

Microlocalization of the Topological Boundary Value Morphism for Regular-Specializable Systems

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Introduction

In microlocal analysis, it is one of the main subjects to give an appropriate formulation of the boundary value problems for hyperfunction or microfunction solutions to a system of linear partial differential equations with analytic coefficients (that is, a coherent (left) $\mathcal{D}$ -Module, here in this article, we shall write *Module* with a capital letter, instead of *sheaf of modules*). If the system is *regular-specializable*, we can define the *nearby-cycle* of the system in the theory of $\mathcal{D}$ -Modules. The definitions of regular-specializable $\mathcal{D}$ -Module and its nearby-cycle are initiated by Kashiwara [Kas], Kashiwara and Kawai [K-K 1] and Malgrange [Mal] for regular-holonomic cases. These definitions extended to the *specializable* $\mathcal{D}$ -Module (see Laurent [L], Laurent and Malgrange [L-Ma] and Mebkhout [Me]). After the results by Kashiwara and Oshima [K-O], Oshima [Os] and Schapira [Sc 3], [Sc 4], for any hyperfunction solutions to regular-specializable system Monteiro Fernandes [MF 1] defined a boundary value morphism (called the *topological boundary value morphism*) which takes values in hyperfunction solutions to the nearby-cycle of the system instead of the *induced* system. This morphism is injective (cf. [MF 2]) and a generalization of the non-characteristic boundary value morphism (for the non-characteristic case, see Komatsu and Kawai [Ko-K], Schapira [Sc 1] and further Kataoka [Kat]). Moreover recently Laurent and Monteiro Fernandes [L-MF 2] reformulated this boundary value morphism and discussed the solvability under a kind of hyperbolicity condition (the *near-hyperbolicity*). However, since this morphism is defined only for hyperfunction solutions, a microlocal boundary value problem is not considered. Therefore in this article, we shall state a microlocalization of their result in the framework of Oaku [Oa 2] and Oaku-Yamazaki [O-Y].

The details of this article will be given in our forthcoming paper [Y].

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1 Notation

In this section, we shall fix the notation used in later sections.

We denote the set of integers, of real numbers and of complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively as usual. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z}; n \geq 1\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In this article, all the manifolds are assumed to be paracompact. Let M be an $(n+1)$ -dimensional real analytic manifold and N a one-codimensional closed real analytic submanifold of M . Let X and Y be complexifications of M and N respectively such that Y is a closed submanifold of X and that $Y \cap M = N$. Moreover in this paper, we assume the existence of a partial complexification of M in X ; that is, there exists a $(2n+1)$ -dimensional real analytic submanifold L of X containing both M and Y such that the triplet (N, M, L) is locally isomorphic to $(\mathbb{R}^n \times \{0\}, \mathbb{R}^{n+1}, \mathbb{C}^n \times \mathbb{R})$ by a local coordinate system $(z, \tau) = (x + \sqrt{-1}y, t + \sqrt{-1}s)$ of X around each point of N . We say such a coordinate system *admissible*. We shall mainly follow the notation in Kashiwara-Schapira [K-S 2]; we denote the normal deformations of N and Y in M and L by $\widetilde{M}_N$ and $\widetilde{L}_Y$ respectively and regard $\widetilde{M}_N$ as a closed submanifold of $\widetilde{L}_Y$. We have the following commutative diagram:

$$\begin{array}{ccccccc}
 T_N M & \xhookrightarrow{s_M} & \widetilde{M}_N & \xleftarrow{j_M} & \Omega_M & & \\
 \downarrow i' & \searrow \tau_N & \downarrow p_M & \swarrow \tilde{p}_M & \downarrow i_M & & \\
 & & N & \xhookrightarrow{\quad} & M & \xhookrightarrow{\quad} & X \\
 & & \downarrow i|_N & \downarrow \tilde{i}' & \downarrow i & \downarrow \tilde{i} & \\
 T_Y L & \xhookrightarrow{s_L} & \widetilde{L}_Y & \xleftarrow{j_L} & \Omega_L & & \\
 \downarrow \tau_Y & \searrow & \downarrow p_L & \swarrow \tilde{p}_L & \downarrow i_L & & \\
 & & Y & \xhookrightarrow{i_Y} & L & \xhookrightarrow{i_L} & X
 \end{array}$$

and by admissible coordinates we have locally the following relation:

$$\begin{array}{ccccc}
 N = \mathbb{R}_x^n \times \{0\} & \hookrightarrow & M = \mathbb{R}_x^n \times \mathbb{R}_t & & \\
 \downarrow & & \downarrow i & \searrow i_M & \\
 Y = \mathbb{C}_z^n \times \{0\} & \xhookrightarrow{i_Y} & L = \mathbb{C}_z^n \times \mathbb{R}_t & \xhookrightarrow{i_L} & X = \mathbb{C}_z^n \times \mathbb{C}_\tau
 \end{array}$$

With these coordinates, we often identify $T_Y X$ and $T_Y L$ with X and L respectively.

The projection $\tau_Y: T_Y L \rightarrow Y$ induces natural mappings:

$$T_N^* Y \xleftarrow{\tau_{Y\pi}} T_N M \times_N T_N^* Y \xrightarrow{\tau_Y'} T_{T_N M}^* T_Y L,$$

and by τ_Y' we identify $T_{T_N M}^* T_Y L$ with $T_N M \times_N T_N^* Y$. Similarly by natural mappings

$$T_{\widetilde{M}_N}^* \widetilde{L}_Y \xleftarrow{s_{L\pi}} T_N M \times_{\widetilde{M}_N} T_{\widetilde{M}_N}^* \widetilde{L}_Y \xrightarrow{s_L'} T_{T_N M}^* T_Y L,$$

we identify $T_N M \times_{\widetilde{M}_N} T_{\widetilde{M}_N}^* \widetilde{L}_Y$ with $T_{T_N M}^* T_Y L$.

$T_Y L \setminus T_Y Y$ has two components with respect to its fiber. We denote one of them by $T_Y L^+$ and represent (at least locally) by fixing an admissible coordinate system

$$T_Y L^+ = \{(z, t) \in T_Y L; t > 0\}.$$

Moreover set $T_N M^+ := T_Y L^+ \cap T_N M$. Note that to define $T_Y L^+$ (or $T_N M^+$) by means of admissible coordinates is equivalent to determining a local isomorphism $or_{Y/L} \simeq \mathbb{Z}_Y$ (or equivalently $or_{N/M} \simeq \mathbb{Z}_N$). Here $or_{Y/L}$ denotes the relative orientation sheaf.

Define open embeddings f and f_N by:

$$\begin{array}{ccc} T_Y L^+ & \xhookrightarrow{f} & T_Y L \\ \cup & \circlearrowleft & \cup \\ T_N M^+ & \xhookrightarrow{f_N} & T_N M. \end{array}$$

Thus we regard $T_N M^+ \times_N T_N^* Y$ as an open set of $T_{T_N M}^* T_Y L$. Moreover f induces mappings:

$$\begin{array}{ccc} T_{T_N M^+}^* T_Y L^+ & \xleftarrow{f'} T_N M^+ \times_{T_N M} T_{T_N M}^* T_Y L & \xrightarrow{f_\pi} T_{T_N M}^* T_Y L \\ & \downarrow \wr & \downarrow \wr \\ & T_N M^+ \times_N T_N^* Y & \xrightarrow{f_N \times \text{id}} T_N M \times_N T_N^* Y. \end{array}$$

Hence we identify $T_{T_N M^+}^* T_Y L^+$ with $T_N M^+ \times_N T_N^* Y$, and f_π with $f_N \times \text{id}$.

2 Several Sheaves Attached to the Boundary

In this section, we recall several sheaves attached to the boundary due to Oaku [Oa 2]. These sheaves will play essential roles for our boundary value problem. We remark that in Oaku [Oa 2] these sheaves are defined on cosphere bundles. So we shall present equivalent but slightly different definitions on cotangent bundles along the line of Oaku-Yamazaki [O-Y]. We refer to Oaku [Oa 2] or Oaku-Yamazaki [O-Y] for the proofs. Note that although the higher-codimensional case is treated in Oaku-Yamazaki [O-Y], the same proofs also work as in the one-codimensional case.

As usual, we denote by $\mathcal{O}_X$, $\mathcal{B}_M$ and $\mathcal{C}_M$ the sheaf of *holomorphic functions* on X , of *hyperfunctions* on M and of *microfunctions* on $T_M^* X$ respectively. Further, we denote by $\mathcal{B}\mathcal{O}_L$ the sheaf of *hyperfunctions with holomorphic parameters* on L ; that is,

$$\mathcal{B}\mathcal{O}_L := \mathcal{H}_L^1(\mathcal{O}_X) \otimes or_{L/X} \simeq i_L^! \mathcal{O}_X \otimes or_{L/X}[1].$$

We denote as usual by ν and μ the Sato specialization and microlocalization functors respectively.

2.1 Definition. We set:

$$\begin{aligned}\mathcal{C}_{N|M} &:= s_{L\pi}^{-1} \mathcal{H}^n(\mu_{\widetilde{M}_N}(j_{L*} \widetilde{p}_L^{-1} \mathcal{B}\mathcal{O}_L)) \otimes or_{M/L}, \\ \mathcal{B}_{N|M} &:= \mathcal{C}_{N|M}|_{T_N M}.\end{aligned}$$

We denote by $\pi_{N|M}$ the natural projection from $T_{T_N M}^* T_Y L$ to $T_N M$. Let $\dot{\pi}_{N|M}$ be the restriction of $\pi_{N|M}$ to $T_{T_N M}^* T_Y L \setminus T_{T_N M}^* T_N M$ as usual. By virtue of the following proposition, we can regard $\mathcal{C}_{N|M}$ as a microlocalization of $\nu_N(\mathcal{B}_M)$:

2.2 Proposition. *There exists the following exact sequence on $T_N M$:*

$$0 \longrightarrow \nu_Y(\mathcal{B}\mathcal{O}_L)|_{T_N M} \longrightarrow \mathcal{B}_{N|M} \longrightarrow \dot{\pi}_{N|M*} \mathcal{C}_{N|M} \longrightarrow 0.$$

Moreover, an isomorphism $\nu_N(\mathcal{B}_M) \simeq \mathcal{B}_{N|M}$ holds.

2.3 Definition. We set:

$$\begin{aligned}\widetilde{\mathcal{C}}_{N|M} &:= \mathcal{H}^n(\mu_{T_N M}(\nu_Y(\mathcal{B}\mathcal{O}_L))) \otimes or_{N/Y}, \\ \widetilde{\mathcal{B}}_{N|M} &:= \widetilde{\mathcal{C}}_{N|M}|_{T_N M} \simeq \mathcal{H}_{T_N M}^n(\nu_Y(\mathcal{B}\mathcal{O}_L)) \otimes or_{N/Y}.\end{aligned}$$

By the following fact, we can regard $\mathcal{C}_{N|M}$ as a subsheaf of $\widetilde{\mathcal{C}}_{N|M}$:

2.4 Proposition. *There exists a natural monomorphism $\mathcal{C}_{N|M} \longrightarrow \widetilde{\mathcal{C}}_{N|M}$.*

3 Regular-Specializable Systems

In this section, we shall recall the basic results concerning the regular-specializable $\mathcal{D}$ -Module and its nearby-cycle.

As usual, we denote by $\mathcal{D}_X$ the sheaf on X of holomorphic differential operators, and by $\{\mathcal{D}_X^{(m)}\}_{m \in \mathbb{N}_0}$ the usual order filtration on $\mathcal{D}_X$. First, let us recall the definition of the V -filtration:

3.1 Definition. Denote by $\mathcal{I}_Y$ the defining Ideal of Y in $\mathcal{O}_X$ with a convention that $\mathcal{I}_Y^j = \mathcal{O}_X$ for $j \leq 0$. The V -filtration $\{V_Y^k(\mathcal{D}_X)\}_{k \in \mathbb{Z}}$ (along Y) is a filtration on $\mathcal{D}_X|_Y$ defined by

$$V_Y^k(\mathcal{D}_X) := \bigcap_{j \in \mathbb{Z}} \{P \in \mathcal{D}_X|_Y; P \mathcal{I}_Y^j \subset \mathcal{I}_Y^{j-k}\}.$$

It is easy to see that by admissible coordinates, this filtration written as

$$V_Y^k(\mathcal{D}_X) = \left\{ \sum_{j-i \leq k} P_{ij}(z; \partial_z) \tau^i \partial_\tau^j \in \mathcal{D}_X|_Y \right\}.$$

For the fundamental properties of this filtration, we refer to Björk [Bj], Sabbah [Sab] and Schapira [Sc 2]).

Let us denote by ϑ the Euler operator. Note that $\vartheta \in V_Y^0(\mathcal{D}_X) \setminus V_Y^{-1}(\mathcal{D}_X)$ and that ϑ can be represented by $\tau \partial_\tau$ by admissible coordinates.

3.2 Definition. A coherent $\mathcal{D}_X$ -Module $\mathcal{M}$ defined on a neighborhood of Y is said to be *regular-specializable (along Y)* if there exist locally a coherent $\mathcal{O}_X$ -sub-Module $\mathcal{M}_0$ of $\mathcal{M}$ and a non-zero polynomial $b(\alpha) \in \mathbb{C}[\alpha]$ such that the following conditions are satisfied:

- (1) $\mathcal{M}_0$ generates $\mathcal{M}$ over $\mathcal{D}_X$; that is, $\mathcal{M} = \mathcal{D}_X \mathcal{M}_0$;
- (2) $b(\vartheta) \mathcal{M}_0 \subset (\mathcal{D}_X^{(m)} \cap V_Y^{-1}(\mathcal{D}_X)) \mathcal{M}_0$, where m is the degree of $b(\alpha)$.

In what follows, we shall omit the phrase “along Y ” since Y is fixed.

3.3 Remark. (1) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ -Module for which Y is non-characteristic. Then, it is easy to see that $\mathcal{M}$ is regular-specializable.

(2) Kashiwara-Kawai [K-K 1] proved that every regular-holonomic $\mathcal{D}_X|_Y$ -Module is regular-specializable.

3.4 Proposition. *If $\mathcal{M}$ is a regular-specializable $\mathcal{D}_X$ -Module, then each cohomology of $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_Y(\mathcal{O}_X))$ and $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_Y(\mathcal{O}_X))$ is a locally $\mathbb{C}^\times$ -conic sheaf.*

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X|_Y$ -Module. Recall that a V -filtration $\{F^k \mathcal{M}\}_{k \in \mathbb{Z}}$ is said to be *good* if there exist locally a system of generators $\{u_j\}_{j=1}^m$ and $k_j \in \mathbb{Z}$ such that for any $k \in \mathbb{Z}$

$$F^k \mathcal{M} = \sum_{j=1}^m V_Y^{k-k_j}(\mathcal{D}_X) u_j$$

holds. The following theorem is proved by Kashiwara [Kas] (cf. also Björk [Bj]):

3.5 Theorem. *Set $G := \{\alpha \in \mathbb{C}; 0 \leq \operatorname{Re} \alpha < 1\}$. Then, for any regular-specializable $\mathcal{D}_X$ -Module $\mathcal{M}$, there exist a unique good V -filtration $\{V_G^k(\mathcal{M})\}_{k \in \mathbb{Z}}$ on $\mathcal{M}$ and a non-zero polynomial $b_G(\alpha) \in \mathbb{C}[\alpha]$ such that $b_G^{-1}(0) \subset G$ and that for any $k \in \mathbb{Z}$ the following holds:*

$$b_G(\vartheta + k) V_G^k(\mathcal{M}) \subset V_G^{k-1}(\mathcal{M}).$$

3.6 Definition. Under the notation of Theorem 3.5, we set:

$$\begin{aligned} \Psi_Y(\mathcal{M}) &:= V_G^0(\mathcal{M})/V_G^{-1}(\mathcal{M}), \\ \Phi_Y(\mathcal{M}) &:= V_G^1(\mathcal{M})/V_G^0(\mathcal{M}), \end{aligned}$$

and call $\Psi_Y(\mathcal{M})$ the *nearby-cycle* of $\mathcal{M}$ and $\Phi_Y(\mathcal{M})$ the *vanishing-cycle* of $\mathcal{M}$ respectively.

3.7 Remark. Laurent [L] extended the definitions of nearby and vanishing cycles to the derived category of bounded complexes with (regular) specializable cohomology by using the theory of second microlocalization.

Let $\iota: Y \longrightarrow X$ be the natural inclusion. Then the *induced system*, or the *inverse image* in the sense of $\mathcal{D}$ -Modules is defined by

$$D\iota^* \mathcal{M} := \mathcal{O}_Y \otimes_{\iota^{-1}\mathcal{O}_X}^L \iota^{-1} \mathcal{M}.$$

Then we have (cf. Laurent [L], Mebkhout [Me] or Sabbah [Sab]):

3.8 Proposition. *If $\mathcal{D}_X$ -Module $\mathcal{M}$ is regular-specializable, then $\Psi_Y(\mathcal{M})$, $\Phi_Y(\mathcal{M})$ and each cohomology of $D\iota^* \mathcal{M}$ are coherent $\mathcal{D}_Y$ -Modules. Moreover, there exists the following distinguished triangle:*

$$\Phi_Y(\mathcal{M}) \xrightarrow{\text{var}} \Psi_Y(\mathcal{M}) \longrightarrow D\iota^* \mathcal{M} \xrightarrow{+1}.$$

As usual, we denote by $\mathcal{C}_{Y|X}^{\mathbb{R}} := \mu_Y(\mathcal{O}_X)[1]$ the sheaf of *real holomorphic microfunctions* on T_Y^*X . Set $\dot{T}_Y X := T_Y X \setminus T_Y Y$ as usual (the definition of $\dot{T}_Y X$ is similar). Using an admissible coordinate system we define a continuous section $\sigma: Y \longrightarrow \dot{T}_Y X$ by $z \longmapsto (z, 1)$. Similarly we define ${}^t\sigma: Y \longrightarrow \dot{T}_Y^* X$ by $z \longmapsto (z, 1)$. Denote by $\mathcal{N}_{X|Y}$ the sheaf of Nilsson class functions on X along Y and regard as a sheaf on Y . Then the following theorem is proved by Laurent [L] (cf. also Kashiwara-Kawai [K-K 2]):

3.9 Theorem. *Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X$ -Module. Then, there exists the following isomorphism of distinguished triangles:*

$$\begin{array}{ccccc} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \sigma^{-1} \nu_Y(\mathcal{O}_X)) & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, {}^t\sigma^{-1} \mathcal{C}_{Y|X}^{\mathbb{R}}) \xrightarrow{+1} \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ R\mathcal{H}om_{\mathcal{D}_Y}(D\iota^* \mathcal{M}, \mathcal{O}_Y) & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{O}_Y) & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{O}_Y) \xrightarrow{+1}. \end{array}$$

Moreover, a natural morphism $\mathcal{N}_{X|Y} \longrightarrow \sigma^{-1} \nu_Y(\mathcal{O}_X)$ induces an isomorphism:

$$R\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{N}_{X|Y}) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \sigma^{-1} \nu_Y(\mathcal{O}_X)).$$

3.10 Remark. (1) The isomorphism (the Cauchy-Kovalevskaja type theorem)

$$R\mathcal{H}om_{\mathcal{D}_Y}(D\iota^* \mathcal{M}, \mathcal{O}_Y) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y$$

holds for Fuchsian systems in the sense of Laurent-Monteiro Fernandes [L-MF 1].

(2) Recently Mandai [Man] extended the definition of boundary values to a general Fuchsian differential equation in the complex domain.

4 Boundary Value Morphism

In this section, we shall define our injective boundary value morphism. Recall the mappings f_π and $\tau_{Y\pi}$ defined in Section 1.

4.1 Theorem. *For any regular-specializable $\mathcal{D}_X$ -Module $\mathcal{M}$, there exists the following isomorphism:*

$$f_\pi^{-1} \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M}) \xrightarrow{\sim} f_\pi^{-1} \tau_{Y\pi}^{-1} \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N).$$

The proof is based on Proposition 3.4 and Theorem 3.9 .

4.2 Definition. For any regular-specializable $\mathcal{D}_X$ -Module $\mathcal{M}$, we define by virtue of Proposition 2.4 and Theorem 4.1:

$$\begin{aligned} \beta: f_\pi^{-1} \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) &\longrightarrow f_\pi^{-1} \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M}) \\ &\xrightarrow{\sim} f_\pi^{-1} \tau_{Y\pi}^{-1} \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N). \end{aligned}$$

By the construction, we can obtain the following Holmgren type theorem:

4.3 Theorem. (1) *The morphism β gives a monomorphism*

$$\beta^0: f_\pi^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \hookrightarrow f_\pi^{-1} \tau_{Y\pi}^{-1} \mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N).$$

(2) *The restriction of β^0 to the zero-section $T_N M^+$ coincides with the topological boundary value morphism in the sense of Monteiro Fernandes [MF 1].*

4.4 Remark. (1) For a general Fuchsian system in the sense of Tahara [T], Oaku [Oa 2] defined an injective boundary value morphism under additional conditions of characteristic exponents by using a detailed study due to Tahara [T].

(2) Let $\mathcal{C}_{N|M}^F \subset \mathcal{C}_{N|M}$ be the subsheaf consisting of F -mild microfunctions, and $\tilde{\mathcal{C}}_{N|M}^A := \mu_N(\mathcal{O}_X|_Y) \otimes \mathcal{O}_{N/Y}[n]$ (see Oaku [Oa 1], [Oa 2], and Oaku-Yamazaki [O-Y]). Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X$ -Module and set $\mathcal{M}_Y := \mathcal{H}^0(\mathbf{D}\iota^* \mathcal{M}) = \mathcal{O}_Y \otimes_{\iota^{-1}\mathcal{O}_X} \mathcal{M}$. Since $\mathcal{M}$ is a Fuchsian system, by the argument in Oaku-Yamazaki [O-Y] we have the following commutative diagram:

$$\begin{array}{ccccccc} f_\pi^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}^F) & \hookrightarrow & f_\pi^{-1} \tau_{Y\pi}^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M}^A) & \xrightarrow{\sim} & f_\pi^{-1} \tau_{Y\pi}^{-1} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N) \\ \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\ f_\pi^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) & \hookrightarrow & f_\pi^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M}) & \xrightarrow{\sim} & f_\pi^{-1} \tau_{Y\pi}^{-1} \mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N), \end{array}$$

that is, the boundary value morphism

$$\gamma^F: f_\pi^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}^F) \hookrightarrow f_\pi^{-1} \tau_{Y\pi}^{-1} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N)$$

for F -mild microfunctions and β^0 are compatible.

5 Solvability

In this section, we shall state the solvability theorem under a kind of hyperbolicity condition. First, let us recall the following (Laurent-Monteiro Fernandes [L-MF 2]):

5.1 Definition. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ -Module on a neighborhood of Y . Then we say $\mathcal{M}$ is *near-hyperbolic* at $x_0 \in N$ (in dt -codirection) if there exist positive constants C and ε_1 such that

$$\begin{aligned} \text{char}(\mathcal{M}) \cap \{ (z, \tau; z^*, \tau^*) \in T^*X; |z - x_0|, |\tau| < \varepsilon_1, \text{Re } \tau > 0 \} \\ \subset \{ (z, \tau; z^*, \tau^*) \in T^*X; |\text{Re } \tau^*| < C(|\text{Im } z^*|(|\text{Im } z| + |\text{Im } \tau|) + |\text{Re } z^*|) \} \end{aligned}$$

holds by an admissible coordinate system.

5.2 Remark. As is shown by Laurent-Monteiro Fernandes [L-MF 2, Lemma 1.3.2], the near-hyperbolicity condition is weaker than the hyperbolicity condition (see also Bony-Schapira [B-S]).

5.3 Theorem. Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X$ -Module. Assume that $\mathcal{M}$ is near-hyperbolic at $x_0 \in N$. Then, for any $p^* = (x_0, t_0; \sqrt{-1} \langle \xi_0, dx \rangle) \in T_{T_N M^+}^* T_Y L^+$

$$\beta: \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M})_{p^*} \longrightarrow \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N)_{\tau_Y \pi(p^*)}$$

is an isomorphism.

5.4 Remark. (1) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ -Module for which Y is non-characteristic. Then, it is known that $\Psi_Y(\mathcal{M}) \xrightarrow{\sim} \mathbf{D}\iota^* \mathcal{M} \simeq \mathcal{M}_Y$. Moreover by virtue of the commutative diagram in Remark 4.4, we see that β^0 is equivalent to the non-characteristic boundary value morphism (see Kataoka [Kat] and Oaku [Oa 2]). In particular, the restriction of β^0 to the zero-section $T_N M^+$ is equivalent to Komatsu-Kawai [Ko-K] and Schapira [Sc 1]. Moreover, if each $\pm dt \in T_N^* M$ is hyperbolic for $\mathcal{M}$, then the nearly-hyperbolic condition is satisfied (cf. Kashiwara-Schapira [K-S 1]).

(2) Assume that $X = \mathbb{C}^{n+1}$ and so on by taking an admissible coordinate system. Let $b(\alpha)$ be a non-zero polynomial with degree m , and $Q \in \mathcal{D}_X^{(m)} \cap V_Y^{-1}(\mathcal{D}_X)$ and set $\mathcal{M} := \mathcal{D}_X / \mathcal{D}_X (b(\vartheta) + Q)$. Then $\mathcal{M}$ is regular-specializable. For simplicity, assume that

$$b(\alpha) = \prod_{j=1}^{\mu} (\alpha - \alpha_j)^{\nu_j} \quad (\alpha_i - \alpha_j \notin \mathbb{Z} \text{ for } 1 \leq i \neq j \leq \mu)$$

(note that $\sum_{j=1}^{\mu} \nu_j = m$). Then a direct calculation shows that $\Psi_Y(\mathcal{M}) \simeq \mathcal{D}_Y^{\oplus m}$, and β^0 is equivalent to γ in Oaku [Oa 2]: Let $p^* = (x_0, t_0; \sqrt{-1} \langle \xi_0, dx \rangle)$ be a point of $T_{T_N M^+}^* T_Y L^+$, and $f(x, t)$ a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M})$ at p^* . Then, since $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}_{X|Y}) \simeq$

$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \sigma^{-1} \nu_Y(\mathcal{O}_X))$ by virtue of Theorem 3.9, we can see that $f(x, t)$ has a defining function

$$F(z, \tau) = \sum_{j=1}^{\mu} \sum_{k=1}^{\nu_j} F_{jk}(z, \tau) \tau^{\alpha_j} (\log \tau)^{k-1}$$

as a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M})$ at p^* . Here each $F_{jk}(z, \tau)$ is holomorphic on a neighborhood of $\{(z, 0) \in X; |x_0 - z| < \varepsilon, \text{Im } z \in \Gamma\}$ with a positive constant ε and an open convex cone Γ such that $\xi_0 \in \text{Int}(\Gamma^\circ)$ (the interior of the dual cone Γ° of Γ). Then, $\beta^0(f)$ is equivalent to $\{\text{sp}_N(F_{jk}(x + \sqrt{-1}\Gamma, 0)); 1 \leq k \leq \nu_j, 1 \leq j \leq \mu\}$. Moreover, if the principal symbol of $b(\vartheta) + Q$ written as $\tau^m P(z, \tau; z^*, \tau^*)$ for a hyperbolic polynomial P at dt -codirection, then the nearly-hyperbolic condition is satisfied. Note that this operator is a special case of Fuchsian hyperbolic operators due to Tahara [T].

5.5 Example. Assume that $X = \mathbb{C}^{n+1}$. Take an operator $A(z; \partial_z) \in \mathcal{D}_Y^{(1)}$ at the origin and set $A^0 := \text{id}$ and $A^{(j)} := \frac{1}{j!} A \circ A^{(j-1)} \in \mathcal{D}_Y^{(j)}$ for $j \geq 1$. Let $p^* = (0, 1; \sqrt{-1} \langle \xi, dx \rangle)$ be a point of $T_{N,M}^* T_Y L^+$ and set $p_0 := (0; \sqrt{-1} \langle \xi, dx \rangle) \in T_N^* Y$. Consider the following differential equations:

$$\begin{aligned} \mathcal{M}_1 &:= \mathcal{D}_X / \mathcal{D}_X (\vartheta(\vartheta - 1) - \tau A(z; \partial_z) \vartheta), \\ \mathcal{M}_2 &:= \mathcal{D}_X / \mathcal{D}_X ((\vartheta - 1)^2 - \tau A(z; \partial_z) \vartheta), \\ \mathcal{M}_3 &:= \mathcal{D}_X / \mathcal{D}_X ((\vartheta - 1)(\vartheta - 2) - \tau A(z; \partial_z) \vartheta). \end{aligned}$$

Let $f_i(x, t)$ be a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_i, \tilde{\mathcal{C}}_{N|M})$ at p^* . Then:

(1) $f_1(x, t)$ has the following defining function as a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M})$ at p^* :

$$F_1(z, \tau) = U_0(z) + \sum_{j=0}^{\infty} \frac{A^{(j)} U_1(z)}{j+1} \tau^{j+1}.$$

In this case, $f_1(x, t)$ is always F -mild. Hence $\beta^0(f_1(x, t))$ is given by $\gamma^F(f_1(x, t)) = \{(\partial_t^l f_1)(x, +0)\}_{l=0,1} = \{\text{sp}_N(U_l)(x)\}_{l=0,1}$ at p_0 . Indeed if $\tau \neq 0$, $\mathcal{M}_1$ is isomorphic to $\mathcal{D}_X / \mathcal{D}_X (\partial_\tau^2 - \partial_\tau A(z; \partial_z))$ for which Y is non-characteristic.

(2) $f_2(x, t)$ has the following defining function as a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M})$ at p^* :

$$F_2(z, \tau) = \sum_{j=0}^{\infty} A^{(j)} U_1(z) \tau^{j+1} - \sum_{j=1}^{\infty} \sum_{k=1}^j \frac{A^{(j)} U_0(z)}{k} \tau^{j+1} + \sum_{j=0}^{\infty} A^{(j)} U_0(z) \tau^{j+1} \log \tau,$$

and $\beta^0(f_1(x, t))$ is given by $\{\text{sp}_N(U_l)(x)\}_{l=0,1}$ at p_0 . Further if $f_1(x, t)$ is F -mild, then $U_0(z) = 0$ and $\gamma^F(f_2(x, t)) = \{(\partial_t^l f_2)(x, +0)\}_{l=0,1} = \{\text{sp}_N(U_1)(x)\}$ at p_0 .

(3) $f_3(x, t)$ has the following defining function as a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M})$ at p^* :

$$F_3(z, \tau) = \sum_{j=0}^{\infty} A^{(j)} U_2(z) \tau^{j+2} + U_1(z) \tau - \sum_{j=2}^{\infty} \sum_{k=1}^{j-1} \frac{j A^{(j)} U_1(z)}{k} \tau^{j+1} \\ + \left(A U_1(z) \tau^2 + \sum_{j=2}^{\infty} j A^{(j)} U_1(z) \tau^{j+1} \right) \log \tau,$$

and $\beta^0(f_3(x, t))$ is given by $\{\mathrm{sp}_N(U_l)(x)\}_{l=1,2}$ at p_0 . In the case where $f_3(x, t)$ is F -mild, we must impose the condition $A U_1(z) = 0$. Under this condition, $\gamma^F(f_3(x, t))$ is given by $\gamma^F(f_3(x, t)) = \{(\partial_t^l f_3)(x, +0)\}_{0 \leq l \leq 2} = \{0, \mathrm{sp}_N(U_1)(x), 2 \mathrm{sp}_N(U_2)(x)\}$ at p_0 with $A(\partial_t f_3)(x, +0) = A \mathrm{sp}_N(U_1)(x) = 0$.

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