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Ergodic properties of the equilibrium processes associated with infinitely many Markovian particles

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Consider a system of independent identically distributed Markov processes which have an invariant measure λ . It is known that if each process starts from each point of a λ -Poisson point process at time zero, these particles are λ -Poisson distributed at every later time $t > 0$. In this paper we are concerned with the ergodic properties of the stationary process obtained from such a system of particles, which is called the equilibrium process. Sinai's ideal gas model is a special example of the equilibrium processes.

Let $(X, \mathcal{B}_X, \lambda)$ be a σ -finite measure space, and denote by $\mathcal{K}(X)$ a family of all the counting measures on X , i.e. each element of $\mathcal{K}(X)$ is an integer-valued measure with a countable set as its support. $\mathcal{K}(X)$ is equipped with a σ -field $\mathcal{G}$ which is generated by $\{\mathcal{J} \in \mathcal{K}(X) ; \mathcal{J}(A) = n\}$, $n \geq 0$, $A \in \mathcal{B}_X$. An element $\mathcal{J}$ of $\mathcal{K}(X)$ is represented by $\mathcal{J} = \sum_i \delta_{x_i}$ where $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$. Let π_λ be a probability measure on $(\mathcal{K}(X), \mathcal{G})$.

π_λ is λ -Poisson point process if it satisfies the following ; for any disjoint system $A_1, \dots, A_n$ of $\mathcal{B}_X$ such that $\lambda(A_i) < +\infty$, $i=1, \dots, n$

$\mathcal{J}(A_1), \dots, \mathcal{J}(A_n)$ are independent random variables on $(\mathcal{K}(X), \mathcal{G}, \pi_\lambda)$, and

$$\pi_\lambda\{\mathcal{J} ; \mathcal{J}(A_i) = n\} = \left[\frac{\lambda(A_i)}{n} \right]^n \exp[-\lambda(A_i)], \quad i=1, \dots, n.$$

Next, we define the equilibrium processes associated with Markovian particles.

Let X be a locally compact separable Hausdorff space and $\mathcal{B}_X$ be the topological Borel field of X . Denote by W the path space of X , that is, each element of W is a X -valued right continuous function with left limit defined on $(-\infty, \infty)$, and define the shift operators $\{\theta_t\}_{-\infty < t < \infty}$ of W as usual ; $(\theta_t f)_s = f_{t+s}$ for each f of W .

Put $S = \mathcal{K}(X)$ and $\Omega = \mathcal{K}(W)$. Denote by $\{\Theta_t\}_{-\infty < t < \infty}$ the shift operators on Ω induced by the shift operators $\{\theta_t\}_{-\infty < t < \infty}$ on W , i.e.

$$\Theta_t \omega = \sum_i \delta_{\theta_t f_i} \quad \text{if } \omega = \sum_i \delta_{f_i}$$

Define S -valued process $\{\xi_t(\omega)\}_{-\infty < t < \infty}$ on Ω as follows ;

$$\xi_t(\omega) = \sum_i f_t^i \quad \text{if } \omega = \sum_i \delta_{f_i}$$

Then $\xi_t(\omega)$ is right continuous in t in a natural topology.

In our situation a motion of one particle is given as a Markov process on X and denote by $\{P_t(x, dy)\}$ its transition probabilities.

Assumption

$P_t(x, dy)$ is a conservative Feller Markov process and have a Radon invariant measure λ , that is, $\{P_t(x, dy)\}$ induces a semi-group of contraction operators $\{T_t\}$ on $C_\infty(X)$, and $\int T_t f(x) \lambda(dx) = \int f(x) \lambda(dx)$ for every f of $C_0(X)$.

Under this assumption $\{T_t\}$ is, also, a semi-group of contraction operators on $L^2(X, \mathcal{B}_X, \lambda)$.

Lemma There is only one σ -finite measure Q on $(W, \mathcal{B}_W)$ such that for $-\infty < t_1 < t_2 < \dots < t_n < +\infty$ and $\{A_i\}_{i=1,2,\dots,n}$

$$\begin{aligned} & Q[f; f_{t_1} \in A_1, f_{t_2} \in A_2, \dots, f_{t_n} \in A_n] \\ &= \int_{A_1} \lambda(dx_1) \int_{A_2} P_{t_2-t_1}(x_1, dx_2) \dots \int_{A_n} P_{t_n-t_{n-1}}(x_{n-1}, dx_n) \end{aligned}$$

In Particular Q is $\{\theta_t\}$ -invariant.

Denote by $\mathbb{B}$ the σ -field generated by $\{\omega \in \Omega; \omega(A)=n\} \quad n \geq 0, A \in \mathcal{B}_X$ and put $\mathbb{P} = \Pi_Q$ (Q -Poisson point process). We consider $(\Omega, \mathbb{B}, \mathbb{P})$ as our basic probability space.

Proposition 1. $\{\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty < t < \infty}\}$ is a right-continuous Markov stationary process with Π_λ as its absolute law.

The Markov process defined above is called the equilibrium process associated with $[\{T_t\}, \lambda]$. Our purpose is to investigate the ergodic properties.

Proposition 2. The following (i), (ii), and (iii) are equivalent.

- (i) $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\})$ is metrically transitive.
- (ii) $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_K P_s(x, K) \lambda(dx) ds = 0$ for every compact subset K of X .
- (iii) $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (T_s f, g)_{L^2(\lambda)} ds = 0$ for all f and g of $L^2(X, \lambda)$.

Proposition 3. The following three statements are equivalent.

- (i) $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty < t < \infty})$ has the strong mixing property.
- (ii) $\lim_{t \rightarrow \infty} \int_K \lambda(dx) P_t(x, K) = 0$ for all f and g of $L^2(X, \lambda)$.
- (iii) $\lim_{t \rightarrow \infty} (T_t f, g)_{L^2(\lambda)} = 0$ for all f and g of $L^2(X, \lambda)$.

Proposition 4. The following three statements are equivalent.

- (i) $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty < t < \infty})$ is purely non-deterministic.
- (ii) $\lim_{t \rightarrow \infty} \int_X \lambda(dx) [P_t(x, K)]^2 = 0$ for every compact subset K of X .
- (iii) $\lim_{t \rightarrow \infty} \|T_t f\|_{L^2(\lambda)} = 0$ for every f of $L^2(X, \lambda)$.

Proposition 5. $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty < t < \infty})$ is purely non-deterministic if and only if $\mathbb{E}[\xi_t | \xi_0]$ converges to λ vaguely in probability.

Next we study the Bernoulli property of the shift flow $\{\Theta_t\}_{-\infty < t < \infty}$

It is easy to see that $\{\Theta_t\}_{-\infty < t < \infty}$ is a flow on the probability space $(\Omega = \mathcal{K}(W), \mathbb{B}, \mathbb{P} = \Pi_Q)$.

So, we define the Bernoulli property in the strong sense.

$(\Omega, \mathcal{B}, \mathbb{P}; \{\Theta_t\}_{-\infty < t < \infty})$ is called a Bernoulli flow if there exists a σ -subfield ζ_0 of $\mathcal{B}$ and $\zeta_t = \Theta_t \cdot \zeta_0$ satisfies the following conditions ;

- (i) $\zeta_t \subset \zeta_s$ for any $t < s$
- (ii) $\bigcap_t \zeta_t = \{\phi, \Omega\} \pmod{\mathbb{P}}$
- (iii) $\bigvee_t \zeta_t = \mathcal{B} \pmod{\mathbb{P}}$
- (iv) for any $t < s$ there exists a σ -subfield ζ_t^s of $\mathcal{B}$ such that $\zeta_s = \zeta_t \vee \zeta_t^s$ and $\zeta_t \perp \zeta_t^s$.

The following lemma is a criterion of the Bernoulli property of our shift flow $\{\Theta_t\}_{-\infty < t < \infty}$

Lemma Suppose that there exists a real measurable function $\tau(f)$ on the σ -finite measure space $(W, \mathcal{B}_W, Q)$ such that for almost all f w.r.t. Q (a) $-\infty < \tau(f) < +\infty$

$$(b) \quad \tau(f) = t + \tau(\Theta_t f) \quad \text{for all } t \text{ of } \mathbb{R}^1.$$

Then, $(\Omega, \mathcal{B}, \mathbb{P}; \{\Theta_t\}_{-\infty < t < \infty})$ is a Bernoulli flow.

We can show the following proposition by appealing to this lemma.

Proposition 6. Suppose that $\{T_t\}$ is transient in the sense that

$$\int_0^\infty (T_t \varphi, \varphi)_{L^2(\lambda)} dt < +\infty \quad \text{for every } \varphi \text{ of } C_0^+(X).$$

Then, $(\Omega, \mathcal{B}, \mathbb{P}; \{\Theta_t\}_{-\infty < t < \infty})$ is a Bernoulli flow.

The equilibrium process $\{\xi_t\}$ induces a factor flow of $\{\Theta_t\}$. Since a Bernoulli flow $\{\Theta_t\}$ in our sense is a Bernoulli flow in the weak sense (i.e. the automorphism $\{\Theta_t\}$ is Bernoulli for each $t \neq 0$), the shift flow induced by $\{\xi_t\}$ is also a Bernoulli flow in the weak sense by the theorem of Ornstein.

Finally we can prove a central limit theorem related to the equilibrium process. Denote by $G\varphi(x) = \int_0^\infty T_t \varphi(x) dt$ if the integral is well-defined.

Proposition 7. Consider any function $\varphi \in L^2(X, \lambda)$ which satisfies $(G|\varphi|, |\varphi|)_{L^2(\lambda)} < +\infty$ and $(G(|\varphi|G|\varphi|), |\varphi|)_{L^2(\lambda)} < +\infty$. Then, we have

$$\lim_{t \rightarrow \infty} \mathbb{P}[\omega; \alpha < \frac{\int_0^t \langle \varphi, \xi_s \rangle ds - t \cdot \langle \varphi, \lambda \rangle}{\sqrt{2(\varphi, G\varphi)_{L^2(\lambda)} \times t}} < \beta] = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \exp(-\frac{x^2}{2}) dx \quad \text{for } \alpha < \beta.$$

Let $\{Q_t(\xi, d\eta)\}$ the transition probability of the equilibrium process defined in Proposition 1.

In general, $\{Q_t(\xi, d\eta)\}$ has many invariant measures besides λ -Poisson point processes. In this paper we treated only the equilibrium processes with λ -Poisson point processes Π_λ as its absolute law. But this is reasonable because of the following proposition.

Proposition 8. Suppose that

$$\lim_{t \rightarrow \infty} \sup_{x \in X} P_t(x, K) = 0 \quad \text{for any compact set } K \subset X.$$

Let $\Pi(d\xi)$ an invariant probability measure with respect to $\{Q_t(\xi, d\eta)\}$. If the stationary process generated by $\{Q_t(\xi, d\eta)\}, \Pi(d\xi)$ is metrically transitive, $\Pi = \Pi_\lambda$ for some P_t -invariant measure λ .

References

- [1] D.S. ORNSTEIN, Factors of Bernoulli shifts are Bernoulli shifts, Adv. in Math. 5, p.349-364, 1970.
- [2] Y.G. SINAI-K.L. VOLKOVISKII, Ergodic properties of the ideal gas with infinitely many degrees of freedom, Funl. Anal. and its Appl., 5, No.3, p.19-21, 1971.
- [3] H. TOTOKI, A class of special flow, Z. Wahr. und verw. Geb. 15, p.157-167, 1970.