i!}$, $[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and $[k]_i! = \prod_{n=1}^k [n]_i$.

For a finite dimensional $U_q(\mathfrak{g})$ -module M , we set for $\lambda \in P = \{\lambda \in \mathfrak{t}^*; \langle h_i, \lambda \rangle \in \mathbb{Z}\}$
 $M_\lambda = \{u \in M; t_i u = q^{2\langle \alpha_i, \lambda \rangle} u\}$. We call M integrable if $M = \oplus M_\lambda$. Then we have

$$(1.1.5) \quad M_\lambda = \bigoplus_{0 \leq k \leq \langle h_i, \lambda \rangle} f_i^{(k)}(M_\lambda \cap \text{Ker } e_i)$$

We define the operators $\tilde{e}_i, \tilde{f}_i$ acting on M by

$$(1.1.6) \quad \tilde{e}_i f_i^{(k)} u = f_i^{(k-1)} u \quad \text{and} \quad \tilde{f}_i f_i^{(k)} u = f_i^{(k+1)} u$$

for $u \in M_\lambda \cap \text{Ker } e_i$ and (λ, k) as above.

Definition 1.1.1. A pair (L, M) is called a crystal base of a finite-dimensional integrable representation M if the following condition are satisfied:

$$(1.1.7) \quad L \text{ is a free sub-} A\text{-module of } M \text{ such that } \mathbb{Q}(q) \otimes_A L \cong M.$$

Here A is the ring of rational functions regular at $q = 0$

$$(1.1.8) \quad B \text{ is a base of the } \mathbb{Q}\text{-vector space } L/qL$$

$$(1.1.9) \quad L = \oplus L_\lambda, \quad B = \bigsqcup B_\lambda$$

where $L_\lambda = L \cap M_\lambda$ and $B_\lambda = B \cap (L_\lambda/qL_\lambda)$

$$(1.1.10) \quad \tilde{f}_i L \subset L, \quad \tilde{e}_i L \subset L$$

$$(1.1.11) \quad \tilde{f}_i B \subset B \cup \{0\} \quad \text{and} \quad \tilde{e}_i B \subset B \cup \{0\}$$

$$(1.1.12) \quad \text{For } u, v \in B \quad \text{and} \quad i \in I, \quad u = \tilde{e}_i v \text{ if and only if } v = \tilde{f}_i u.$$

Then the following results are proved in [K] when $\mathfrak{g} = A_n, B_n, C_n$ and D_n and announce in [K] in general case. Let $\lambda \in P_+ = \{\lambda \in \mathfrak{t}^*; \langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq 0}\}$ and $V(\lambda)$ be the irreducible integrable $U_q(\mathfrak{g})$ -module generated by the highest weight vector u_λ of weight λ . Let $L(\lambda)$ be the sub A -module generated by the vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\lambda$ and let $B(\lambda)$ be the subset of the $L(\lambda)/qL(\lambda)$ consisting of the non-zero vector of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\lambda \bmod qL(\lambda)$.

Theorem 1.1.2. $(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.

Theorem 1.1.3. Let $M \in \mathcal{O}_{\text{int}}$ and (L, B) is a crystal base of M . Then there is an isomorphism

$$M \cong \oplus_j V(\lambda_j) \text{ by which } (L, B) \cong \oplus_j (L(\lambda_j), B(\lambda_j)).$$

Theorem 1.1.4. Let (L_j, B_j) be a crystal base of an integrable $U_q(\mathfrak{g})$ -module M_j ($j = 1, 2$). Set $L = L_1 \otimes_A L_2 \subset M_1 \otimes M_2$ and $B = \{b_1 \otimes b_2; b_j \in B_j (j = 1, 2)\} \subset L/qL$. Then we have

$$\begin{aligned} \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if there exists } n \geq 1 \text{ such that } f_i^n b_1 \neq 0 \text{ and } e_i^n b_2 = 0. \\ b_1 \otimes \tilde{f}_i b_2 & \text{otherwise.} \end{cases} \\ \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{e}_i b_2 & \text{if there exists } n \geq 1 \text{ such that } e_i^n b_1 \neq 0 \text{ and } f_i^n b_2 = 0. \\ \tilde{e}_i b_1 \otimes b_2 & \text{otherwise.} \end{cases} \end{aligned}$$

Defintion 1.1.5. A crystal graph of a crystal base (L, B) is the colored oriented graph B , by the rule:

$$u \xrightarrow{i} v \iff v = \tilde{f}_i u.$$

Example 1.1.6

Let V_i is the $i + 1$ -dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$.

(i) For $\mathfrak{g} = \mathfrak{sl}_2$, the crystal graph of V_2 is given by as follows;

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

(ii) For $\mathfrak{g} = \mathfrak{sl}_2$, the crystal graph of V_3 is given by as follows;

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

(iii) By Theorem 1.1.4, the crystal graph of $V_3 \otimes V_2$ is described by as follows;

$$\begin{array}{c} V_3 \\ \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \\ V_2 \\ \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \\ \downarrow \qquad \qquad \qquad \downarrow \\ \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \\ \downarrow \qquad \qquad \qquad \downarrow \qquad \downarrow \\ \bullet \longrightarrow \bullet \qquad \qquad \bullet \qquad \bullet \end{array}$$

Here, we get that the crystal graph of $V_3 \otimes V_2$ are decomposed into three connected components. They correspond to V_5 , V_3 and V_1 respectively.

§2. Remarks on the crystal graphs

2.1 Let us investigate first the crystal graph of the tensor product of the vector representation of $U_q(sl_2)$. The crystal graph of the vector representation is $u_+ \longrightarrow u_-$. The crystal graph of the trivial representation is u_0 . We shall calculate $\tilde{e}(u_{i_1} \otimes \cdots \otimes u_{i_N})$ and $\tilde{f}(u_{i_1} \otimes \cdots \otimes u_{i_N})$, where e and f are generators of $U_q(sl_2)$, and $i_1, \dots, i_N = +, -, 0$.

Proposition 2.1.1. *For $u = u_{i_1} \otimes \cdots \otimes u_{i_N}$ ($i_j = +, -, 0$), the actions of $\tilde{e}$ and $\tilde{f}$ are given by the following three steps;*

- (I) We neglect u_0
- (II) If there is $u_+ \otimes u_-$ in u , then we neglect such a pair.
- (III) Then $\tilde{e}$ changes the u_- in the most right to u_+ and $\tilde{f}$ changes the u_+ in the most left to u_- . If there is no u_- (resp. u_+), then $\tilde{e}u = 0$ (resp. $\tilde{f}u = 0$).

Example

$$\text{For } u = u_- \otimes u_0 \otimes \underbrace{u_+ \otimes u_+ \otimes u_- \otimes u_-}_{\text{neglected}} \otimes u_+,$$

$$\tilde{e}u = u_+ \otimes u_0 \otimes u_+ \otimes u_+ \otimes u_- \otimes u_- \otimes u_+$$

$$\tilde{f}u = u_- \otimes u_0 \otimes u_+ \otimes u_+ \otimes u_- \otimes u_- \otimes u_-$$

2.2 Now let $U_q(g)$ be the q -analogue as in §1. and let $\lambda_1, \dots, \lambda_N \in P_+$ and $\lambda = \sum_i \lambda_i$. Then there is a unique embedding $V(\lambda) \hookrightarrow V(\lambda_1) \otimes \cdots \otimes V(\lambda_N)$ sending u_λ to $u_{\lambda_1} \otimes \cdots \otimes u_{\lambda_N}$. Hence $B(\lambda)$ is embedded into $\otimes_{j=1}^N B(\lambda_j)$.

Proposition 2.2.1. *Assume the following condition for any k ($1 \leq k < N$).*

(2.2.1) *If $u \in B(\lambda_{k+1})$ satisfies*

$$(i) \ u_{\lambda_k} \otimes u \in B(\lambda_k + \lambda_{k+1})$$

$$(ii) \ \tilde{e}u = 0 \text{ for any } i \text{ such that } \langle h_i, \lambda_\nu \rangle = 0 \text{ for } \nu \leq k,$$

$$\text{then } u = u_{\lambda_{k+1}}$$

Then we have

$$(2.2.2) \quad V(\lambda) \cong \bigcap_{k=1}^{N-1} V(\lambda_1) \otimes \cdots \otimes V(\lambda_{k-1}) \otimes V(\lambda_k + \lambda_{k+1}) \otimes V(\lambda_{k+2}) \otimes \cdots \otimes V(\lambda_N),$$

$$(2.2.3) \quad B(\lambda) \cong \bigcap_{k=1}^{N-1} B(\lambda_1) \otimes \cdots \otimes B(\lambda_{k-1}) \otimes B(\lambda_k + \lambda_{k+1}) \otimes B(\lambda_{k+2}) \otimes \cdots \otimes B(\lambda_N).$$

§3. Crystal Graphs for $U_q(C_n)$ -modules

3.1 Notation

In the rest of this paper we shall treat the C_n -case. Let $(\varepsilon_1, \dots, \varepsilon_n)$ be the orthonormal base of the dual of the Cartan subalgebra of C_n such that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ ($1 \leq i < n$) and $\alpha_n = 2\varepsilon_n$ form the set of simple roots. Hence, α_n is the long roots and $\alpha_1, \dots, \alpha_{n-1}$ are short roots. Let $\{\Lambda_i\}_{1 \leq i \leq n}$ be the dual base of $\{h_i\}_{1 \leq i \leq n}$. Hence $\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i$ ($1 \leq i \leq n$).

3.2 The crystal graph of the vector representation.

First let us consider the vector representation $V(\Lambda_1) = V_\square$. Letting $\boxed{i}, \boxed{\bar{i}}$ ($1 \leq i \leq n$) be the base of $\mathbb{Q}(q)^{\oplus 2n}$, the vector representation of $U_q(C_n)$ is explicitly constructed as follows;

$$(3.2.1) \quad \begin{aligned} t_i \boxed{j} &= q^{2(h_i, \varepsilon_j)} \boxed{j} & , t_i \boxed{\bar{j}} &= q^{2(h_i, -\varepsilon_j)} \boxed{\bar{j}} \\ e_i \boxed{j} &= \delta_{i+1, j} \boxed{j-1} & , e_i \boxed{\bar{j}} &= \delta_{i, j} \boxed{\bar{j}+1} \\ f_i \boxed{j} &= \delta_{i, j} \boxed{j+1} & , f_i \boxed{\bar{j}} &= \delta_{i+1, j} \boxed{\bar{j}-1} \end{aligned} \quad (1 \leq i < n, 1 \leq j \leq n)$$

and

$$(3.2.2) \quad \begin{aligned} e_n \boxed{j} &= 0, & e_n \boxed{\bar{j}} &= \delta_{j, n} \boxed{n} \\ f_n \boxed{j} &= \delta_{j, n} \boxed{\bar{n}} & , f_n \boxed{\bar{j}} &= 0 \end{aligned} \quad (1 \leq j \leq n)$$

Here, we understand $\boxed{i} = \boxed{\bar{i}} = 0$ unless $1 \leq j \leq n$. Then the crystal base $(L(V_\square), B(V_\square))$ is given by

$$(3.2.3) \quad \begin{aligned} L(V_\square) &= \bigoplus_{i=1}^n (A \boxed{i} \oplus A \boxed{\bar{i}}) \\ B(V_\square) &= \{ \boxed{i}, \boxed{\bar{i}} ; 1 \leq i \leq n \}, \end{aligned}$$

and the crystal graph of $V_\square$ is given by;

$$(3.2.4) \quad \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

Remark that we have

$$(3.2.5) \quad \tilde{e}_i^2 = \tilde{f}_i^2 = 0 \quad \text{on } B(V_\square)$$

Hence, the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $B(V_\square)^{\otimes m}$ is given by Proposition 2.1.1.

3.3 The crystal graph of the fundamental representations

The representation $V(\Lambda_N)$ with highest weight Λ_N ($1 \leq N \leq n$) is embedded into $V_\square^{\otimes N}$. Similarly to the A_n -case, the connected component of the crystal graph of $B(V_\square)^{\otimes N}$ containing $\boxed{1} \otimes \cdots \otimes \boxed{N}$ is that of $B(\Lambda_N)$.

$$\text{We write } \begin{array}{|c|} \hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \quad \text{for } \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}$$

We denote by u_{Λ_N} the highest weight vector $\boxed{1} \otimes \cdots \otimes \boxed{N}$. We give the linear order on $\{i, \bar{i}; 1 \leq i \leq n\}$ by

$$(3.3.1) \quad 1 \prec 2 \prec \cdots \prec n \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}.$$

This ordering is derived by the crystal graph (3.2.4) of $V_\square$. We set

$$(3.3.2) \quad I_N^{(C)} = \left\{ \begin{array}{|c|} \hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(V_\square)^{\otimes N}; \begin{array}{l} (1) \quad 1 \preceq i_1 \prec \cdots \prec i_N \preceq \bar{1}, \\ (2) \quad \text{if } i_k = p \text{ and } i_l = \bar{p}, \\ \quad \text{then } k + (N - l + 1) \leq p \end{array} \right\}.$$

Proposition 3.3.2. $B(\Lambda_N)$ coincides with $I_N^{(C)}$

Remark 3.3.3.

- (i) $\tilde{e}_i^3 = \tilde{f}_i^3 = 0$ for $1 \leq i \leq n$ and $\tilde{e}_n^2 = \tilde{f}_n^2 = 0$ on $B(\Lambda_N)$.
- (ii) If $u \in B(\Lambda_N)$ satisfies $\tilde{f}_i^2 u \neq 0$, then u contains i and $\overline{i+1}$ but neither $i+1$ nor $\bar{i}$. If $u \in B(\Lambda_N)$ satisfies $\tilde{e}_i^2 u \neq 0$, then u contains $i+1$ and $\bar{i}$ but neither i nor $\overline{i+1}$.
- (iii) If $u \in B(\Lambda_N)$ satisfies $\tilde{f}_i u \neq 0$ and $\tilde{e}_i u \neq 0$, then u contains $i+1$, $\overline{i+1}$ but neither i nor $\bar{i}$.

3.4 The crystal graph of $V(\Lambda_M + \Lambda_N)$

Now, we shall investigate the crystal graph of $V(\Lambda_M + \Lambda_N)$ with $1 \leq M \leq N \leq n$. By embedding $V(\Lambda_M + \Lambda_N)$ into $V(\Lambda_M) \otimes V(\Lambda_N)$, $B(\Lambda_M + \Lambda_N)$ is the connected component of $B(\Lambda_M) \otimes B(\Lambda_N)$ containing $u_{\Lambda_M} \otimes u_{\Lambda_N}$.

$$\text{For } u = \begin{bmatrix} j_1 \\ \vdots \\ j_M \end{bmatrix} \in B(\Lambda_M) \text{ and } v = \begin{bmatrix} i_1 \\ \vdots \\ i_N \end{bmatrix} \in B(\Lambda_N),$$

$$\text{the vector } u \otimes v \in B(\Lambda_M) \otimes B(\Lambda_N) \text{ will be denoted by } \begin{bmatrix} i_1 & j_1 \\ \vdots & \vdots \\ i_N & j_M \end{bmatrix}$$

Definition 3.3.1. For $1 \leq i \leq j \leq n$, we say that $w \in B(\Lambda_M) \otimes B(\Lambda_N)$ is in (i, j) -configuration if w holds the following; (3.4.1) There exist $1 \leq p \leq q \leq r \leq s$ such that $i_p = i$, $j_q = j$, $j_r = \bar{j}$, $j_s = \bar{i}$ or $i_p = i$, $i_q = j$, $i_r = \bar{j}$, $j_s = \bar{i}$. Remark that when $i = j$, we understand that $p = q$ and $r = s$. Now, we define $p(i, j; w) = (q - p) + (s - r)$, remark that if there exist another $1 \leq p' \leq q' < r' \leq s'$ such that gives (i, j) -configuration on w , we take the largest one as $p(i, j; w)$. Let us set

$$(3.4.2) \quad I_{(M,N)}^{(C)} = \left\{ w = \begin{bmatrix} i_1 & j_1 \\ \vdots & \vdots \\ i_N & j_M \end{bmatrix} \in B(\Lambda_M) \otimes B(\Lambda_N); \begin{array}{l} w \text{ satisfies the conditions} \\ (M.N.1) \text{ and } (M.N.2) \end{array} \right\}$$

$$(M.N.1) \quad i_k \leq j_k \text{ for } 1 \leq k \leq M$$

$$(M.N.2) \quad \text{if } w \text{ is in the } (i, j)\text{-configuration, then } p(i, j; w) < j - i.$$

Remark that any vector of $I_{(M,N)}^{(C)}$ is not in the (i, i) -configuration, because $p(i, i; w) \geq 0$.

Proposition 3.4.3. $B(\Lambda_M + \Lambda_N)$ coincides with $I_{(M,N)}^{(C)}$.

3.5 The crystal graph of $V(\lambda)$

Let $\lambda = \sum_{i=1}^p \Lambda_{l_i}$ ($1 \leq l_1 \leq l_2 \leq \dots \leq n$) be a dominant integral weight. Let us consider the crystal graph of $B(\lambda)$. By Lemma 3.4.4, we can apply Proposition 2.2.1 and hence

$$B(\lambda) = \left\{ u_1 \otimes \dots \otimes u_p \in B(V_{\mathbb{Z}}^{\Lambda_{l_1}}) \otimes \dots \otimes B(V_{\mathbb{Z}}^{\Lambda_{l_p}}); u_i \otimes u_{i+1} \in B(\Lambda_{l_i} + \Lambda_{l_{i+1}}) \text{ for } 1 \leq i \leq p \right\}$$

For the Young diagram Y with the columns of l_i ($1 \leq i \leq p$), we define a C -semi-standard tableau with shape Y with elements $\{1, 2, \dots, n, \bar{n}, \dots, \bar{2}, \bar{1}\}$ in each boxes of Y satisfying the following conditions;

(3.5.1) Letting $t_{i,j}$ be the element of $\{1, 2, \dots, n, \bar{n}, \dots, \bar{2}, \bar{1}\}$ at the i -th column and j -th row, we have

$$t_{i,j} \leq t_{i+1,j} \quad \text{and} \quad t_{i,j} < t_{i,j+1}$$

(3.5.2) For $1 \leq p \leq q \leq n$, if $t_{i,j} = p$, $t_{i+1,j'} = \bar{p}$ and if $t_{i,k} = q$, $t_{i,k'} = \bar{q}$ (resp. $t_{i+1,k} = q$, $t_{i+1,k'} = \bar{q}$) then $(k - j) + (j' - k') < q - p$.

Theorem 3.5.1. $B(\lambda)$ coincides with the set of the C -semi-standard tableaux with shape Y . The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are described by Proposition 2.1.2 by identifying i and $\overline{i+1}$ with u_+ , $i+1$ and $\bar{i}$ with u_- , and others with u_0 .

§4. Littlewood Richardson rule for C_n .

In this section, we give the rule to decompose $V_Y \otimes V_{Y'}$ (Y and Y' are Young diagrams with depth n) in terms of crystal graph.

4.1 The following lemma plays a significant role in this rule.

Lemma 4.1.1. For $u \in B(V_Y)$ and $v \in B(V_{Y'})$,

$$\tilde{e}_i(u \otimes v) = 0 \text{ (for any } i) \iff \begin{cases} \bullet \tilde{e}_i u = 0 \\ \bullet \tilde{e}_i^{<h_i, \lambda>+1} v = 0 \text{ (for any } i) \end{cases}$$

where λ is the weight of Y .

4.2. Decomposition of $V_Y \otimes V_\square$

Lemma 4.2.1. For a Young diagram $Y = (l_1, l_2, \dots, l_n)$, when we identify Y and the highest element u_Y of $B(V_Y)$, where u_Y is the following;

$$\begin{array}{c} 111 \dots \dots \dots 111 \\ 222 \dots \dots \dots 222 \\ \dots \dots \dots \\ ii \dots \dots ii \\ \dots \dots \dots \\ n \dots \dots n \end{array} \quad \# \{i \in u_Y\} = l_i$$

a) For $\quad \in B(V_\square)$ ($j = 1, 2, \dots, n$), $Y \otimes \quad$ is the highest element of $B(V_Y \otimes V_\square)$ if and only if $l_{j-1} - l_j > 0$. b) For $\quad \in B(V_\square)$ ($j = 1, 2, \dots, n$), $Y \otimes \quad$ is the highest element of $B(V_Y \otimes V_\square)$ if and only if $l_j - l_{j+1} > 0$

Remark that we assume $l_0 = \infty$ and $l_{n+1} = 0$.

Proof We can easily obtain the result by Lemma 4.1.1 and the following facts;

For any i ,

$$\begin{aligned} \tilde{e}_i^{<h_i, \lambda>+1} \boxed{j} &= 0 \iff <h_{j-1}, \lambda> = l_{j-1} - l_j > 0 \\ \tilde{e}_i^{<h_i, \lambda>+1} \boxed{\bar{j}} &= 0 \iff <h_j, \lambda> = l_j - l_{j+1} > 0 \end{aligned}$$

q.e.d.

Now, we get the following proposition.

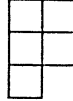
Proposition 4.2.2. Let $Y = (l_1, l_2, \dots, l_n)$ be a Young diagram and V_Y be a finite dimensional irreducible C_n -module characterized by Y ,

$$V_Y \otimes V_{\square} \cong \bigoplus_{j=1}^n V_{(Y \leftarrow j)} \oplus \bigoplus_{j=1}^n V_{(Y \leftarrow \bar{j})}$$

where $(Y \leftarrow j) = (l_1, \dots, l_j + 1, \dots, l_n)$ and $(Y \leftarrow \bar{j}) = (l_1, \dots, l_j - 1, \dots, l_n)$.

Remark 4.2.3. If Y is not a Young diagram, then V_Y means a 0-dimensional vector space.

Proof By Lemma 4.1.1 and Lemma 4.2.1 we can identify $u \otimes \boxed{j}$ (resp. $u \otimes \boxed{\bar{j}}$) with highest condition with a Young diagram $(Y \leftarrow j)$ (resp. $(Y \leftarrow \bar{j})$). Hence, $Y \otimes \boxed{j}$ (resp. $u \otimes \boxed{\bar{j}}$) is the highest element of $B(V_Y \otimes V_{\square})$ if and only if $(Y \leftarrow j)$ (resp. $(Y \leftarrow \bar{j})$) is a Young diagram. Since both $Y \otimes \boxed{j}$ (resp. $Y \otimes \boxed{\bar{j}}$) and $(Y \leftarrow j)$ (resp. $(Y \leftarrow \bar{j})$) have the same weight, $u \otimes \boxed{j}$ (resp. $u \otimes \boxed{\bar{j}}$) ($1 \leq j \leq n$) with the highest condition can be identified with a Young diagram $(Y \leftarrow j)$ (resp. $(Y \leftarrow \bar{j})$). q.e.d.

Example 4.2.4. For $\mathfrak{g} = C_3$ and $Y = (2, 2, 1) =$ 

$$B(V_{\square}) = \{ \boxed{1}, \boxed{2}, \boxed{3}, \boxed{\bar{3}}, \boxed{\bar{2}}, \boxed{\bar{1}} \}$$

$$Y \otimes \boxed{1} = (Y \leftarrow 1) = (3, 2, 1) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & & \\ \hline \end{array} \quad Y \otimes \boxed{\bar{3}} = (Y \leftarrow \bar{3}) = (2, 2, 0) = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$$

$$Y \otimes \boxed{2} = (Y \leftarrow 2) = (2, 3, 1) \times \quad Y \otimes \boxed{\bar{2}} = (Y \leftarrow \bar{2}) = (2, 1, 1) = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array}$$

$$Y \otimes \boxed{3} = (Y \leftarrow 3) = (2, 2, 2) = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad Y \otimes \boxed{\bar{1}} = (Y \leftarrow \bar{1}) = (1, 2, 1) \times$$

Then, we get

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \\ \hline \end{array} \otimes \square = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array}$$

4.3. Decomposition of $V_Y \otimes V_{Y'}$

We shall treat a general case. Let Y and Y' be Young diagrams. We give a combinatorial description for irreducible decomposition of $V_Y \otimes V_{Y'}$. By the following lemma and the way of the construction of the crystal graph, we know that the previous elementary case plays a significant role in a general case.

Lemma 4.3.1. *Let $J = \{1, 2, \dots, p\}$ be a finite index set and V_j ($j \in J$) be a finite dimensional irreducible representation of $U_q(C_n)$. For $u_1 \otimes u_2 \otimes \dots \otimes u_p \in B(\bigotimes_{j \in J} V_j)$, following two assertions are equivalent;*

(A) $u_1 \otimes u_2 \otimes \dots \otimes u_p$ is the highest element of $B(\bigotimes_{j \in J} V_j)$

(B) For any $j \in J$, $u_1 \otimes u_2 \otimes \dots \otimes u_p$ is the highest element of $B(V_1 \otimes \dots \otimes V_j)$

Proof First assuming (B), we get (A) easily. Next we assume (A). For any $j \in J$ we can consider $u_1 \otimes u_2 \otimes \dots \otimes u_p = (u_1 \otimes \dots \otimes u_j \otimes u_{j+1} \otimes \dots \otimes u_p) \in B(V_1 \otimes \dots \otimes V_j) \otimes B(V_{j+1} \otimes \dots \otimes V_p)$. By Lemma 4.1.1, if $(u_1 \otimes \dots \otimes u_j \otimes u_{j+1} \otimes \dots \otimes u_p)$ satisfies the highest condition, $u_1 \otimes \dots \otimes u_j$ also satisfies the highest condition. Hence, we get (B). q.e.d.

Here, by Proposition 4.2.2 and Lemma 4.3.1, we obtain the following theorem.

Theorem 4.3.2. *Let Y and Y' be Young diagrams. Let m be $\#Y'$. Then we obtain the following;*

$$V_Y \otimes V_{Y'} \cong \bigoplus_{\boxed{j_1} \otimes \dots \otimes \boxed{j_m} \in B(V_{Y'})} V_{(((Y \leftarrow j_1) \leftarrow j_2) \dots) \dots \leftarrow j_m)}$$

where $V_{(((Y \leftarrow j_1) \leftarrow j_2) \dots) \dots \leftarrow j_m)}$ is a 0-dimensional vector space if there exists $k \in \{1, \dots, m\}$ such that $((((Y \leftarrow j_1) \leftarrow j_2) \dots) \dots \leftarrow j_k)$ is not a Young diagram.

Example 4.3.3. For $\mathfrak{g} = C_2$, $Y = \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} = (2, 2)$ and $Y' = \begin{smallmatrix} \square \\ \square \end{smallmatrix} = (1, 1)$.

$$B(V_{Y'}) = \left\{ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 1 \\ \bar{2} \end{smallmatrix}, \begin{smallmatrix} 2 \\ \bar{2} \end{smallmatrix}, \begin{smallmatrix} 2 \\ \bar{1} \end{smallmatrix}, \begin{smallmatrix} \bar{2} \\ \bar{1} \end{smallmatrix} \right\}$$

$$Y \otimes \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} = ((\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \leftarrow 1) \leftarrow 2) = (\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \leftarrow 2) = \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}$$

$$Y \otimes \begin{smallmatrix} 1 \\ \bar{2} \end{smallmatrix} = ((\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \leftarrow 1) \leftarrow \bar{2}) = (\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \leftarrow \bar{2}) = \begin{smallmatrix} \square & \square & \square \\ \square & & \square \end{smallmatrix}$$

$$Y \otimes \begin{smallmatrix} 2 \\ \bar{2} \end{smallmatrix} = ((\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \leftarrow 2) \leftarrow \bar{2}) = (\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \leftarrow \bar{2}) \quad \times$$

$$Y \otimes \begin{smallmatrix} 2 \\ \bar{1} \end{smallmatrix} = ((\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \leftarrow 2) \leftarrow \bar{1}) = (\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \leftarrow \bar{1}) \quad \times$$

$$Y \otimes \begin{smallmatrix} \bar{2} \\ \bar{1} \end{smallmatrix} = ((\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \leftarrow \bar{2}) \leftarrow \bar{1}) = (\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \leftarrow \bar{1}) = \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$$

Then we get (we omit "V")

$$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \otimes \begin{smallmatrix} \square \\ \square \end{smallmatrix} = \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \oplus \begin{smallmatrix} \square & \square & \square \\ \square & & \square \end{smallmatrix} \oplus \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$$

Remark We have already obtained the similar conclusions for A_n , B_n and D_n .

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