

On some estimations of the trigonometrical sums
in an algebraic number field

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§ 1 Introduction

We have discussed the Waring Problem in an algebraic number field⁽¹⁾ as an application of Vinogradov's mean value theorem⁽²⁾. This is another application of our previous paper⁽²⁾. Using the Korobov's method we shall prove the estimations of some trigonometrical sums in minor arcs in an algebraic number field.⁽³⁾ Our results will be given as Theorem 1 and Theorem 2 in §3. In the rational number field the estimations of this type are very interesting and important in the analytic theory of numbers. We use a letter c to denote a positive constant depending on K alone. It is not necessarily the same one each time it occurs. The constant c may well depend on another parameter $*$. In this case we write as $c(*)$ to explain the meaning. We use the same terminology and notations as before.^(1,2)

§ 2 Lemmata

Let $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{k-1})$ and $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{k-1})$ be $k+2$ dimensional complex row vectors and let $\mu^* = (\mu_0, \mu_1, \dots, \mu_{k+1})^*$ be a $k+2$ dimensional complex column vector (we call μ^* the transposed of μ). Further let A be a $k+2$ dimensional quadratic (symmetric)

matrix such that

$$A = \begin{bmatrix} \binom{0}{0} \alpha_0, & \binom{1}{1} \alpha_1, & \dots, & \binom{k}{k} \alpha_k, & \binom{k+1}{k+1} \alpha_{k+1} \\ \binom{1}{0} \alpha_1, & \binom{2}{1} \alpha_2, & \dots, & \binom{k+1}{k} \alpha_{k+1}, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \binom{k}{0} \alpha_k, & \binom{k+1}{1} \alpha_{k+1}, & \dots & \dots & \dots \\ \binom{k+1}{0} \alpha_{k+1}, & 0, & \dots & \dots & 0 \end{bmatrix}$$

where $\binom{i}{j}$ are binomial coefficients and $\binom{0}{0} = 0$. We define two $k+2$ dimensional complex vectors $\underline{\beta} = (\beta_0, \beta_1, \dots, \beta_{k+1})$ and $\underline{\gamma}^* = (\gamma_0, \gamma_1, \dots, \gamma_{k+1})^*$ by the following relations:

$$\underline{\lambda} A \underline{\mu} = \underline{\beta} \underline{\mu}^* = \underline{\lambda} \underline{\gamma}^*.$$

Then we have

$$\beta_i = \binom{i}{1} \alpha_1 \lambda_0 + \binom{i+1}{1} \alpha_{i+1} \lambda_1 + \dots + \binom{i+1}{i} \alpha_{k+1} \lambda_{k+1-i} \quad (0 \leq i \leq k+1)$$

and

$$\gamma_i = \binom{i}{0} \alpha_i \mu_0 + \binom{i+1}{1} \alpha_{i+1} \mu_1 + \dots + \binom{k+1}{k+1-i} \alpha_{k+1} \mu_{k+1-i} \quad (0 \leq i \leq k+1).$$

Let $N_s(T; \nu_1, \dots, \nu_k)$ be the number of solutions of the system of k ($k \geq 2$) diophantine equations

$$\begin{cases} \lambda_1 + \lambda_2 + \dots + \lambda_s = \mu_1 + \mu_2 + \dots + \mu_s + \nu_1 \\ \lambda_1^2 + \lambda_2^2 + \dots + \lambda_s^2 = \mu_1^2 + \mu_2^2 + \dots + \mu_s^2 + \nu_2 \\ \dots \dots \dots \\ \lambda_1^k + \lambda_2^k + \dots + \lambda_s^k = \mu_1^k + \mu_2^k + \dots + \mu_s^k + \nu_k, \end{cases}$$

where $\theta \ni \lambda_j$, $\lambda_j < T$, ($1 \leq j \leq s$) and $\nu_1, \dots, \nu_k$ are arbitrarily fixed integers in θ . Let $N_s(T; 0, \dots, 0) = N_s(T) = N_{s,k}(T)$.

We have now to describe two lemmas. They are essential to prove the following theorems.

Lemma 1 $k \geq 2$ denotes a positive integer. R is a sufficiently large positive number. T is a positive real number with the condition that

$$(2R)^{k(1 + 1/(k-1))^{\tau-1}} < T$$

with a positive integer $\tau \geq 1$. Further let s be a positive integer with

$$s \geq \frac{n}{2(n-1)} k(k+1) + \tau k - 1.$$

Then we have the following estimate:

$$N_s(T) \leq c \frac{(k^2 + 2s + \frac{(\tau-1)}{k}s - \frac{1}{4}(\tau-1)(k+1))^{\tau-2k}}{s^{\tau-2k}} \frac{2sn - \frac{1}{2}k(k+1)n + \delta n}{T},$$

where

$$\delta = \delta(k, \tau) = \frac{1}{2}k(k+1)(1 - 1/k)^{\tau}.$$

This is the Vinogradov's mean value theorem in an algebraic number field.

Lemma 2 Let m, m_1 be two natural numbers and let T_1, T be real numbers such that $0 < T_1 \leq T$. We define a trigonometrical sum S as follows:

$$S = S(\alpha_1, \dots, \alpha_{k+1}) = \sum_{\lambda < T} E(f(\lambda))$$

where $f(\lambda) = \alpha_1 \lambda + \dots + \alpha_{k+1} \lambda^{k+1}$ and the summation means that λ runs through all $\lambda \in \theta$ such that $\lambda < T$. Let

$$\tilde{f}_\nu = \binom{\nu+1}{\nu} \alpha_{\nu+1} \tilde{\lambda}_1 + \dots + \binom{k+1}{\nu} \alpha_{k+1} \tilde{\lambda}_{k+1-\nu} \quad (1 \leq \nu \leq k).$$

Further, we write

$$V_{m,m_1} = \sum_{\substack{\tilde{\lambda}_1, \dots, \tilde{\lambda}_k \\ \tilde{\mu}_1, \dots, \tilde{\mu}_k}} N_m(T; \tilde{\lambda}_1, \dots, \tilde{\lambda}_k) N_{s_1}(T_1; \tilde{\mu}_1, \dots, \tilde{\mu}_k) E(\tilde{\beta}_1 \tilde{\mu}_1 + \dots + \tilde{\beta}_k \tilde{\mu}_k)$$

where the summation means that $\tilde{\lambda}_i, \tilde{\mu}_i$ ($1 \leq i \leq k$) run through all

$\tilde{\lambda}_i, \tilde{\mu}_i \in \mathcal{O}$ such that

$$|\lambda_{\nu}^{(p)}| < mT^{\nu}, \quad |\mu_{\nu}^{(p)}| \leq m_1 T_1^{\nu} \quad (1 \leq p \leq n_1),$$

$$|\lambda_{\nu}^{(q)}| \leq 2mT^{\nu}, \quad |\mu_{\nu}^{(q)}| \leq 2m_1 T_1^{\nu} \quad (n_1+1 \leq q \leq n_1+n_2) (1 \leq \nu \leq k).$$

Then we have

$$\left| \frac{1}{2} S \right|^{l_{mm_1}} \leq T^{l_{m_1 m n} - 2mn} T_1^{-2m_1 n} V_{mm_1} + (c T_1 T^{n-1})^{l_{mm_1}}.$$

This is fundamental lemma of Korobov in an algebraic number field.

§3 Theorems

Theorem 1. Let T be a sufficiently large number and $k \geq 2$ be a positive integer. We may suppose that $k^{\frac{1}{2}} \log k < r < k - k^{\frac{1}{2}} \log k$ and we define $\bar{s}$ and $s = \underline{s}$ by $\bar{s} = \max(k-r, r)$ and $s = \underline{s} = \min(k-r, r)$. Let $f(u) = \alpha_1 u + \alpha_2 u^2 + \dots + \alpha_{k+1} u^{k+1}$, $\alpha_i = x_{i1} \zeta_1 + \dots + x_{in} \zeta_n$, $(x_{i1}, \dots, x_{in}) \in U$, ($1 \leq i \leq k+1$). We put a number α_{k+1} of $E(h, t)$ which is defined by the Farey division with respect to (h, t) , where $h = c^{n_{mk}} s D^2 T^{\bar{s}}$, $c \geq 32$, $m = [s^2 \log 2 e^2 n]$ and $t = T^{\underline{s}}$. Then we have

$$|S| \leq c T^{\frac{1}{n - 96(\log n + 3)^2 k^2 \log k}},$$

where c is a suitable positive constant depending only on K .

Theorem 2 Let ε be a positive number and assume that $\varepsilon n < r < (1 - \varepsilon)n$ ($0 < \varepsilon < 1$) in theorem 1. Then we have

$$|S| \leq c(\varepsilon, n) T^{n - \frac{c(\varepsilon, n)}{k^2}}$$

where

$$c(\varepsilon, n) = 1/(32 \log(2e^2 n) \log(2e^2 n / \varepsilon^2)).$$

To obtain these results we must use some results of Siegel and Mitsui of the estimations of trigonometrical sums in an algebraic number field ^(4,5,6).

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References

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