

ON THE CONNES-CONSANI-SOULE TYPE ZETA FUNCTION FOR $\mathbb{F}_1$ -SCHEMES

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PART I

§1, Background....

Manin (Denninger, Kurokawa, Kapranov-Smironov...) suggested $\exists$ a curve $C = \overline{\text{Spec } \mathbb{Z}}$ “defined over” $\mathbb{F}_1$ whose “zeta function” $\zeta_C(s)$ is the complete Riemann zeta function $\zeta_{\mathbb{Q}}(s)$:

$$\begin{aligned}\zeta_C(s) &= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) =: \zeta_{\mathbb{Q}}(s) \\ \zeta(s) &:= \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p: \text{ primes}} \frac{1}{1-p^{-s}} \quad (\Re(s) > 1) \\ \Gamma(s) &:= \int_0^{\infty} x^{s-1} e^{-x} dx \quad (\Re(s) > 0)\end{aligned}$$

Furthermore, they suggested the Riemann hypothesis may be solved in a fashion similar to the Weil conjecture for smooth schemes defined over a finite field $\mathbb{F}_q$ ($q \rightarrow 1$).

Kato, Kurokawa-Ochiai-Wakayama, Deitmar, Toen-Vaquie, Haran, Durov, Soulé, Connes-Conani... proposed some similar notions of $\mathbb{F}_1$ -schemes.

(commutative rings $\rightarrow$ commutative monoid with 0)

Deitmar-Kurokawa-Koyama, Kurokawa-Ochiai, Soule, Connes-Consani proposed different kinds of zeta functions of $\mathbb{F}_1$ -schemes.

$$\begin{aligned}
|X(\mathbb{F}_{q^n})| &\rightarrow |Y(\mathbb{Z}/n)| \\
|(\operatorname{Spec} R)(\mathbb{F}_{q^n})| &= |\operatorname{Hom}_{\mathbf{rings}}(R, \mathbb{F}_{q^n})| \\
|(\operatorname{Spec} A)(\mathbb{Z}/n)| &= |\operatorname{Hom}_{\mathbf{groups}}(A, \mathbb{Z}/n)|
\end{aligned}$$

§2, The plan of the paper

PART I: (Soule, Connes-Consani) Rough idea of the $\mathbb{F}_1$ -schem and the zeta function for some class of $\mathbb{F}_1$ -scheme.

PART II: (Connes-Consani) A similarity between “the counting functions” of the “hypothetical $C = \overline{\operatorname{Spec} \mathbb{Z}}$ ”, and an irreducible smooth projective algebraic curve defined over a finite field.

PART III: (Connes-Consani, Deitmar-Kurokawa-Koyama, M) $\mathbb{F}_1$ -zeta functions of Deitmar-Kurokawa-Koyama and Kurokawa-Ochiai, some invariants for finite abelian groups, and an expression of the Soule-Connes-Consani zeta function for general, not necessarily torsion free, Noetherian $\mathbb{F}_1$ -schemes.

§3, A rough idea of the $\mathbb{F}_1$ -scheme

There is a very general theory of $\mathbb{F}_1$ -scheme, e.g.

[CC] Alain Connes and Caterina Consani,
“Schemes over F_1 and zeta functions”, ArXiv0903.2024

which employs the functor-of-points philosophy for the category $\mathcal{R}ing \cup_{\text{adjoint}} \mathcal{M}onoid_0$.

$$\begin{aligned}
\mathcal{M}onoid_0 &\rightleftarrows \mathcal{R}ing \\
\operatorname{Hom}_{\mathcal{R}ing}(\mathbb{Z}[M], R) &\cong \operatorname{Hom}_{\mathcal{M}onoid_0}(M, R) \\
M &\mapsto \mathbb{Z}[M] \quad (0_M \mapsto 0_{\mathbb{Z}[M]}) \\
R &\leftarrow R
\end{aligned}$$

$$\text{Ob}(\mathcal{R}ing \cup_{\text{adjoint}} \mathcal{M}onoid_0) = \text{Ob}(\mathcal{R}ing) \coprod \text{Ob}(\mathcal{M}onoid_0)$$

$$\text{Hom}_{\mathcal{R}ing \cup_{\text{adjoint}} \mathcal{M}onoid_0}(X, Y) = \begin{cases} \text{Hom}_{\mathcal{R}ing}(X, Y) & \text{if } X, Y \in \mathcal{R}ing \\ \text{Hom}_{\mathcal{M}onoid_0}(X, Y) & \text{if } X, Y \in \mathcal{M}onoid_0 \\ \emptyset & \text{if } X \in \mathcal{R}ing, Y \in \mathcal{M}onoid_0 \\ \text{Hom}_{\mathcal{R}ing}(\mathbb{Z}[X], Y) & \text{if } X \in \mathcal{M}onoid_0, Y \in \mathcal{R}ing \\ \cong \text{Hom}_{\mathcal{M}onoid_0}(X, Y) & \end{cases}$$

A $\mathbb{F}_1$ -functor is by definition a functor

$$\mathcal{R}ing \cup_{\text{adjoint}} \mathcal{M}onoid_0 \rightarrow \mathcal{S}et,$$

which is equivalent to the following data:

- $\underline{X} : \mathcal{M}onoid_0 \rightarrow \mathcal{S}et$
- $X_{\mathbb{Z}} : \mathcal{R}ing \rightarrow \mathcal{S}et$
- $e : \underline{X} \rightarrow X_{\mathbb{Z}} \circ \beta$, where $\beta : \mathcal{M}onoid_0 \rightarrow \mathcal{R}ing$, $M \mapsto \mathbb{Z}[M]$ ($0_M \mapsto 0_{\mathbb{Z}[M]}$)
($\iff e : \underline{X} \circ \beta^* \rightarrow X_{\mathbb{Z}}$, where $\beta^* : \mathcal{R}ing \rightarrow \mathcal{M}onoid_0$, $R \mapsto R$)

Connes-Consani defined a $\mathbb{F}_1$ -scheme $\mathcal{X}$ to be a $\mathbb{F}_1$ -functor $\mathcal{R}ing \cup_{\text{adjoint}} \mathcal{M}onoid_0 \rightarrow \mathcal{S}et$ s.t.

- $X_{\mathbb{Z}}$, its restriction to $\mathcal{R}ing$, is a $\mathbb{Z}$ -scheme.
- $\underline{X}$, its restriction to $\mathcal{M}onoid_0$, is a $\mathcal{M}_0$ -scheme.
- the natural transformation $e : \underline{X} \circ \beta^* \rightarrow X_{\mathbb{Z}}$, associated to a field, is a bijection of sets. In particular,

$$\begin{array}{ccc} X_{\mathbb{Z}}(\mathbb{F}_q) & \xleftarrow[e]{\cong} & (\underline{X} \circ \beta^*)(\mathbb{F}_q) = \underline{X}(\mathbb{F}_q) = \underline{X}(\mathbb{F}_1[\mathbb{Z}/(q-1)]) = \underline{X}(\mathbb{F}_{1(q-1)}) \\ \parallel & & \parallel \\ \text{Hom}_{\mathbb{Z}\text{-sch}}(\text{Spec } \mathbb{F}_q, X_{\mathbb{Z}}) & \xrightarrow{\quad\quad\quad} & \text{Hom}_{\mathcal{M}_0\text{-sch}}(\text{Spec } \mathbb{F}_{1(q-1)}, \underline{X}) \end{array}$$

$$\text{Here, } (\lim_{q \rightarrow 1} \mathbb{F}_{q^n} \simeq) \mathbb{F}_{1^n} := \mathbb{F}_1[\mathbb{Z}/n\mathbb{Z}] := \mathbb{Z}/n\mathbb{Z} \cup \{0\}$$

For Noetherian $\mathbb{F}_1$ -scheme $\mathcal{X}$ (both $X_{\mathbb{Z}}$ and $\underline{X}$ admit a finite open cover by Noetherian affine representables in each category),

- (1) there are just finitely many “points” in $\underline{X}$.
- (2) at each such a point $x \in \underline{X}$, the “residue field” $\kappa(x) = \mathbb{F}_1[\mathcal{O}_x^\times]$ is a finitely generated abelian group $\mathcal{O}_x^\times = \mathbb{Z}^{n(x)} \times \prod_j \mathbb{Z}/m_j(x)\mathbb{Z}$

(3) $\underline{X}(\mathbb{F}_{1^n}) = \text{Hom}_{\mathcal{M}_0\text{-sch}}(\text{Spec } \mathbb{F}_{1^n}, \underline{X}) = \coprod_{x \in \underline{X}} \text{Hom}_{\mathfrak{A}b}(\mathcal{O}_x^\times, \mathbb{Z}/n\mathbb{Z})$,
where

$$(\lim_{q \rightarrow 1} \mathbb{F}_{q^n} \simeq) \mathbb{F}_{1^n} := \mathbb{F}_1[\mathbb{Z}/n\mathbb{Z}] := \mathbb{Z}/n\mathbb{Z} \cup \{0\}$$

In general, a $\mathcal{M}_0$ -scheme $\underline{X}$ is called torsion free, if $\mathcal{O}_x^\times$ is a torsion free abelian group for any $x \in \underline{X}$.

§4, The zeta function for some class of $\mathbb{F}_1$ -scheme by Soule, Connes-Consani

(Deitmar, Connes-Consani) For a Noetherian $\mathbb{F}_1$ -scheme $\mathcal{X}$ with $\underline{X}$ torsion free, $\exists N(u+1) \in \mathbb{Z}_{\geq}[u]$ s.t.

$$|\underline{X}(\mathbb{F}_{1^n})| = N(n+1), \quad \forall n \in \mathbb{N}$$

In particular,

$$|X_{\mathbb{Z}}(\mathbb{F}_q)| = |\underline{X}(\mathbb{F}_{1(q-1)})| = N(q), \quad \forall q, \text{ a prime power}$$

$$\therefore |\underline{X}(\mathbb{F}_{1^n})| = \sum_{x \in \underline{X}} |\text{Hom}_{\mathfrak{A}b}(\mathcal{O}_x^\times, \mathbb{Z}/n\mathbb{Z})| = \sum_{x \in \underline{X}} |\text{Hom}_{\mathfrak{A}b}(\mathbb{Z}^{n(x)}, \mathbb{Z}/n\mathbb{Z})| = \sum_{x \in \underline{X}} n^{n(x)}$$

So, set $N(u+1) := \sum_{x \in \underline{X}} u^{n(x)} \in \mathbb{Z}_{\geq}[u]$. $\square$

So, we are naively lead to define the zeta faunction of $\mathcal{X}$ as the Hasse zeta function of $X_{\mathbb{Z}}$, as our first attempt:

$$\zeta(s, X_{\mathbb{Z}}) := \prod_{p:\text{prime}} \zeta(s, X_{\mathbb{Z}}/\mathbb{F}_p),$$

where $\zeta(s, X_{\mathbb{Z}}/\mathbb{F}_p)$ is the congruence zeta function

$$\zeta(s, X_{\mathbb{Z}}/\mathbb{F}_p) = \exp \left(\sum_{m=1}^{\infty} \frac{|X_{\mathbb{Z}}(\mathbb{F}_{p^m})|}{m} p^{-ms} \right)$$

Bad News. (Soule, Deitmar, Kurokawa) When $N(v) = N(u+1) = \sum_{x \in \underline{X}} u^{n(x)} = \sum_{x \in \underline{X}} v^{n(x)} = \sum_{x \in \underline{X}} (v-1)^{n(x)} = \sum_{k=0}^d a_k v^k$, ($a_k \in \mathbb{Z}$),

$$\zeta(s, X_{\mathbb{Z}}) = \prod_{k=0}^d \zeta(s-k)^{a_k}, \quad \zeta(s, X_{\mathbb{Z}}/\mathbb{F}_p) = \prod_{k=0}^d (1-p^{k-s})^{-a_k}$$

(Too complicated and redundanct for such simple (comparing with $C = \overline{\text{Spec } \mathbb{Z}}$) $\mathcal{X}$!)

Good News. (Soule, predicted by Manin, corected by Kurokawa) For a Noetherian $\mathbb{F}_1$ -scheme $\mathcal{X}$ with $\underline{X}$ torsion free (so, $\exists N(v) = \sum_{k=0}^d a_k v^k \in \mathbb{Z}[v]$ s.t. $|\underline{X}(\mathbb{F}_{1^n})| = N(n+1)$, $\forall n \in \mathbb{N}$),

$$\zeta_{\mathcal{X}}(s) := \lim_{p \rightarrow 1} \zeta(s, X_{\mathbb{Z}}/\mathbb{F}_p) (p-1)^{N(1)} = \prod_{k=0}^d (s-k)^{-a_k}$$

(Kurokawa) In an ideal case, for the l -th Betti number b_l of $X_{\mathbb{Z}}/\mathbb{F}_p$,

$$\begin{aligned} \prod_{k=0}^d (1 - p^k p^{-s})^{-a_k} &= \zeta(s, X_{\mathbb{Z}}/\mathbb{F}_p) \stackrel{Weil \text{ conj.}}{=} \prod_{l=0}^m \left(\prod_{j=1}^{b_l} (1 - \alpha_{l,j} p^{-s}) \right)^{(-1)^{l+1}} \\ &\quad (|\alpha_{l,j}| = p^{l/2}) \\ &= \prod_{l=0}^m (1 - p^{l/2} p^{-s})^{-(-1)^l b_l} \implies b_l = \begin{cases} a_{l/2} & l: \text{ even} \\ 0 & l: \text{ odd} \end{cases} \end{aligned}$$

Thus, $N(v) = \sum_{k=0}^d a_k \in \mathbb{Z}_{\geq 0}[v]$, $N(1) = \sum_{k=0}^d a_k = \sum_{l=0}^m (-1)^l b_l$, the Euler characterisitic of $X_{\mathbb{Z}}/\mathbb{F}_p$.

Example (Toric variety)

fan picture: lattice N : a group $N \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$.

convex cone σ in $N_{\mathbb{R}}$: a convex subset $\sigma \subseteq N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathbb{R}_{\geq 0} \sigma = \sigma$.

A convex cone σ is called:

polyhedral:: if it is finitely generated,

rational:: if the generators lie in the lattice N ,

proper:: if it does not contain a non-zero sub vector space of $N_{\mathbb{R}}$.

fan Δ in N : a finite collection Δ of proper convex rational polyhedral cones σ in the real vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ s.t.

- every face of a cone in Δ is in Δ ,
 - the intersection of two cones in Δ is a face of each.
- (Here zero is considered a face of every cone.)

monoid picture: dual lattice M : $M := \text{Hom}(N, \mathbb{Z})$

dual cone $\check{\sigma}$ in $M_{\mathbb{R}} := \text{Hom}(N, \mathbb{R})$: $\check{\sigma} := \{\alpha \in M_{\mathbb{R}} \mid \alpha(\sigma) \geq 0\}$

monoid A_{σ} : $A_{\sigma} := \check{\sigma} \cap M$; face inclusion $\tau \subseteq \sigma \implies A_{\tau} \supseteq A_{\sigma}$

affine open U_{σ} : $U_{\sigma} := \text{Spec}(\mathbb{C}[A_{\sigma}]) = \text{Spec}(\mathbb{C}[\check{\sigma} \cap M])$

toriv variety X_{Δ} : X_{Δ} is obtained by glueing $U_{\sigma} = \text{Spec}(\mathbb{C}[A_{\sigma}])$ along $U_{\tau} \rightarrow U_{\sigma}$ for each face inclusion $\tau \subseteq \sigma$

This construction allows us to define a $\mathbb{F}_1$ -scheme $\mathcal{X}_\Delta$.

(Deitmar) Given a fan $\Delta \subseteq N \cong \mathbb{Z}^n$, for $j = 0, 1, 2, \dots, n$, let

f_j be the number of cones in Δ of dimension j , and set $c_j := \sum_{k=j}^n f_{n-k} (-1)^{k+j} \binom{k}{j}$. Then,

$$\zeta_{\mathcal{X}_\Delta}(s) = \prod_{j=0}^n (s-j)^{-c_j}$$

$$\begin{aligned} \therefore N(u+1) &= \sum_{x \in \underline{X}_\Delta} u^{n(x)} = \sum_{k=0}^n f_k u^{n-k} = \sum_{k=0}^n f_{n-k} u^k \\ \Rightarrow N(v) &= \sum_{k=0}^n f_{n-k} (v-1)^k = \sum_{k=0}^n f_{n-k} \sum_{j=0}^k \binom{k}{j} x^j (-1)^{k-j} \\ &= \sum_{j=0}^n x^j \sum_{k=j}^n f_{n-k} \binom{k}{j} (-1)^{k-j} \quad \square \end{aligned}$$

Question Can we define $\zeta_{\mathcal{X}}(s)$ for more general $\mathbb{F}_1$ -scheme $\mathcal{X}$?

Good News. Connes-Consani proposed two solutions.

Solution 1: This proceeds as follows:

- Extend “canonically” $N(n+1) := |\underline{X}(\mathbb{F}_{1^n})|$, ($n \in \mathbb{N}$) to

$$N : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}, \quad \text{s.t. } \exists C > 0, \exists k \in \mathbb{N}, \text{ s.t. } |N(u)| \leq Cu^k$$

- As far as zero points and poles concerns, can characterize $\zeta_N(s)$ (which is supposed to be $\zeta_{\mathcal{X}}(s)$) by

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = - \int_1^\infty N(u) u^{-s} d^*u, \quad d^*u = du/u$$

$$\begin{aligned}
\therefore \quad \zeta_N(s) &:= \lim_{q \rightarrow 1} \exp \left(\sum_{r \geq 1} N(q^r) \frac{(q^{-s})^r}{r} \right) (q-1)^\chi, \quad \chi = N(1) \\
\Rightarrow \quad \frac{\partial_s \zeta_N(s)}{\zeta_N(s)} &= \lim_{q \rightarrow 1} \partial_s \left(\sum_{r \geq 1} N(q^r) \frac{(q^{-s})^r}{r} \right) \\
&= \lim_{q \rightarrow 1} \partial_s \left(\sum_{r \geq 1} N(q^r) \frac{(q^{-r})^s}{r} \right) \\
&= \lim_{q \rightarrow 1} \sum_{r \geq 1} N(q^r) \frac{(q^{-r})^s}{r} \log(q^{-r}) \\
&= - \lim_{q \rightarrow 1} \sum_{r \geq 1} N(q^r) (q^r)^{-s} \log q \\
&= - \lim_{q \rightarrow 1} \sum_{r \geq 1} N(q^r) (q^r)^{-s} (\log(q^r) - \log(q^{r-1})) \\
&= - \int_1^\infty N(u) u^{-s} d \log u = - \int_1^\infty N(u) u^{-s} du / u
\end{aligned}$$

Solution 2: Rather than extending to $N : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, consider $\zeta_N^{\text{disc}}(s)$, whose zero points and poles are characterized by

$$\frac{\partial_s \zeta_N^{\text{disc}}(s)}{\zeta_N^{\text{disc}}(s)} = - \sum_{n \geq 1} N(n) n^{-s-1}$$

Good News. (Connes-Consani) For any Noetherian $\mathbb{F}_1$ -scheme $\mathcal{X}$,

$\exists h(z)$, an entire function, s.t.

$$\zeta_N^{\text{disc}}(s) = \zeta_N(s) \exp(h(z))$$

Therefore, $\zeta_N^{\text{disc}}(s)$ and $\zeta_N(s)$ have the same zero points and poles including multiplicities.

$\zeta_N^{\text{disc}}(s)$ may be defined for more general, not necessarily Noetherian, $\mathbb{F}_1$ -schemes...

PART II

§5, Hypothetical computation of $N(n+1) = |(\overline{\text{Spec } \mathbb{Z}})(\mathbb{F}_{1^n})|$ (Connes-Consani)

Using results of Ingham, Connes-Consani observed:

- Regard $w(u) = \sum_{\rho \in \mathbb{Z}} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1}$ as a distribution on $[1, \infty)$.
- Then,

$$(1) \quad N(u) := u - \frac{d}{du} w(u) + 1 = u - \frac{d}{du} \left(\sum_{\rho \in \mathbb{Z}} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1} \right) + 1,$$

where the derivative is in the sense of distributions, enjoys

$$(2) \quad -\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = \int_1^\infty N(u) u^{-s} d^*u$$

- The evaluation $\omega(1) = \lim_{s \rightarrow 1} w(s) = \sum_{\rho \in \mathbb{Z}} \text{order}(\rho) \frac{1}{\rho+1} = \frac{1}{2} + \frac{\gamma}{2} + \frac{\log 4\pi}{2} - \frac{\zeta'(-1)}{\zeta(-1)}$, plays an essential role in establishing (2).

Connes-Consani further pointed out the following analogue:

- X , an irreducible, smooth projective algebraic curve over $\mathbb{F}_p$,
- If X comes from $\mathbb{F}_1$ by “scalar extension” ,

$$N(q) = |X(\mathbb{F}_q)| = q - \sum_{\alpha} \alpha^r + 1, \quad q = p^r,$$

where α 's are the complex roots of the characteristic polynomial of the Frobenius on $H_{\text{et}}^1(X \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$ ($\ell \neq p$)

- Expressing these roots in the form $\alpha = p^\rho$, for $\rho \in Z'$, the set of zeros of the Hasse-Weil zeta function of X ,

$$(3) \quad N(q) = |X(\mathbb{F}_q)| = q - \sum_{\rho \in Z'} \text{order}(\rho) q^\rho + 1.$$

- Now, compare (3) with the formal differantiation of (1):

$$N(u) \sim u - \sum_{\rho \in \mathbb{Z}} \text{order}(\rho) u^\rho + 1$$

PART III

§6, Invariants $\mu_r(A)$ for an abelian group A

For a finite abelian group A , define the r -th μ -invariant $\mu_r(A)$ ($r \in \mathbb{N}$) by

$$\mu_r(A) := \frac{1}{|A|^r} \sum_{k_1, \dots, k_r=1}^{|A|} \left| \text{Hom}_{\mathfrak{Ab}}(A, \mathbb{Z}/(k_1 k_2 \cdots k_r)\mathbb{Z}) \right|.$$

$\mu_r(A)$ is essentially the average of the random variable

$$\begin{aligned} \tilde{X}_r(A) : \tilde{\Omega} := \mathbb{N}^r &\rightarrow \mathbb{N} \\ (k_1, k_2, \dots, k_r) &\mapsto \left| \text{Hom}_{\mathfrak{Ab}}(A, \mathbb{Z}/(k_1 k_2 \cdots k_r)\mathbb{Z}) \right|, \end{aligned}$$

when the infinite set $\tilde{\Omega} = \mathbb{N}^r$ is given the homogeneous measure.

$$\Rightarrow \mu_r(A) = E \left[\tilde{X}_r(A) \right], \quad E \left[\tilde{X}_r(A)^w \right] = \mu_r(A^w) \quad (w \in \mathbb{N})$$

The invariants $\mu_r(A)$ were first considered by Deitmar-Kurokawa-Koyama and Kurokawa-Ochiai, through their study of, what they call, multiplicative Igusa-type zeta functions of $\mathbb{F}_1$ -scheme, which we review by comparing with the Connes-Consani modified zeta function.

- (i) The modified zeta function $\zeta_{\mathcal{X}}^{\text{disc}}(s)$ for a Noetherian $\mathbb{F}_1$ -scheme $\mathcal{X}$, defined and studied by Connes-Consani [CC] is characterized by the following property:

$$\begin{cases} -\frac{\zeta_{\mathcal{X}}^{\text{disc}}(s)'}{\zeta_{\mathcal{X}}^{\text{disc}}(s)} & \equiv \sum_{x \in \underline{X}} \sum_{m \geq 1} \left| \text{Hom}_{\mathfrak{Ab}}(\mathcal{O}_x^*, \mathbb{Z}/m\mathbb{Z}) \right| (m+1)^{-s-1} \pmod{\text{constant } N(1)} \\ \zeta_{\mathcal{X}}(s) & = e^{h(z)} \zeta_{\mathcal{X}}^{\text{disc}}(s) \quad (\zeta_{\mathcal{X}}(s) : \text{Soulé zeta function, } h(z) : \text{entire}) \end{cases}$$

- (ii) The multivariable (r variable) Igusa type zeta function $Z_{\mathcal{X}}^{\text{Igusa}}(s_1, \dots, s_r)$ for a Noetherian $\mathbb{F}_1$ -scheme $\mathcal{X}$ ([DKK] for $r = 1$ and [KO] for general $r \in \mathbb{N}$) is given by

$$Z_{\mathcal{X}}^{\text{Igusa}}(s_1, \dots, s_r) := \sum_{x \in \underline{X}} \sum_{m_1, \dots, m_r \geq 1}^{\infty} \left| \text{Hom}_{\mathfrak{Ab}}(\mathcal{O}_x^*, \mathbb{Z}/m_1 \cdots m_r \mathbb{Z}) \right| m_1^{-s_1} \cdots m_r^{-s_r}$$

- [DKK] Anton Deitmar, Shin-ya Koyama and Nobushige Kurokawa,
“Absolute zeta functions.” Proc. Japan Acad. Ser.
 A Math. Sci. 84 (2008), no. 8, 138–142
- [KO] Nobushige Kurokawa and Hiroyuki Ochiai,
*“A multivariable Euler product of Igusa type and its ap-
 plications,”* Journal of Number Theory, 12 pages,
 Available online 10 March 2009.

Analyzing analytic properties of

$$Z_{\text{Spec } \mathbb{F}_1[A]}^{\text{Igusa}}(s_1, \dots, s_r) = \sum_{m_1, \dots, m_r \geq 1}^{\infty} \left| \text{Hom}_{\mathfrak{Ab}}(A, \mathbb{Z}/m_1 \cdots m_r \mathbb{Z}) \right| m_1^{-s_1} \cdots m_r^{-s_r},$$

some very mysterious looking identity of elementary number theory, which expresses $\mu_r(A)$ in two different ways, was obtained in the following two cases:

- [DKK] $r = 1$ and arbitrary finite abelian group A .
 [KO] Cyclic groups $A = \mathbb{Z}/n\mathbb{Z}$ and arbitrary $r \in \mathbb{N}$.

I reported a purely elementary proof of some slight generalization of these identities at the Vanderbilt conference in May, 2009:

- [M1] Norihiko Minami,
“On the random variable $\mathbb{N}^r \ni (k_1, k_2, \dots, k_r) \mapsto \gcd(n, k_1 k_2 \dots k_r) \in \mathbb{N}$,” arXiv:0907.0916.
- [M2] Norihiko Minami, *“On the random variable $\mathbb{N} \ni l \mapsto \gcd(l, n_1) \gcd(l, n_2) \dots \gcd(l, n_k) \in \mathbb{N}$,”* arXiv:0907.0918.

Theorem of [DKK] type. For a finite abelian group $A = \prod_{j=1}^k (\mathbb{Z}/n_j\mathbb{Z})$,

$$\begin{aligned} \mu_1(A) &= \mu_1 \left(\prod_{j=1}^k (\mathbb{Z}/n_j\mathbb{Z}) \right) \\ &:= \frac{1}{\text{lcm}(n_1, n_2, \dots, n_k)} \sum_{l=1}^{\text{lcm}(n_1, n_2, \dots, n_k)} \gcd(l, n_1) \gcd(l, n_2) \cdots \gcd(l, n_k) \\ &= \prod_{p \mid \text{lcm}(n_1, n_2, \dots, n_k)} \left[p^{\nu_{p,0} + \dots + \nu_{p,k-1}} \right. \\ &\quad \left. + \left(1 - \frac{1}{p} \right) \sum_{j=0}^{k-1} p^{\nu_{p,0} + \dots + \nu_{p,j}} \sum_{l=\nu_{p,j}}^{\nu_{p,j+1}-1} p^{(k-j)\nu_{p,j}-l} \right] \end{aligned}$$

Here, for each prime $p \mid n$,

$$\begin{aligned} \{\nu_{p,1}, \nu_{p,2}, \dots, \nu_{p,k-1}, \nu_{p,k}\} &= \{\text{ord}_p(n_1), \text{ord}_p(n_2), \dots, \text{ord}_p(n_{k-1}), \text{ord}_p(n_k)\} \\ \nu_{p,0} &:= 0 \leq \nu_{p,1} \leq \nu_{p,2} \leq \dots \leq \nu_{p,k-1} \leq \nu_{p,k} \end{aligned}$$

Set ${}_n H_r := {}_{n+r-1} C_r$. Then, we have:

Theorem of [KO] type. For $n, r \in \mathbb{N}$, $w \in \mathbb{C}$,

$$\begin{aligned} &\frac{1}{n^r} \sum_{k_1, \dots, k_r=1}^n \gcd(n, k_1 \cdots k_r)^w \\ &= \begin{cases} \prod_{p \mid n} \left[\left(\frac{1-p^{-1}}{1-p^{w-1}} \right) + p^{\text{ord}_p(n)(w-1)} \sum_{l=0}^{r-1} {}_{\text{ord}_p(n)} H_l \left\{ (1-p^{-1})^l - \left(\frac{1-p^{-1}}{1-p^{w-1}} \right)^r (1-p^{w-1})^l \right\} \right] & (\text{if } w \neq 1) \\ \prod_{p \mid n} \left[\sum_{l=0}^r {}_{\text{ord}_p(n)} H_l \left(1 - \frac{1}{p} \right)^l \right] & (\text{if } w = 1) \end{cases} \\ &= \begin{cases} \prod_{p \mid n} \left[\left(\frac{1-p^{-1}}{1-p^{w-1}} \right) + p^{\text{ord}_p(n)(w-1)} (1-p^{-1})^r \sum_{l=0}^{r-1} {}_{\text{ord}_p(n)} H_l \left\{ (1-p^{-1})^{l-r} - (1-p^{w-1})^{l-r} \right\} \right] & (\text{if } w \neq 1) \\ \prod_{p \mid n} \left[\sum_{l=0}^r {}_{\text{ord}_p(n)} H_l \left(1 - \frac{1}{p} \right)^l \right] & (\text{if } w = 1) \end{cases} \end{aligned}$$

Corollary [KO]. For $n, r \in \mathbb{N}$,

$$\mu_r(\mathbb{Z}/n\mathbb{Z}) = \prod_{p \mid n} \left[\sum_{l=0}^r {}_{\text{ord}_p(n)} H_l \left(1 - \frac{1}{p} \right)^l \right]$$

§7 Motivation of the rest of talk

When we play with $\mu_r(A)$, the following questions seem to be very natural:

- Is there any more conceptual interpretation or description of $\mu_r(A)$?
- Is $\mu_r(A)$, whose origin is the Igusa-type zeta functions for $\mathbb{F}_1$ -schemes of Kurokawa and his collaborators, useful to study $\mathbb{F}_1$ -scheme?
- Is there any relationship between the zeta functions of Soullé, Connes-Consani, and the Igusa-type zeta functions, which was the origin of $\mu_r(A)$?

§8 $\mu_1(A)$ and the zeta functions of Soullé, Connes-Consani.

The logarithmic derivative of the deformed modified zeta function of Soulé type $\zeta_\chi^{\text{disc}}(s; w)$:

$$\frac{\partial_s \zeta_\chi^{\text{disc}}(s; w)}{\zeta_\chi^{\text{disc}}(s; w)} \equiv - \sum_{x \in X} \sum_{m \geq 1} \left| \text{Hom}_{\mathfrak{A}_b}(\mathcal{O}_x^*, \mathbb{Z}/m\mathbb{Z}) \right|^w (m+1)^{-s-1} \pmod{\text{constant}}$$

is a meromorphic function of s with all of its poles simple.

This gives us the following expression of the deformed modified zeta function of Soulé type:

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$$\zeta_{\mathcal{X}}^{\text{disc}}(s; w) = e^{h(s; w)} \prod_{x \in \underline{X}} \left(\left(\prod_{j=0}^{n(x)w} (s - j)^{(-\binom{n(x)w}{j})(-1)^{n(x)w-j}} \right)^{\left(\frac{\sum_{k=1}^{l(x)} |\text{Hom}_{\mathcal{O}_b}(A_x, \mathbb{Z}/k\mathbb{Z})|}{l(x)} \right)^w} \right),$$

where, for each $x \in \underline{X}$, $\mathcal{O}_x^* = \mathbb{Z}^{n(x)} \times \prod_j \mathbb{Z}/m_j(x)\mathbb{Z} =: \mathbb{Z}^{n(x)} \times A_x$, $l(x) = \text{lcm}\{m_j(x)\}$, and $h(s; w)$ is some entire function of s depending upon $w \in \mathbb{N}$.

Restricting to the case $w = 1$ further, we obtain the following:

For a Noetherian F_1 -scheme $\mathcal{X}$, there are some entire functions $h_1(s), h_2(s)$ s.t.

$$\begin{aligned} \zeta_{\mathcal{X}}(s) &= e^{h_1(s)} \zeta_{\mathcal{X}}^{\text{disc}}(s) \\ &= e^{h_2(s)} \prod_{x \in \underline{X}} \left(\left(\prod_{j=0}^{n(x)} (s - j)^{(-\binom{n(x)}{j})(-1)^{n(x)-j}} \right)^{\mu_1(A_x)} \right), \end{aligned}$$

where, for each $x \in \underline{X}$, $\mathcal{O}_x^* = \mathbb{Z}^{n(x)} \times \prod_j \mathbb{Z}/m_j(x)\mathbb{Z} =: \mathbb{Z}^{n(x)} \times A_x$,

Message:

- μ_1 measures “local contribution of ramification”!.
- locally, torsion does not creat any new singularity.

An outline of the proof of the $\zeta_\chi^{\text{disc}}(s; w)$ formula.

$$\begin{aligned}
 & - \sum_{x \in \underline{X}} \sum_{m=1}^{\infty} \left| \text{Hom}_{\mathfrak{A}b}(\mathcal{O}_x^*, \mathbb{Z}/m\mathbb{Z}) \right|^w (m+1)^{-s-1} \\
 (4) \quad & = - \sum_{x \in \underline{X}} \sum_{j=0}^{n(x)w} \binom{n(x)w}{j} (-1)^{n(x)w-j} l(x)^{-(s+1-j)} \\
 & \quad \times \sum_{k=1}^{l(x)} \left| \text{Hom}_{\mathfrak{A}b}(A_x, \mathbb{Z}/k\mathbb{Z}) \right|^w \zeta \left(s+1-j, \frac{k+1}{l(x)} \right),
 \end{aligned}$$

where the Hurwitz zeta function

$$\zeta(s, q) := \sum_{n \geq 0} (n+q)^{-s} \quad (\Re(s) > 1, \Re(q) > 0)$$

only has a pole of residue 1 at $s = 1$. Thus, the singularities of (4) are poles at $s = j \in \cup_{x \in X} \{0, \dots, n(x)\}$ with residue

$$\begin{aligned}
 & - \sum_{x \in \underline{X}} \sum_{j=0}^{n(x)w} \binom{n(x)w}{j} (-1)^{n(x)w-j} l(x)^{-(1)} \sum_{k=1}^{l(x)} \left| \text{Hom}_{\mathfrak{A}b}(A_x, \mathbb{Z}/k\mathbb{Z}) \right|^w \\
 & = \sum_{x \in \underline{X}} \sum_{j=0}^{n(x)w} \left(- \binom{n(x)w}{j} (-1)^{n(x)w-j} \frac{\sum_{k=1}^{l(x)} \left| \text{Hom}_{\mathfrak{A}b}(A_x, \mathbb{Z}/k\mathbb{Z}) \right|^w}{l(x)} \right) \\
 & = \sum_{x \in \underline{X}} \sum_{j=0}^{n(x)w} \left(- \binom{n(x)w}{j} (-1)^{n(x)w-j} \right) \frac{\sum_{k=1}^{l(x)} \left| \text{Hom}_{\mathfrak{A}b}(A_x, \mathbb{Z}/k\mathbb{Z}) \right|^w}{l(x)} \\
 & = \sum_{x \in \underline{X}} \sum_{j=0}^{n(x)w} \left(- \binom{n(x)w}{j} (-1)^{n(x)w-j} \right) \mu_1(A_x^w)
 \end{aligned}$$

Now the claim follows immediately. $\square$

§9. The conceptual interpretation of $\mu_1(A)$.

For any finite abelian group A ,

$$(5) \quad \mu_1(A) := \frac{1}{|A|} \sum_{k=1}^{|A|} \left| \text{Hom}_{\mathfrak{A}b}(A, \mathbb{Z}/k\mathbb{Z}) \right| = \sum_{a \in A} \frac{1}{|a|}$$

If we interpret that $\frac{1}{|a|} = \frac{1}{\infty} = 0$ for an element a of infinite order, we may generalize the definition of $\mu_1(A)$ to finitely generalized abelian groups, as well as to finite (not necessary commutative) groups.

Proof of $\mu_1(A) = \sum_{a \in A} \frac{1}{|a|}$.

$$\begin{aligned}
 & \frac{1}{|A|} \sum_{l=1}^{|A|} \left| \text{Hom}_{\mathfrak{Ab}}(A, \mathbb{Z}/l\mathbb{Z}) \right| = \frac{1}{|A|} \sum_{l=1}^{|A|} \left| \text{Hom}_{\mathfrak{Ab}}(\mathbb{Z}/l\mathbb{Z}, A) \right| \\
 &= \frac{1}{|A|} \sum_{l=1}^{|A|} \sum_{\text{cyclic } C \subset A} \left| \text{Epi}_{\mathfrak{Ab}}(\mathbb{Z}/l\mathbb{Z}, C) \right| = \frac{1}{|A|} \sum_{\text{cyclic } C \subset A} \sum_{l=1}^{|A|} \left| \text{Epi}_{\mathfrak{Ab}}(\mathbb{Z}/l\mathbb{Z}, C) \right| \\
 &= \frac{1}{|A|} \sum_{\text{cyclic } C \subset A} \sum_{l=1}^{|A|} \left| \text{Mono}_{\mathfrak{Ab}}(C, \mathbb{Z}/l\mathbb{Z}) \right| = \frac{1}{|A|} \sum_{\text{cyclic } C \subset A} \frac{|A|}{|C|} \phi(|C|) \\
 &= \sum_{\text{cyclic } C \subset A} \frac{\phi(|C|)}{|C|} = \sum_{h \in \text{Hom}(\mathbb{Z}, A)} \frac{1}{|h(1)|} = \sum_{a \in A} \frac{1}{|a|}
 \end{aligned}$$

□

§10, $\mu_r(A)$ for general $r \in \mathbb{N}$.

For any abelian group A and $r \in \mathbb{N}$, we have

$$\begin{aligned}
 \mu_r(A) &= \sum_{a \in A} \frac{KO_{r-1}(|a|)}{|a|} = \sum_{a \in A} \frac{\prod_{p|a} \left[\sum_{l=0}^{r-1} \frac{H_l \left(1 - \frac{1}{p}\right)^l}{\text{ord}_p(|a|)} \right]}{|a|} \\
 &= \prod_{p|A} \sum_{a \in A_p} \frac{KO_{r-1}(|a|)}{|a|} = \prod_{p|A} \left(\sum_{a \in A_p} \frac{\sum_{l=0}^{r-1} \frac{H_l \left(1 - \frac{1}{p}\right)^l}{\text{ord}_p(|a|)}}{|a|} \right)
 \end{aligned}$$

where KO stands for Kurokawa-Ochiai [KO]:

$$KO_r(n) := \begin{cases} 1 & (r = 0) \\ \mu_r(\mathbb{Z}/n\mathbb{Z}) = \prod_{p|n} \left[\sum_{l=0}^r \frac{H_l \left(1 - \frac{1}{p}\right)^l}{\text{ord}_p(n)} \right] & (r \geq 1) \end{cases}$$

§11, Connes-Consani modified Soulé type zeta function, again

To recap, let us combine the two theorem:

For a Noetherian F_1 -scheme $\mathcal{X}$, there are some entire functions $h_1(s), h_2(s)$ s.t.

$$\begin{aligned} \zeta_{\mathcal{X}}(s) &= e^{h_1(s)} \zeta_{\mathcal{X}}^{\text{disc}}(s) \\ &= e^{h_2(s)} \prod_{x \in \underline{X}} \left(\left(\prod_{j=0}^{n(x)} (s-j)^{(-\binom{n(x)}{j})(-1)^{n(x)-j}}) \right)^{\sum_{a \in A_x} \frac{1}{|a|}} \right) \end{aligned}$$

where, for each $x \in \underline{X}$, $\mathcal{O}_x^* = \mathbb{Z}^{n(x)} \times \prod_j \mathbb{Z}/m_j(x)\mathbb{Z} =: \mathbb{Z}^{n(x)} \times A_x$,

Once again, the above result is in the following:

[M3] Norihiko Minami,

“Meromorphicity of some deformed multivariable zeta functions for F_1 -schemes,” arXiv:0910.3879

I would like to end this paper with the following question to transformation group theorists:

Is there any application of the invariants $\mu_{\tau}(A)$ to the transformation group theory?