

Predual of Campanato spaces and Riesz potentials

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1. INTRODUCTION

This is an announcement of my recent work.

Let $X = (X, \delta, \mu)$ be a space of homogeneous type (SHT), i.e. X is a topological space endowed with a quasi-distance δ and a nonnegative measure μ such that

$$\begin{aligned} \delta(x, y) &\geq 0 \quad \text{and} \quad \delta(x, y) = 0 \text{ if and only if } x = y, \\ \delta(x, y) &= \delta(y, x), \\ (1.1) \quad \delta(x, y) &\leq K_1 (\delta(x, z) + \delta(z, y)), \end{aligned}$$

the balls $B(x, r) = \{y \in X : \delta(x, y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x , μ is defined on a σ -algebra of subsets of X which contains all balls, and

$$(1.2) \quad 0 < \mu(B(x, 2r)) \leq K_2 \mu(B(x, r)) < \infty,$$

where $K_i \geq 1$ ($i = 1, 2$) are constants independent of $x, y, z \in X$ and $r > 0$.

If there are constants θ ($0 < \theta \leq 1$) and $K_3 \geq 1$ such that

$$(1.3) \quad |\delta(x, z) - \delta(y, z)| \leq K_3 (\delta(x, z) + \delta(y, z))^{1-\theta} \delta(x, y)^\theta, \quad x, y, z \in X,$$

then the balls are open sets. The number θ is called the order of the SHT.

We shall say that a SHT is normal if there are constants $K_4 > 0$ and $K_5 > 0$

$$(1.4) \quad K_4 r \leq \mu(B(x, r)) \leq K_5 r \quad \text{for } x \in X \text{ and } \mu(\{x\}) < r < \mu(X).$$

We note that, for any SHT (X, d, μ) , there exists a quasi-distance δ such that (X, δ, μ) is normal and of some order θ , and that the topologies induced on X by d and δ coincide (Macías and Segovia (1979)).

Let $X = \mathbb{R}^n$, $d(x, y) = |x - y|$ and μ be the Lebesgue measure. If $\delta(x, y) = |x - y|^n$, then $(\mathbb{R}^n, \delta, \mu)$ is normal and of order $1/n$.

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In this talk we always assume that (X, δ, μ) is normal and of order θ and that $\mu(\{x\}) = 0$ for all x in X .

We consider Riesz potentials

$$I_\alpha f(x) = \int_X \frac{f(y)}{\delta(x, y)^{1-\alpha}} d\mu(y),$$

for $0 < \alpha < \theta$. It is known that the operator I_α is bounded from $L^p(X)$ to $L^q(X)$ if $1 < p < q < \infty$ and $-1/p + \alpha = -1/q$ (Gatto and Vagi(1990)). This boundedness is well known as the Hardy-Littlewood-Sobolev theorem in $\mathbb{R}^n$ case.

In this report, we define a generalized Hardy space $H_U^{[\phi, q]}(X)$ and investigate continuity of I_α on $H_U^{[\phi, q]}(X)$. We show

$$\left(H_U^{[\phi, q]}(X)\right)^* = \mathcal{L}_{q', \phi}(X),$$

where $\mathcal{L}_{q', \phi}(X)$ is a Campanato space. Campanato spaces are Banach spaces modulo constants, which include $\text{BMO}(X)$ and $\text{Lip}_\alpha(X)$ as special cases.

We first define I_α for functions $f \in \mathcal{L}_{q', \phi}(X)$. To do this we define the modified version of I_α as follows;

$$\tilde{I}_\alpha f(x) = \int_X f(y) \left(\frac{1}{\delta(x, y)^{1-\alpha}} - \frac{1 - \chi_{B_0}(y)}{\delta(x_0, y)^{1-\alpha}} \right) dy,$$

where $B_0 = B(x_0, r_0)$ is a fixed ball. We can show that $\tilde{I}_\alpha f(x)$ converges absolutely for all x and therefore changing B_0 in the definition above results in adding a constant. We assume that δ satisfies the cancellation property;

$$(1.5) \quad \int_X \left(\frac{1}{\delta(x, y)^{1-\alpha}} - \frac{1}{\delta(x', y)^{1-\alpha}} \right) d\mu(y) = 0 \quad \text{for any } x, x' \text{ in } X.$$

In case of $X = \mathbb{R}^n$ or $\mathbb{T}^n$, (1.5) holds for $\delta(x, y) = |x - y|^n$ and for $0 < \alpha < 1$. For other examples of spaces of homogeneous type with the property (1.5), see [3]. We note that, for all normal spaces (X, δ, μ) with $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in X$, we can find a quasi-distance δ_α equivalent to δ , such that (1.5) holds (see [2]).

2. CAMPANATO SPACES $\mathcal{L}_{p, \phi}(X)$ AND HÖLDER SPACES $\Lambda_\phi(X)$

Let $1 \leq p < \infty$ and $\phi : X \times (0, \infty) \rightarrow (0, \infty)$. For a ball $B = B(x, r)$, we shall write $\phi(B)$ in place of $\phi(x, r)$. The function spaces $\mathcal{L}_{p, \phi}(X)$ and $\Lambda_\phi(X)$ are defined

to be the sets of all f such that $\|f\|_{\mathcal{L}_{p,\phi}} < \infty$ and $\|f\|_{\Lambda_\phi} < \infty$, respectively, where

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p d\mu(x) \right)^{1/p},$$

$$\|f\|_{\Lambda_\phi} = \sup_{x,y \in X, x \neq y} \frac{2|f(x) - f(y)|}{\phi(x, \delta(x, y)) + \phi(y, \delta(y, x))},$$

and

$$f_B = \mu(B)^{-1} \int_B f(x) d\mu(x).$$

Then $\mathcal{L}_{p,\phi}(X)$ and $\Lambda_\phi(X)$ are Banach spaces modulo constants with the norms $\|f\|_{\mathcal{L}_{p,\phi}}$ and $\|f\|_{\Lambda_\phi}$, respectively. If $p = 1$ and $\phi \equiv 1$, then $\mathcal{L}_{1,\phi}(X) = \text{BMO}(X)$.

Let $\mathcal{G}_*$ be the set of all functions $\phi : X \times (0, \infty) \rightarrow (0, \infty)$ such that

$$(2.1) \quad \frac{1}{A_1} \leq \frac{\phi(x, s)}{\phi(x, r)} \leq A_1, \quad \frac{1}{2} \leq \frac{s}{r} \leq 2,$$

$$(2.2) \quad \phi(x, r) \leq A_2 \phi(y, s), \quad B(x, r) \subset B(y, s),$$

where A_1 and $A_2 > 0$ are independent of $r, s > 0$, $x, y \in X$.

Theorem 2.1. *Let $\phi \in \mathcal{G}_*$. Then*

$$\mathcal{L}_{p,\phi}(X) = \mathcal{L}_{1,\phi}(X)$$

with equivalent norms for every $1 \leq p < \infty$.

Theorem 2.2. *Let $\phi \in \mathcal{G}_*$ and there exists $C > 0$ such that*

$$(2.3) \quad \int_0^{\delta(x,y)} \frac{\phi(x, t)}{t} dt \leq C \phi(x, \delta(x, y)), \quad x, y \in X.$$

Then

$$\Lambda_\phi(X) = \mathcal{L}_{p,\phi}(X)$$

with equivalent norms for every $1 \leq p < \infty$.

We say that $\alpha(\cdot) : X \rightarrow [0, \infty)$ is log-Hölder continuous if there exists $C_0 > 0$ such that

$$(2.4) \quad |\alpha(x) - \alpha(y)| \leq \frac{C_0}{\log(1/\delta(x, y))} \quad \text{for } \delta(x, y) < 1/2.$$

Let $\alpha_- = \inf_{x \in X} \alpha(x)$ and $\alpha_+ = \sup_{x \in X} \alpha(x)$.

Example 2.1. Let $\alpha(\cdot)$ be log-Hölder continuous and

$$\phi(x, r) = r^{\alpha(x)} \quad \text{with } 0 < \alpha_- \leq \alpha_+ \leq \theta.$$

Then $\phi \in \mathcal{G}_*$ and satisfies (2.3). In this case we denote $\Lambda_\phi(X)$ by $\text{Lip}_{\alpha(\cdot)}(X)$ and

$$\|f\|_{\text{Lip}_{\alpha(\cdot)}} = \sup_{x,y \in X, x \neq y} \frac{2|f(x) - f(y)|}{\delta(x,y)^{\alpha(x)} + \delta(y,x)^{\alpha(y)}}.$$

If $\alpha(x) \equiv \alpha$, then $\text{Lip}_{\alpha(\cdot)}(X) = \text{Lip}_\alpha(X)$.

3. GENERALIZED HARDY SPACES $H_U^{[\phi,q]}(X)$

Let $\phi : X \times (0, \infty) \rightarrow (0, \infty)$, $1 < q \leq \infty$ and $1/q + 1/q' = 1$.

Definition 3.1 ($[\phi, q]$ -atom). A function a on X is called a $[\phi, q]$ -atom if there exists a ball B such that

- (i) $\text{supp } a \subset B$,
- (ii) $\|a\|_q \leq \frac{1}{\mu(B)^{1/q'} \phi(B)}$,
- (iii) $\int_X a(x) d\mu(x) = 0$,

where $\|a\|_q$ is the L^q norm of a . We denote by $A[\phi, q]$ the set of all $[\phi, q]$ -atoms.

Let $\mathcal{F}$ be the set of all continuous, increasing and bijective functions $\Phi : [0, \infty) \rightarrow [0, \infty)$. Then $\Phi(0) = 0$ and $\lim_{r \rightarrow \infty} \Phi(r) = \infty$ for all $\Phi \in \mathcal{F}$.

Let $\mathcal{F}_X$ be the set of all functions $\Phi : X \times [0, \infty) \rightarrow [0, \infty)$ such that

- (i) $\Phi(x, \cdot) \in \mathcal{F}$ for every $x \in X$, and
- (ii) $\Phi(\cdot, r)$ is measurable on X for all $r \in [0, \infty)$.

We denote by $\Phi^{-1}(x, \cdot)$ the inverse of $\Phi(x, \cdot)$ with respect to $r \in [0, \infty)$.

For $\Phi \in \mathcal{F}_X$ and $B = B(x, r)$, let

$$(3.1) \quad \phi(x, r) = \phi(B) = \frac{1}{\mu(B) \Phi^{-1}(x, 1/\mu(B))}.$$

Then

$$\frac{1}{\mu(B)^{1/q'} \phi(B)} = \mu(B)^{1/q} \Phi^{-1} \left(x, \frac{1}{\mu(B)} \right).$$

If $\Phi(x, r) = r^{p(x)}$, $p(\cdot) : X \rightarrow (0, 1]$, then

$$\frac{1}{\mu(B)^{1/q'} \phi(B)} = \mu(B)^{1/q-1/p(x)}.$$

If $\Phi(x, r) = r^p$, $0 < p \leq 1$, then

$$\frac{1}{\mu(B)^{1/q'} \phi(B)} = \mu(B)^{1/q-1/p}.$$

In this case, $[\phi, q]$ -atoms are the usual (p, q) -atoms.

We define $H_U^{[\phi,q]}(X)$ as a subspace of the dual of $\mathcal{L}_{q',\phi}(X)$. We can see $A[\phi,q] \subset (\mathcal{L}_{q',\phi}(X))^*$ as follows. If a is a $[\phi,q]$ -atom and a ball B satisfies (i)–(iii), then

$$\begin{aligned}
 (3.2) \quad \left| \int_X a(x)g(x) d\mu(x) \right| &= \left| \int_B a(x)(g(x) - g_B) d\mu(x) \right| \\
 &\leq \|a\|_q \left(\int_B |g(x) - g_B|^{q'} d\mu(x) \right)^{1/q'} \\
 &\leq \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |g(x) - g_B|^{q'} d\mu(x) \right)^{1/q'} \\
 &\leq \|g\|_{\mathcal{L}_{q',\phi}}.
 \end{aligned}$$

That is, the mapping $g \mapsto \int_X ag d\mu$ is a bounded linear functional on $\mathcal{L}_{q',\phi}(X)$ with norm not exceeding 1.

Definition 3.2 ($H_U^{[\phi,q]}(X)$). Let $\phi : X \times (0, \infty) \rightarrow (0, \infty)$, $1 < q \leq \infty$, $1/q + 1/q' = 1$ and $U \in \mathcal{F}$ be concave. We define the space $H_U^{[\phi,q]}(X) \subset (\mathcal{L}_{q',\phi}(X))^*$ as follows:

$f \in H_U^{[\phi,q]}(X)$ if and only if there exist sequences $\{a_j\} \subset A[\phi,q]$ and positive numbers $\{\lambda_j\}$ such that

$$(3.3) \quad f = \sum_j \lambda_j a_j \text{ in } (\mathcal{L}_{q',\phi}(X))^* \text{ and } \sum_j U(\lambda_j) < \infty.$$

From $U(0) = 0$ and the concavity of U it follows that

$$(3.4) \quad U(Cr) \leq CU(r), \quad 1 \leq C < \infty, \quad 0 \leq r < \infty,$$

$$(3.5) \quad U(r+s) \leq U(r) + U(s), \quad 0 \leq r, s < \infty.$$

Then $H_U^{[\phi,q]}(X)$ is a linear space.

In general, the expression (3.3) is not unique. We define

$$\|f\|_{H_U^{[\phi,q]}} = \inf \left\{ U^{-1} \left(\sum_j U(\lambda_j) \right) \right\},$$

where the infimum is taken over all expressions as in (3.3). We note that $\|f\|_{H_U^{[\phi,q]}}$ is not a norm in general. Let $m(f, g) = U(\|f - g\|_{H_U^{[\phi,q]}})$ for $f, g \in H_U^{[\phi,q]}(X)$. Then $m(f, g)$ is a metric and $H_U^{[\phi,q]}(X)$ is complete with respect to this metric.

If $\phi(B) = \mu(B)^{1/p-1}$ and $U(r) = r^p$, then $H_U^{[\phi,q]}(X)$ coincides $H^{p,q}(X)$ defined by Coifman and Weiss (1977). They showed $H^{p,q}(X) = H^{p,\infty}(X)$ with equivalent metrics when $0 < p \leq 1 < q \leq \infty$ and denoted this space by $H^p(X)$. We extend this result to $H_U^{[\phi,q]}(X) = H_U^{[\phi,\infty]}(X)$ in the next section.

Let $I(r) = r$. Then $\|f\|_{H_I^{[\phi,q]}}$ is a norm and $H_I^{[\phi,q]}(X)$ is a Banach space, which was defined by Zorko (1986) in the case $X = \mathbb{R}^n$. Therefore, our definition is a generalization of both definitions.

From the definition we have the following relations.

Proposition 3.1. (i) If $1 < q_1 < q_2 \leq \infty$, then

$$H_U^{[\phi,q_2]}(X) \subset H_U^{[\phi,q_1]}(X).$$

(ii) If $\psi(B) \leq C\phi(B)$ for all balls B , then

$$H_U^{[\phi,q]}(X) \subset H_U^{[\psi,q]}(X).$$

(iii) If $V(r) \leq CU(r)$ for $0 \leq r \leq 1$, then

$$H_U^{[\phi,q]}(X) \subset H_V^{[\phi,q]}(X).$$

(iv) For any concave function $U \in \mathcal{F}$,

$$H_U^{[\phi,q]}(X) \subset H_I^{[\phi,q]}(X).$$

In the above, the inclusion mapping are continuous.

4. EQUIVALENCE $H_U^{[\phi,q]}(X) = H_U^{[\phi,\infty]}(X)$

Theorem 4.1. Let $\phi \in \mathcal{G}_*$. If there exists $C_* > 0$ such that

$$(4.1) \quad U(rs) \leq C_* U(r)U(s) \quad \text{for } 0 < r, s \leq 1,$$

$$(4.2) \quad U\left(\frac{\mu(B_1)\phi(B_1)}{\mu(B_2)\phi(B_2)}\right) \leq C_* \frac{\mu(B_1)}{\mu(B_2)} \quad \text{for } B_1 \subset B_2,$$

then

$$H_U^{[\phi,q]}(X) = H_U^{[\phi,\infty]}(X),$$

with equivalent topologies.

For $\Phi(x, r) \in \mathcal{F}_X$, let

$$\phi(x, r) = \phi(B) = \frac{1}{\mu(B)\Phi^{-1}(x, 1/\mu(B))}.$$

Example 4.1. Assume that $\mu(X) < \infty$. Let $p(\cdot)$ be log-Hölder continuous and

$$\Phi(x, r) = r^{p(x)}, \quad U(r) = r^{p_+} \quad \text{with } 0 < p_- \leq p_+ \leq 1.$$

Then the assumption of Theorem 4.1 holds. Therefore

$$H_U^{[\phi,q]}(X) = H_U^{[\phi,\infty]}(X).$$

In this case we denote $H_U^{[\phi,q]}(X)$ by $H^{p(\cdot)}(X)$. If $p(\cdot) \equiv p$, then $H^{p(\cdot)}(X) = H^p(X)$, the usual Hardy space.

5. DUALITY

Let $L_c^q(X)$ be the set of all L^q -functions with bounded support, and let

$$L_c^{q,0}(X) = \left\{ f \in L_c^q(X) : \int_X f d\mu = 0 \right\}.$$

Then, for $1 < q \leq \infty$, $L_c^{q,0}(X)$ is dense in $H_U^{[\phi,q]}(X)$.

If $g \in \mathcal{L}_{q',\phi}(X)$ and $f \in L_c^{q,0}(X)$, then $f(g+c)$ is integrable for all constants c and $\int_X f(g+c) d\mu$ is independent of c .

Theorem 5.1. *If U satisfies*

$$(5.1) \quad \sup_{0 < s \leq 1} \frac{U(rs)}{U(s)} \rightarrow 0 \quad (r \rightarrow 0),$$

then

$$\left(H_U^{[\phi,q]}(X) \right)^* = \mathcal{L}_{q',\phi}(X).$$

More precisely, if $g \in \mathcal{L}_{q',\phi}(X)$, then the mapping $\ell : f \mapsto \int_X f(g+c) d\mu$, for $f \in L_c^{q,0}(X)$, can be extended to a continuous linear functional on $H_U^{[\phi,q]}(X)$. Conversely, if ℓ is a continuous linear functional on $H_U^{[\phi,q]}(X)$, then there exists $g \in \mathcal{L}_{q',\phi}(X)$ such that $\ell(f) = \int_X f(g+c) d\mu$ for $f \in L_c^{q,0}(X)$. The norm $\|\ell\|$ is equivalent to $\|g\|_{\mathcal{L}_{q',\phi}}$.

Corollary 5.2. *Let $\phi \in \mathcal{G}_*$. Then, for any $q \in (1, \infty]$ and for any concave function $U \in \mathcal{F}$ with (5.1),*

$$\left(H_U^{[\phi,q]}(X) \right)^* = \mathcal{L}_{1,\phi}(X).$$

Corollary 5.3. *Let $\phi \equiv 1$. Then, for any $q \in (1, \infty]$ and for any concave function $U \in \mathcal{F}$ with (5.1),*

$$\left(H_U^{[\phi,q]}(X) \right)^* = \text{BMO}(X).$$

Corollary 5.4. *Let $\phi \in \mathcal{G}_*$ and there exists $C > 0$ such that*

$$\int_0^{\delta(x,y)} \frac{\phi(x,t)}{t} dt \leq C\phi(x, \delta(x,y)), \quad x, y \in X.$$

Then, for any $q \in (1, \infty]$ and for any concave function $U \in \mathcal{F}$ with (5.1),

$$\left(H_U^{[\phi,q]}(X) \right)^* = \Lambda_\phi(X).$$

Example 5.1. Under the assumption of Example 4.1, let $\alpha(x) = 1/p(x) - 1$. Then

$$(H^{p(\cdot)}(X))^* = \text{Lip}_{\alpha(\cdot)}(X).$$

$$6. \text{ EQUIVALENCE } H_U^{[\phi, q]}(X, d, \mu) = H_U^{[\psi, q]}(X, \delta, \mu)$$

For a space of homogeneous type (X, d, μ) such that the balls are open sets, let

$$(6.1) \quad \delta(x, y) = \begin{cases} \inf\{\mu(B^d) : B^d \ni x, y\} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

where B^d denotes a ball by the quasi-distance d . Then (X, δ, μ) is normal and the topologies induced on X by d and δ coincide.

Theorem 6.1. Suppose that $\psi : X \times (0, \infty) \rightarrow (0, \infty)$ satisfies (2.1). Let $\tilde{\phi}(x, r) = \phi(x, \mu(B^d(x, r)))$. Then

$$\begin{aligned} \mathcal{L}_{p, \tilde{\phi}}(X, d, \mu) &= \mathcal{L}_{p, \phi}(X, \delta, \mu), \\ H_U^{[\tilde{\phi}, q]}(X, d, \mu) &= H_U^{[\phi, q]}(X, \delta, \mu), \end{aligned}$$

with equivalent topologies, respectively.

Example 6.1. Let $X = \mathbb{R}^n$, $d(x, y) = |x - y|$ and μ be the Lebesgue measure. Then

$$\begin{aligned} \delta(x, y) &= \frac{v_n}{2^n} |x - y|^n, \\ \tilde{\phi}(x, r) &= \phi(x, v_n r^n), \end{aligned}$$

where v_n is the volume of the unit ball. Therefore, $(\mathbb{R}^n, \delta, \mu)$ is of order $1/n$ and, for $0 < \alpha < \theta = 1/n$,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\delta(x, y)^{1-\alpha}} d\mu(y) = \int_{\mathbb{R}^n} \frac{f(y)}{(\frac{v_n}{2^n} |x - y|^n)^{1-\alpha}} d\mu(y).$$

7. RIESZ POTENTIALS ON $\mathcal{L}_{p, \phi}(X)$

Theorem 7.1. Let $0 < \alpha < \theta$, $1 \leq p < \infty$ and $\phi, \psi \in \mathcal{G}_*$. Assume that there exists a constant $A > 0$ such that, for all $x \in X$ and $r > 0$,

$$(7.1) \quad r^\theta \int_r^\infty \frac{t^\alpha \phi(x, t)}{t^{1+\theta}} dt \leq A\psi(x, r).$$

Then $\tilde{I}_\alpha$ is bounded from $\mathcal{L}_{p, \phi}(X)$ to $\mathcal{L}_{p, \psi}(X)$.

Corollary 7.2. *Let $\mu(X) < \infty$, $0 < \alpha < \theta$. Assume that $\beta(\cdot)$ and $\gamma(\cdot)$ are log-Hölder continuous and*

$$\alpha + \beta(x) = \gamma(x) \quad \text{with} \quad 0 < \beta_- < \gamma_+ < \theta.$$

Then $\tilde{I}_\alpha$ is bounded from $\text{Lip}_{\beta(\cdot)}(X)$ to $\text{Lip}_{\gamma(\cdot)}(X)$.

8. RIESZ POTENTIALS ON $H_U^{[\phi, \infty]}(X)$

Theorem 8.1. *Let $0 < \alpha < \theta$, $\phi, \psi \in \mathcal{G}_*$ and $U, V \in \mathcal{F}$ be concave. Assume that there exist $0 < \epsilon < 1$, $0 < \tau \leq 1$ and $A > 0$ such that*

$$(8.1) \quad \psi(x, r)r^\alpha \leq A\phi(x, r), \quad r > 0,$$

$$(8.2) \quad s^{\alpha-\theta-1} (s\psi(x, s))^{1/\epsilon} \leq Ar^{\alpha-\theta-1} (r\psi(x, r))^{1/\epsilon}, \quad 0 < r \leq s,$$

$$(8.3) \quad V(r) \leq Ar^\tau, \quad r \in (0, 1],$$

$$(8.4) \quad V(rs) \leq AV(r)U(s), \quad 0 \leq r, s \leq 1.$$

Then there exists $C > 0$ such that

$$\|I_\alpha a\|_{H_V^{[\psi, \infty]}} \leq C \quad \text{for all } a \in A[\phi, \infty],$$

and I_α extends to a continuous linear map from $H_U^{[\phi, \infty]}(X)$ to $H_V^{[\psi, \infty]}(X)$.

Corollary 8.2. *Let $\mu(X) < \infty$, $0 < \alpha < \theta$. Assume that $p(\cdot)$ and $q(\cdot)$ are log-Hölder continuous and*

$$(8.5) \quad -\frac{1}{p(x)} + \alpha = -\frac{1}{q(x)} \quad \text{with} \quad \frac{1}{1+\theta} < p_- < q_+ \leq 1.$$

Then there exists $C > 0$ such that

$$\|I_\alpha a\|_{H^{q(\cdot)}} \leq C \quad \text{for all } a \in A(p(\cdot), \infty),$$

and I_α extends to a continuous linear map from $H^{p(\cdot)}(X)$ to $H^{q(\cdot)}(X)$.

In the above, $a \in A(p(\cdot), \infty)$ means that there exists $B = B(x, r)$ such that

- (i) $\text{supp } a \subset B$,
- (ii) $\|a\|_q \leq \mu(B)^{1/q-1/p(x)}$,
- (iii) $\int_X a(x) d\mu(x) = 0$.

REFERENCES

- [1] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.
- [2] A. E. Gatto, C. Segovia and S. Vági, *On fractional differentiation and integration on spaces of homogeneous type*, Rev. Mat. Iberoamericana 12 (1996), no. 1, 111–145.
- [3] A. E. Gatto and S. Vági, *Fractional integrals on spaces of homogeneous type*, Analysis and partial differential equations, 171–216, Dekker, New York, 1990.
- [4] R. A. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. Math. **33** (1979), 257–270.
- [5] E. Nakai, *The Campanato, Morrey and Hölder spaces on spaces of homogeneous type*, Studia Math., **176** (2006), 1–19.
- [6] E. Nakai, *A generalization of Hardy spaces H^p by using atoms*, Acta Math. Sinica, **24** (2008), 1243–1268.
- [7] C. T. Zorko, *Morrey space*, Proc. Amer. Math. Soc. **98** (1986), 586–592.

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