

Γ - PERIODIC WAVELETS AND $L^2(\mathbb{R}^n/\Gamma)$

YASUHIRO ASOO(麻生泰弘)

ABSTRACT. In this paper, we introduce Γ -Periodic Wavelets and give a decomposition of $L^2(\mathbb{R}^n/\Gamma)$.

1. NOTATIONS AND SOME PRELIMINARIES

Let $A \in GL(n, \mathbb{R})$ and $A^* = (A^t)^{-1}$.

Define

$$\Gamma = \{\gamma = Ak; k \in \mathbb{Z}^n\}$$

and

$$\Gamma^* = \{\gamma^* = A^*k; k \in \mathbb{Z}^n\}$$

We call Γ *the lattice with basis A* and Γ^* its dual lattice.

The set $\Omega = \Omega_\Gamma = \{x \in \mathbb{R}^n : x = At, t \in \mathbb{T}^n\}$ is called *the fundamental domain*, where $\mathbb{T} = [0, 1]$.

Definition 1. A Multiresolution Analysis with lattice basis(MRALB) of $L^2(\mathbb{R}^n)$ is a family of closed subspaces, $V_j (j \in \mathbb{Z})$ of $L^2(\mathbb{R}^n)$ such that:

(1) $V_j (j \in \mathbb{Z})$ is an increasing sequence such that

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \quad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$$

(2) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$

(3) $f(x) \in L^2(\mathbb{R}^n)$ belongs to V_0 if and only if $f(x - \gamma) \in V_0$ for all $\gamma \in \Gamma$

(4) There exists $g \in V_0$ such that $\{g(x - \gamma); \gamma \in \Gamma\}$ is a Riesz basis of V_0 . Assume that $\{\varphi(x - \gamma) : \gamma \in \Gamma\}$ is an orthonormal basis of

V_0 , then the Fourier transform of the function φ satisfies

$$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi), \quad \xi \in \mathbb{R}^n$$

with $2\pi\Gamma^*$ - periodic function $m_0(\xi)$, a *filtering function*.

For $f, g \in L^2(\mathbb{R}^n)$, define

$$C(f, g)(\xi) = \sum_{\gamma \in \Gamma^*} \hat{f}(\xi + 2\pi\gamma^*) \overline{\hat{g}(\xi + 2\pi\gamma^*)}$$

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, we call it *the correlation function*. For the following Theorem 1 and Theorem 2, see Y.Asoo[1], and [2].

Theorem 1. Assume $\varphi \in L^2(\mathbb{R}^n)$. Then a system

$$\{2^{\frac{nj}{2}}\varphi(2^jx - \gamma) : \gamma \in \Gamma\}$$

is an orthonormal basis of $V_j(j \in \mathbb{Z})$ if and only if

$$C(f, f)(\xi) = |\det(A)|, \quad \text{a.a. } \xi \in \mathbb{R}^n.$$

Now, for a Riesz basis g of V_0 , define the function φ so that

$$\varphi(\xi) = \sqrt{|\det(A)|} \frac{\hat{g}(\xi)}{\sqrt{C(g, g)(\xi)}},$$

then $\{2^{\frac{nj}{2}}\varphi(2^jx - \gamma) : \gamma \in \Gamma\}$ is an orthonormal basis of V_j . Let $E = \{0, 1\}^n$ and $\psi_\epsilon \in V_1(\epsilon \in E)$ be such that

$$(1.1) \quad \hat{\psi}_\epsilon(2\xi) = m_\epsilon(\xi)\hat{\varphi}(\xi)$$

, where $\psi_0 = \varphi$ and m_ϵ is $2\pi\Gamma^*$ -periodic.

Theorem 2. The system $\{\psi_\epsilon(x - \gamma) : \gamma \in \Gamma, \epsilon \in E\}$ is an orthonormal basis of V_1 if and only if the matrix

$$U(\xi) = (m_\epsilon(\xi + \pi A^* \eta))_{(\epsilon, \eta) \in E^2}$$

is unitary for almost all $\xi \in \mathbb{R}^n$.

For $j \in \mathbb{Z}, \epsilon \in \tilde{E} \equiv E \setminus \{0\}$, and $\gamma \in \Gamma$,

$$(1.2) \quad \psi_{j, \epsilon, \gamma}(x) \equiv 2^{\frac{jn}{2}} \psi_\epsilon(2^jx - \gamma)$$

Define $W_{(j, \epsilon)} = \overline{\langle \psi_{(j, \epsilon, \gamma)} : \gamma \in \Gamma \rangle}$ and $W_j = \bigoplus_{\epsilon \in \tilde{E}} W_{(j, \epsilon)}$.

Then

$$(1.3) \quad L^2(\mathbb{R}^n) = V_0 \bigoplus_{k=0}^{\infty} W_k = \bigoplus_{k=-\infty}^{\infty} W_k$$

Definition 2. We call the system $\{\psi_{(j, \epsilon, \gamma)}; j \in \mathbb{Z}, \epsilon \in \tilde{E}, \gamma \in \Gamma\}$ wavelets basis of $L^2(\mathbb{R}^n)$, and $\{\psi_{(0, \epsilon, \gamma)}; \epsilon \in \tilde{E}, \gamma \in \Gamma\}$ mother wavelets.

In the next section, we define Γ -Periodic Wavelets and study an orthogonal decomposition of $L^2(\mathbb{R}^n/\Gamma)$.

TSUCHIDA 134-6 OKAYAMA, 703-8214, JAPAN

E-mail address: rosage@po.harenet.ne.jp

2. Gamma-PERIODIC WAVELETS AND $L^2(\mathbb{R}^n/\Gamma)$

In the following, let $\{V_j; j \in \mathbb{Z}\}$ be a MRALB with $A \in GL^+(n; \mathbb{R})$, $\{\varphi(x - \gamma); \gamma \in \Gamma\}$ be an orthonormal basis of V_0 , and $\int_{\mathbb{R}^n} \varphi(x) dx = \sqrt{\det(A)}$.

A function $f \in L^2(\mathbb{R}^n)$ is called Γ -periodic if

$$f(x) = f(x + \gamma) \quad \text{for } x \in \mathbb{R}^n, \gamma \in \Gamma.$$

Put

$$(1) \quad P_j = P_j(\gamma) = \{f \in V_j; f \text{ is } \Gamma\text{-periodic}, j \in \mathbb{Z}\}$$

Assume that

- (1) P_j 's are closed subspaces, and $P_j \subset P_{j+1}$;
- (2) $f(x) \in P_j$ if and only if $f(2x) \in P_{j+1}$.

Proposition 1. For $j \leq 0$, $\dim(P_j) = 1$, and for $j \geq 1$, $\dim(P_j) = 2^{nj}$.

Proof. Note that $\sum_{\gamma \in \Gamma} \varphi(x - \gamma) = \frac{1}{\sqrt{\det(A)}} \in P_0$.

For $j \leq 0$ let $f \in P_j$ be

$$f(x) = \sum_{\gamma \in \Gamma} c(\gamma) \varphi(x - \gamma)$$

Then, for any $\gamma \in \Gamma$, $c(\gamma) = c(f) = \text{const.}$ and $f(x) = \frac{c(f)}{\sqrt{\det(A)}}$.

For $j \geq 1$ let $f \in P_j$ then $g(x) = f(2^j x) \in P_0$.

Put $g(x) = \sum_{\gamma \in \Gamma} c(\gamma) \varphi(x - \gamma)$, then for any $\gamma, \gamma_0 \in \Gamma$,

$$c(\gamma + 2^j \gamma_0) = c(\gamma), \quad \text{thus } \dim(P_j) = 2^{nj}. \quad \square$$

For $j \in \mathbb{N}$,

$$(2) \quad \mathbb{Z}_{2^j}^n \equiv (\mathbb{Z} \bmod 2^j)^n$$

$$(3) \quad \Gamma^{(j)} \equiv \left\{ \frac{A k}{2^j}; k \in \mathbb{Z}_{2^j}^n \right\}$$

$\Gamma^{(j)}$ is a finite additive group of order 2^{nj} and $[\Gamma^{(j+1)} : \Gamma^{(j)}] = 2^n$.

For $f \in P_j$ and $\gamma \in \Gamma^{(j)}$, define

$$(\gamma f)(x) = f(x - \gamma).$$

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We see that the space P_j is of dimension 2^{nj} , $\Gamma^{(j)}$ - invariant closed subspace of $L^2(\mathbb{R}^n/\Gamma) = L^2(\Omega_\Gamma)$.

For $j \in \mathbb{N}$,

$$(4) \quad \varphi_j(x) \equiv \sum_{\gamma \in \Gamma} 2^{\frac{nj}{2}} \varphi(2^j(x - \gamma))$$

The function φ_j 's are Γ -periodic.

Theorem 1. *The system $\{\gamma\varphi_j; \gamma \in \Gamma^{(j)}\}$, is an orthonormal basis of the space P_j .*

Proof. Let $j = 0$, then $\varphi_0(x) = \frac{1}{\sqrt{\det(A)}}$ and

$$\int_{\Omega} |\varphi_0(x)|^2 dx = 1.$$

In general, since $\dim(P_j) = |\Gamma^{(j)}| = 2^{nj}$, it is sufficient to show that, for $\gamma_1, \gamma_2 \in \Gamma^{(j)}$,

$$\langle \gamma_1 \varphi_j, \gamma_2 \varphi_j \rangle \equiv \int_{\Omega} (\gamma_1 \varphi_j)(x) \overline{(\gamma_2 \varphi_j)(x)} dx = \delta(\gamma_1, \gamma_2).$$

Put $\gamma_l = \frac{A}{2^j} k_l, k_l \in \mathbb{Z}_{2^j}^n, l = 1, 2$, then

$$\begin{aligned} \langle \gamma_1 \varphi_j, \gamma_2 \varphi_j \rangle &= \int_{\Omega} \varphi_j(x - \frac{A}{2^j} k_1) \overline{\varphi_j(x - \frac{A}{2^j} k_2)} dx \\ &= \sum_{l, m \in \mathbb{Z}^n} 2^{nj} \int_{\Omega} \varphi(2^j[x - Al] - Ak_1) \overline{\varphi(2^j[x - Am] - Ak_2)} dx. \end{aligned}$$

Put $x - Al = 2^{-j}y, y \in \mathbb{R}^n$, then we have

$$\begin{aligned} \langle \gamma_1 \varphi_j, \gamma_2 \varphi_j \rangle &= \sum_{m \in \mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(y - Ak_1) \overline{\varphi(y - A[k_2 + 2^j m])} dy \\ &= \delta(k_1, k_2) = \delta(\gamma_1, \gamma_2). \end{aligned}$$

□

Let $T(\gamma)f = \gamma f, \gamma \in \Gamma^{(j)}, f \in P_j$, then $(T(\gamma), P_j)$ is a unitary representation of the group Γ^j .

Next, we consider the Fourier series expansion of the function φ_j and Poisson summation formula. Let

$$\varphi_j(x) = \sum_{l \in \mathbb{Z}^n} c(l) \exp(2\pi i A^* l \cdot x).$$

Then, we get

$$c(l) = \frac{1}{\det(A)} \int_{\Omega} \varphi_j(x) \exp(-2\pi i A^* l \cdot x) dx.$$

On the other hand,

$$\begin{aligned}
 & \int_{\Omega} \varphi_j(x) \exp(-2\pi i A^* l \cdot x) dx \\
 &= 2^{\frac{n_j}{2}} \int_{\mathbb{R}^n} \varphi(2^j x) \exp(-2\pi i A^* l \cdot x) dx \\
 &= 2^{-\frac{n_j}{2}} \hat{\varphi}\left(2\pi \frac{A^*}{2^j} l\right).
 \end{aligned}$$

Thus, we get

Proposition 2. *The Fourier series expansion of the function φ_j , $j \in \mathbb{N}$ is*

$$(5) \quad \varphi_j(x) = \frac{1}{2^{\frac{n_j}{2}} \det(A)} \sum_{\gamma^* \in \Gamma^*} \hat{\varphi}\left(\frac{2\pi}{2^j} \gamma^*\right) \exp(2\pi i \gamma^* \cdot x)$$

In particular, taking $x = 0$,

$$(6) \quad 2^{\frac{n_j}{2}} \sum_{\gamma \in \Gamma} \varphi(2^j \gamma) = \frac{1}{2^{\frac{n_j}{2}} \det(A)} \sum_{\gamma^* \in \Gamma^*} \hat{\varphi}\left(\frac{2\pi}{2^j} \gamma^*\right)$$

(Poisson Summation Formula)

Now define $\psi_{(0,0)}(x)$ as $\frac{1}{\sqrt{\det(A)}} (= \varphi_0(x))$ and

for $j \in \mathbb{N}$ and $\epsilon \in \tilde{E}$, define $\psi_{j,\epsilon}$ as

$$(7) \quad \psi_{j,\epsilon}(x) = \frac{n_j}{2} \sum_{\gamma \in \Gamma} \psi_{\epsilon}(2^j(x - \gamma))$$

Let $Q_j(\Gamma)$ be the orthogonal complement of $P_j(\Gamma)$ in $P_{j+1}(\Gamma)$, $\dim(Q_j(\Gamma)) = 2^{n_j}(2^n - 1)$.

Theorem 2. *For $j \in \mathbb{N}$ and $\epsilon \in \tilde{E}$, let $Q_{j,\epsilon}(\Gamma)$ be the closure of linear span of $\{\gamma \psi_{j,\epsilon} ; \gamma \in \Gamma^{(j)}\}$.*

Then, $\{\gamma \psi_{j,\epsilon} : \gamma \in \Gamma^{(j)}\}$ is an orthonormal basis of $Q_{j,\epsilon}(\Gamma)$ and

$$(8) \quad Q_j(\Gamma) = \bigoplus_{\epsilon \in \tilde{E}} Q_{j,\epsilon}(\Gamma)$$

$$(9) \quad P_{j+1}(\Gamma) = P_j(\Gamma) \bigoplus_{\epsilon \in \tilde{E}} Q_{j,\epsilon}(\Gamma)$$

$$(10) \quad L^2(\Omega_{\Gamma}) = P_0(\Gamma) \bigoplus_{j \in \mathbb{N}} \bigoplus_{\epsilon \in \tilde{E}} Q_{j,\epsilon}(\Gamma)$$

Proof. For $\epsilon_1 \neq \epsilon_2$, $\{\gamma \psi_{\epsilon_1}(x) ; \gamma \in \Gamma\}$ and $\{\gamma \psi_{\epsilon_2}(x) ; \gamma \in \Gamma\}$ are orthogonal, so that it is sufficient to prove for $Q_{j,\epsilon}(\Gamma)$.

The rest of the proof is done in the same way to Theorem 1 of this

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Definition 1. We call $\{\gamma\psi_{j,\epsilon} ; \gamma \in \Gamma^{(j)}, j \in \mathbb{N}, \epsilon \in \tilde{E}\}$
a Γ -periodic wavelets.

Proposition 3. We have the Fourier series expansion

$$(11) \quad \psi_{j,\epsilon}(x) = \frac{1}{2^{\frac{n_j}{2}} \det(A)} \sum_{\gamma^* \in \Gamma^*} m_\epsilon\left(\frac{\pi\gamma^*}{2^j}\right) \hat{\varphi}\left(\frac{\pi\gamma^*}{2^j}\right) \exp(2\pi i \gamma^* \cdot x)$$

and in particular, taking $x = 0$,

$$(12) \quad 2^{\frac{n_j}{2}} \sum_{\gamma \in \Gamma} \psi_\epsilon(2^j \gamma) = \frac{1}{2^{\frac{n_j}{2}} \det(A)} \sum_{\gamma^* \in \Gamma^*} m_\epsilon\left(\frac{\pi\gamma^*}{2^j}\right) \hat{\varphi}\left(\frac{\pi\gamma^*}{2^j}\right)$$

(Poisson Summation Formula)

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TSUCHIDA 134-6 OKAYAMA, 703-8214, JAPAN

E-mail address: rosage@po.harenet.ne.jp