

On (α, q) -Poisson Operators and Their Distributions

Nobuhiro ASAI (浅井暢宏)
 Department of Mathematics,
 Aichi University of Education,
 Kariya 448-8542, Japan.
 nasai[at]auecc.aichi-edu.ac.jp

Abstract

We propose an (α, q) -analogue of the Poisson operator on the Fock space of Bożejko-Ejsmont-Hasebe [6] and discuss a probability law of the operator. We show that the probability law is expressed by the q -Meixner distribution. Our results contain not only symmetric distributions as in [6], but also non-symmetric ones such as free Poisson, q and q^2 -deformations of Poisson, Pascal, Gamma, and Meixner distributions. This is the summary of the paper [4].

1 Introduction

A deformation of the full Fock space with two parameters $\alpha, q \in (-1, 1)$, namely, the (α, q) -Fock space (or the Fock space of type B) $\mathcal{F}_{\alpha, q}(\mathcal{H})$ over a complex Hilbert space $\mathcal{H}$ is considered by Bożejko-Ejsmont-Hasebe [6]. The $(0, q)$ -Fock space is the q -Fock space in the sense of [7][8]. Their starting point is to replace the Coxeter group of type A, that is, symmetric group $\mathfrak{S}_n$ for the q -Fock space by the Coxeter group of type B, $B(n) := \mathbb{Z}_2^n \rtimes \mathfrak{S}_n$, to construct $\mathcal{F}_{\alpha, q}(\mathcal{H})$ equipped with the (α, q) -inner product $\langle \cdot, \cdot \rangle_{\alpha, q}$. This replacement allows us to define, for $f \in \mathcal{H}$, the (α, q) -creation $B_{\alpha, q}^\dagger(f)$ and annihilation $B_{\alpha, q}(f)$ operators acting on $\mathcal{F}_{\alpha, q}(\mathcal{H})$ and leads us to the problem finding a probability distribution $\nu_{\alpha, q}$ on $\mathbb{R}$ of the (α, q) -Gaussian operator (the Gaussian operator of type B) with respect to the vacuum state.

We propose an (α, q) -analogue of the Poisson operator on $\mathcal{F}_{\alpha, q}(\mathcal{H})$ and discuss a probability law of this operator. We remark that the Poisson operator of type B by Ejsmont [11] is essentially different from ours if $\alpha \neq 0$. In Section 3, we introduce a weighted $(-q, q^2)$ -Poisson operator $\mathbf{Y}_{-q, q^2}$ after relationships between q -Meixner operator $\mathbf{X}_q$ of [17] and our (α, q^2) -Poisson operator are explained. We show in our main Theorem 3.1 that the probability law of $\mathbf{Y}_{-q, q^2}$ is equal to that of the scaled Meixner operator $\mathbf{Y}_q = \frac{\mathbf{X}_q}{1+q}$ with respect to appropriate vacuum states. As a result, one can treat not only symmetric distributions as in [6], but also non-symmetric ones such as free Poisson, q and q^2 -deformations of Poisson, Pascal, Gamma, and Meixner distribution. Consult our paper [4] in detail.

2 (α, q) -operators and their distributions

2.1 q -analogue of the Meixner class of orthogonal polynomials

Let $[n]_q$ denote the q -number given by $[n]_q := 1 + q + \cdots q^{n-1}$ for $n \geq 1$. For given constants $q, \kappa_1, \kappa_2, \gamma, \delta$ with $0 \leq q < 1$, $\kappa_2 > 0$, $\delta \geq 0$, let $\mathbf{m}_q$ denote the probability measure

$\mu(q; \kappa_1, \kappa_2, \gamma, \delta)$ on $\mathbb{R}$ such that the sequence of monic polynomials $\{Q_n^{(q)}(x)\}$ given by the recurrence relation,

$$\begin{cases} Q_0^{(q)}(x) = 1, & Q_1^{(q)}(x) = x - \kappa_1, \\ xQ_n^{(q)}(x) = Q_{n+1}^{(q)}(x) + (\kappa_2 + \delta[n-1]_q)[n]_q Q_{n-1}^{(q)}(x) + (\kappa_1 + \gamma[n]_q) Q_n^{(q)}(x), & n \geq 1. \end{cases} \quad (2.1)$$

is orthogonal with respect to the $L^2(\mathbf{m}_q)$ -inner product. We shall refer the measure $\mathbf{m}_q$ as the q -Meixner distribution. The existence of probability measure $\mathbf{m}_q$ is guaranteed by Favard's theorem, for example, in [10][12]. It is known [10] that the classical Meixner class of orthogonal polynomials and distributions ($q = 1$) can be classified into five types by parameters,

$$\begin{cases} \theta := \frac{\gamma}{\sqrt{\kappa_2}}, & \tau := \frac{\delta}{\kappa_2}, \\ D := \theta^2 - 4\tau. \end{cases}$$

A q -analogue of the classical case is discussed in [2] and characterized as well as $q = 1$ case by the same parameters [9][16]. See also [1][5][15] for the free case $q = 0$. More precisely, the q -Meixner distribution is classified into five types as follows:

- (i) q -Gaussian: $\tau = 0, \theta = 0$.
- (ii) q -Poisson: $\tau = 0, \theta \neq 0$.
- (iii) q -Pascal: $\tau > 0, D > 0$.
- (iv) q -Gamma: $\tau > 0, D = 0$.
- (v) q -Meixner : $D < 0$.

Remark 2.1. For $\alpha, q \in (-1, 1)$, let $\nu_{\alpha, q}$ be the orthogonalizing probability measure of the sequence of monic polynomials $\{P_n^{\alpha, q}(x)\}$ defined by the recurrence relation,

$$\begin{cases} P_0^{\alpha, q}(x) = 1, & P_1^{\alpha, q}(x) = x, \\ xP_n^{\alpha, q}(x) = P_{n+1}^{\alpha, q}(x) + (1 + \alpha q^{n-1})[n]_q P_{n-1}^{\alpha, q}(x), & n \geq 1. \end{cases} \quad (2.2)$$

The measure $\nu_{\alpha, q}$ is symmetric and its explicit expression can be found in [3][6][13]. Since the equality $1 + \alpha q^{n-1} = 1 + \alpha - \alpha(1 - q)[n-1]_q$ holds, $\{P_n^{\alpha, q}(x)\}$ for $\alpha \in (-1, 0]$ can be considered as a special case of $\{Q_n^{(q)}(x)\}$ in the sense of Definition 2.1. Hence the measure $\nu_{\alpha, q}$ for $\{P_n^{\alpha, q}(x)\}$ coincides with $\mu(q; 0, 1 + \alpha, 0, -\alpha(1 - q))$ for $\alpha \in (-1, 0]$.

2.2 (α, q) -Poisson operator on $\mathcal{F}_{\alpha, q}(\mathcal{H})$

To make our point clearer, we put basic facts on the (α, q) -creation $B_{\alpha, q}^\dagger$ and annihilation $B_{\alpha, q}$ operators acting on $\mathcal{F}_{\alpha, q}(\mathcal{H})$. One can see $B_{\alpha, q} = (B_{\alpha, q}^\dagger)^*$ with respect to the (α, q) -inner product $\langle \cdot, \cdot \rangle_{\alpha, q}$. For those who are not familiar with the (α, q) -deformation, refer Appendix A and also [6] written in detail.

Let us consider a self-adjoint operator $\mathbf{P}_{\alpha, q}(f)$ for $f \in \mathcal{H}$ defined by the form,

$$\mathbf{P}_{\alpha, q}(f) := B_{\alpha, q}^\dagger(f) + B_{\alpha, q}(f) + c_1 N_q(f) + c_2 \mathbf{1}, \quad c_1 \geq 0, c_2 \in \mathbb{R}$$

where N_q is the q -number operator having $\mathcal{H}^{\otimes n}$ as the eigenspace with eigenvalue $[n]_q$. Moreover, we find the probability distribution of this operator with respect to the vacuum state $\langle \Omega, \cdot \rangle_{\alpha, q}$. We call $\mathbf{P}_{\alpha, q}(f)$ the (α, q) -Poisson operator (the Poisson operator of type B). In the case $c_1 = 0$, $\mathbf{P}_{\alpha, q}(f)$ is denoted simply by $\mathbf{G}_{\alpha, q}(f)$, called the (α, q) -Gaussian operator (the Gaussian operator of type B).

Remark 2.2. (1) It is easy to see that the operators $B_{0,q}^\dagger$ and $B_{0,q}$ are the same as the q -creation operator $b_q^\dagger$ and q -annihilation operator b_q on the q -Fock space $\mathcal{F}_q(\mathcal{H}) := \mathcal{F}_{0,q}(\mathcal{H})$, respectively. That is, $b_q = (b_q^\dagger)^*$ with respect to the inner product $\langle \cdot, \cdot \rangle_q := \langle \cdot, \cdot \rangle_{0,q}$. (see [8]). In [14], the q -Poisson operator (the Poisson operator of type A) is examined as the sum of $b_q^\dagger, b_q$ and $b_q^\dagger b_q$ and its distribution is identified with the q -Poisson distribution with $\delta = 0$ ($\tau = 0$) of the Meixner's classification.

(2) In [11], Ejsmont introduced the Poisson operator of type B, but his operator is essentially different from ours if $\alpha \neq 0$.

Definition 2.3. For $s \in \mathbb{R}$, we define the translation T_s of a probability measure μ by $T_s\mu(\cdot) = \mu(\cdot - s)$. For $\lambda \in \mathbb{R}, \lambda \neq 0$, we define the dilation D_λ of μ by $D_\lambda\mu(\cdot) = \mu(\cdot/\lambda)$.

Theorem 2.4. Suppose $\alpha, q \in (-1, 1)$ and $f \in \mathcal{H}$ with $\|f\| = 1$. Let $\rho_{\alpha,q,f}$ be the probability distribution of $\mathbf{P}_{\alpha,q}(f)$ with respect to the vacuum state $\langle \Omega, \cdot \Omega \rangle_{\alpha,q}$.

(1) If $q \in (-1, 1)$ and $-1 < \alpha\langle f, \bar{f} \rangle \leq 0$, then $\rho_{\alpha,q,f}$ is

$$\mu(q; c_2, 1 + \alpha\langle f, \bar{f} \rangle, c_1, -\alpha(1 - q)\langle f, \bar{f} \rangle).$$

(2) If $c_1 = 0$, $q \in (-1, 1)$ and $-1 < \alpha\langle f, \bar{f} \rangle < 1$, then $\rho_{\alpha,q,f}$ is equal to $T_{c_2}\nu_{\alpha\langle f, \bar{f} \rangle, q}$, where it is the probability distribution of $\mathbf{G}_{\alpha,q}(f) - c_2\mathbf{1}$.

2.3 q -Meixner operator on $\mathcal{F}_q(\mathcal{H})$

Let us recall the double q -creation and annihilation operators, $(b_q^\dagger)^2$ and $(b_q)^2$ respectively, acting on the q -Fock space $\mathcal{F}_q(\mathcal{H}) := \mathcal{F}_{0,q}(\mathcal{H})$ for $f \in \mathcal{H}$, defined by

$$\begin{cases} (b_q^\dagger)^2 f^{\otimes 2n} = f^{\otimes 2(n+1)}, & n \geq 0, \\ (b_q)^2 f^{\otimes 2n} = [2n]_q [2n-1]_q f^{\otimes 2(n-1)}, & n \geq 1, \\ b_q^\dagger b_q f^{\otimes 2n} = [2n]_q f^{\otimes 2n}, & n \geq 1. \end{cases} \quad (2.3)$$

Yoshida [16] considered a self-adjoint operator $\mathbf{X}_q(c_3, c_4)$ on $\mathcal{F}_q(\mathcal{H})$ given by

$$\mathbf{X}_q(c_3, c_4) = (b_q^\dagger)^2 + (b_q)^2 + c_3 b_q^\dagger b_q + c_4 \mathbf{1}, \quad c_3 \geq 0, c_4 \in \mathbb{R},$$

and the probability distribution of this operator denoted by $\mu_{\mathbf{X}_q}$ with respect to the vacuum state $\langle \Omega, \cdot \Omega \rangle_q$. In this paper, $\mathbf{X}_q$ is called the q -Meixner operator acting on $\mathcal{F}_q(\mathcal{H})$.

It is our main concern in this section to clarify the relationship between probability distributions of $\mathbf{X}_q$ and $\mathbf{P}_{\alpha,q}$ in a sense. Let us first recall the following Theorem.

Theorem 2.5. The probability distribution $\mu_{\mathbf{X}_q}$ of the operator $\mathbf{X}_q(c_3, c_4)$ with respect to the vacuum state $\langle \Omega, \cdot \Omega \rangle_q$ is given as follows:

(1) If $c_3 > 0$, then

$$\mu_{\mathbf{X}_q} = \mu(q^2; c_4, 1 + q, c_3(1 + q), q(1 + q)^2),$$

for $q \in [0, 1)$.

(2) If $c_3 = 0$, then $\mu_{\mathbf{X}_q} = T_{c_4}\nu_{-q,q^2}$ for $q \in (-1, 1)$.

Proof. Due to the equality,

$$(1 + \alpha q^{2(n-1)})[n]_{q^2} = \frac{1}{q(1+q)} (\alpha + q - \alpha(1 - q)[2n - 1]_q) [2n]_q,$$

$\alpha = -q$ implies

$$[2n - 1]_q [2n]_q = (1 + q + q(1 + q)^2 [n - 1]_{q^2}) [n]_{q^2}. \quad (2.4)$$

Due to this identity, we get our assertion. $\square$

3 Relationship between (α, q^2) -Poisson and q -Meixner operators

3.1 The case $\alpha = -q$

To see relationships between the operator $\mathbf{X}_q$ and with $(-q, q^2)$ -operator, we shall define a scaled operator $\mathbf{Y}_q$ of $\mathbf{X}_q$ for $0 \leq q < 1$,

$$\mathbf{Y}_q := \frac{1}{1+q} \mathbf{X}_q, \quad (3.1)$$

and a weighted Poisson type operator $\mathbf{Y}_{-q, q^2}$ defined by

$$\mathbf{Y}_{-q, q^2} := \frac{1}{1+q} \left\{ B_{-q, q^2}^\dagger + \frac{1+q}{1-q} B_{-q, q^2} + c_1(1+q)N_{q^2} + c_2 \mathbf{1} \right\}. \quad (3.2)$$

We remark here that if $c_1 = c_3 = 0$, the condition on q can be relaxed to $q \in (-1, 1)$.

Since $\mathbf{Y}_{-q, q^2}$ is not self-adjoint with respect to $\langle \cdot, \cdot \rangle_{-q, q^2}$ due to the second term in RHS of (3.2), which is a counterpart of $(b_q)^2$ in (3.1), we need to modify (α, q) -creation and annihilation operators by adding a weight $\beta > 0$ as follows:

Let $B_{\beta, \alpha, q}^\dagger(f)$ be the β -weighted (α, q) -creation defined as the (α, q) -creation operator $B_{\alpha, q}^\dagger(f)$ and $B_{\beta, \alpha, q}(f)$ be the β -weighted (α, q) -annihilation operator given by

$$B_{\beta, \alpha, q}(f) := \beta B_{\alpha, q}(f), \quad \beta > 0.$$

The above two β -weighted operators are adjoint each other with respect to the β -weighted (α, q) -inner product,

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\beta, \alpha, q} := \delta_{m, n} \beta^n \langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\alpha, q}, \quad f_k, g_k \in \mathcal{H}.$$

By setting $\beta = \frac{1+q}{1-q}$, the operator $\mathbf{Y}_{-q, q^2}$ can be expressed as

$$\mathbf{Y}_{-q, q^2} = \frac{1}{1+q} \left\{ B_{\beta, -q, q^2}^\dagger + B_{\beta, -q, q^2} + c_1(1+q)N_{q^2} + c_2 \mathbf{1} \right\}$$

and hence $\mathbf{Y}_{-q, q^2}$ is the self-adjoint operator with respect to the inner product $\langle \cdot, \cdot \rangle_{\beta, -q, q^2}$.

Then we can clarify the relationship between probability distributions of $\mathbf{Y}_q$ and $\mathbf{Y}_{-q, q^2}$ with respect to the vacuum state.

Theorem 3.1. *Suppose $c_1 = c_3$ and $c_2 = c_4$. Then the probability law of $\mathbf{Y}_q$ with respect to $\langle \Omega, \cdot \rangle_q$ is equal to that of $\mathbf{Y}_{-q, q^2}$ with respect to $\langle \Omega, \cdot \rangle_{\beta, -q, q^2}$ with $\beta = \frac{1+q}{1-q}$. In fact, the probability distribution $\rho_{\mathbf{Y}}$ of these operators is given as follows:*

(1) If $c_1 > 0$, then $\rho_{\mathbf{Y}}$ is

$$D_a \mu_{\mathbf{x}_q} = \mu \left(q^2; \frac{c_2}{1+q}, \frac{1}{1+q}, c_1, q \right), \quad a = \frac{1}{1+q}$$

for $q \in [0, 1)$.

(2) If $c_1 = 0$, then $\rho_{\mathbf{Y}}$ is $D_a T_{c_2} \nu_{-q, q^2}$ for $q \in (-1, 1)$.

Moreover, the classification parameters θ and τ under $c_1 = c_3$ for the Meixner class are given by

$$\begin{cases} \theta = c_1 \sqrt{1+q}, \\ \tau = q(1+q) \geq 0, \\ D = (1+q)(c_1^2 - 4q). \end{cases} \quad (3.3)$$

Therefore, we get

Proposition 3.2. (1) If $q = 0$ ($\tau = 0$) and $c_1 = c_3 = 0$ ($\theta = 0$), then $\mu_{\mathbf{x}_0} = T_{c_2}\nu_{0,0}$ and $D_a\mu_{\mathbf{x}_0} = D_aT_{c_2}\nu_{0,0}$ are the free Gaussian.
 (2) If $q = 0$ ($\tau = 0$) and $c_1 = c_3 \neq 0$ ($\theta \neq 0$), then $\mu_{\mathbf{x}_0}$ and $D_a\mu_{\mathbf{x}_0}$ are the free Poisson.
 (3) If $0 < q < 1$ ($\tau > 0$) and $c_1 = c_3 > 2\sqrt{q}$ ($D > 0$), then $\mu_{\mathbf{x}_q}$ and $D_a\mu_{\mathbf{x}_q}$ are the q^2 -Pascal.
 (4) If $0 < q < 1$ ($\tau > 0$) and $c_1 = c_3 = 2\sqrt{q}$ ($D = 0$), then $\mu_{\mathbf{x}_q}$ and $D_a\mu_{\mathbf{x}_q}$ are the q^2 -Gamma.
 (5) If $0 < q < 1$ ($\tau > 0$) and $0 \neq c_1 = c_3 < 2\sqrt{q}$ ($D < 0$), then $\mu_{\mathbf{x}_q}$ and $D_a\mu_{\mathbf{x}_q}$ are the q^2 -Meixner.

We have shown by introducing the (α, q^2) -Poisson and the q -Meixner operators that non-symmetric probability distributions such as (2)(3)(4)(5) in Proposition 3.2 can be treated within the framework of the Fock space of type B. In [6], non-symmetric cases are not treated.

3.2 The case $\alpha = q$

Due to Proposition 3.2, $\mathbf{X}_q$ and $\mathbf{Y}_{-q,q^2}$ do not provide the q^2 -Gaussian and q^2 -Poisson laws. In particular, the $\mathbf{Y}_{q,q}$ -operator given by

$$\mathbf{Y}_{q,q} := a\mathbf{P}_{q,q}, \quad a = \frac{1}{1+q}$$

has these probability laws for $q \in (-1, 1)$. It is the self-adjoint operator with respect to the inner product $\langle \cdot, \cdot \rangle_{1+q,0,q^2}$. It is easy to see that if $c_1 = 0$, then $\mathbf{Y}_{q,q} = a(\mathbf{G}_{q,q} + c_2\mathbf{1})$. Hence we have

Proposition 3.3. For $q \in (-1, 1)$, the probability law of $\mathbf{Y}_{q,q}$ with respect to $\langle \Omega, \cdot \Omega \rangle_{1+q,0,q^2}$ is as follows;

- (1) $c_1 = 0 \implies$ the q^2 -Gaussian, $D_aT_{c_2}\nu_{0,q^2}$.
 (2) $c_1 > 0 \implies$ the q^2 -Poisson, $\mu\left(q^2; \frac{c_2}{1+q}, \frac{1}{1+q}, c_1, 0\right)$.

Acknowledgments

This work was supported by Grant-in-Aid for Scientific Research (C), No.16K05175, JSPS. The author would like to thank the organizer, Professor F. Hiroshima, for his kind invitation to RIMS workshop on Mathematical Aspects of Quantum Field Theory and Related Topics.

A Appendix

This appendix is referred from [4]. Let $B(n)$ be the set of bijections σ of the $2n$ points $\{\pm 1, \pm 2, \dots, \pm n\}$ with $\sigma(-k) = -\sigma(k)$. Equipped with the composition operation as a product, $B(n)$ becomes what is called a Coxeter group of type B. It is generated by $\pi_0 := (1, -1)$ and $\pi_i := (i, i+1)$, $1 \leq i \leq n-1$, which satisfy the generalized braid relations

$$\begin{cases} \pi_i^2 = e, & 0 \leq i \leq n-1, \\ (\pi_0\pi_{n-1})^4 = (\pi_i\pi_{i+1})^3 = e, & 1 \leq i \leq n-1, \\ (\pi_i\pi_j)^2 = e, & |i-j| \geq 2, 0 \leq i, j \leq n-1. \end{cases} \quad (\text{A.1})$$

An element $\sigma \in B(n)$ expresses an irreducible form,

$$\sigma = \pi_{i_1} \cdots \pi_{i_k}, \quad 0 \leq i_1, \dots, i_k \leq n-1,$$

and in this case

$$\begin{aligned}\ell_1(\sigma) &:= \text{the number of } \pi_0 \text{ in } \sigma, \\ \ell_2(\sigma) &:= \text{the number of } \pi_i, \ 1 \leq i \leq n-1, \text{ in } \sigma\end{aligned}$$

are well defined.

Let $\mathcal{H}$ be a complex Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, where the inner product is linear on the right and conjugate linear on the left. For a given self-adjoint involution $f \mapsto \bar{f}$ for $f \in \mathcal{H}$, an action of $B(n)$ on $\mathcal{H}^{\otimes n}$ is defined by

$$\begin{cases} \pi_0(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes f_2 \otimes \cdots \otimes \bar{f}_n, & n \geq 1, \\ \pi_i(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \otimes \cdots \otimes f_n, & n \geq 2, \ 1 \leq i \leq n-1. \end{cases}$$

Throughout this paper, we assume the involution $\bar{f}$ of $f \in \mathcal{H}$ is defined in such a way that $\langle f, \bar{f} \rangle \in \mathbb{R}$ holds and $\langle f, \bar{f} \rangle = 0$ is equivalent to $f = 0$.

Let $\mathcal{F}_{\text{fin}}(\mathcal{H})$ denote the algebraic full Fock space over $\mathcal{H}$,

$$\mathcal{F}_{\text{fin}}(\mathcal{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n},$$

where Ω denotes the vacuum vector. We note that elements of $\mathcal{F}_{\text{fin}}(\mathcal{H})$ are expressed as finite linear combinations of the elementary vectors $f_1 \otimes \cdots \otimes f_n \in \mathcal{H}^{\otimes n}$. We equip $\mathcal{F}_{\text{fin}}(\mathcal{H})$ with the inner product

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{0,0} := \delta_{m,n} \prod_{k=1}^n \langle f_k, g_k \rangle, \quad f_k, g_k \in \mathcal{H}.$$

For $\alpha, q \in (-1, 1)$, define the symmetrization operator of type B on $\mathcal{H}^{\otimes n}$ as

$$\begin{aligned}P_{\alpha,q}^{(n)} &= \sum_{\sigma \in B(n)} \alpha^{\ell_1(\sigma)} q^{\ell_2(\sigma)} \sigma, \quad n \geq 1, \\ P_{0,q}^{(n)} &= \sum_{\sigma \in \mathfrak{S}_n} q^{\ell_2(\sigma)} \sigma, \quad n \geq 1, \\ P_{\alpha,q}^{(0)} &= I_{\mathcal{H}^{\otimes 0}}, \quad P_{0,0}^{(n)} = I_{\mathcal{H}^{\otimes n}},\end{aligned}$$

where we put $0^0 = 1$ and $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ by convention and

$$P_{\alpha,q} = \bigoplus_{n=0}^{\infty} P_{\alpha,q}^{(n)}$$

be the symmetrization operator of type B on $\mathcal{F}_{\text{fin}}(\mathcal{H})$. Since $P_{\alpha,q}^{(n)}$ is known to be strictly positive,

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\alpha,q} := \langle f_1 \otimes \cdots \otimes f_m, P_{\alpha,q}(g_1 \otimes \cdots \otimes g_n) \rangle_{0,0}$$

becomes an inner product and $\langle \cdot, \cdot \rangle_{\alpha,q}$ is called the (α, q) -inner product with the convention $0^0 = 1$ and $g_{-k} = \overline{g_k}$, $k = 1, 2, \dots, n$.

Definition A.1. (1) For $\alpha, q \in (-1, 1)$, the (algebraic) full Fock space $\mathcal{F}_{\text{fin}}(\mathcal{H})$ with respect to $\langle \cdot, \cdot \rangle_{\alpha, q}$ is called the (α, q) -Fock space (the Fock space of type B) denoted by $\mathcal{F}_{\alpha, q}(\mathcal{H})$. In this paper, we do not take completion. In particular, $\mathcal{F}_{0, q}(\mathcal{H})$ is nothing but the q -Fock space (the Fock space of type A) $\mathcal{F}_q(\mathcal{H})$ equipped with the q -inner product $\langle \cdot, \cdot \rangle_q := \langle \cdot, \cdot \rangle_{0, q}$ of Bożejko-Speicher [8].

(2) Let $B_{\alpha, q}^\dagger(f)$ be defined as the usual left creation operator,

$$\begin{aligned} B_{\alpha, q}^\dagger(f)\Omega &= f, \\ B_{\alpha, q}^\dagger(f)(f_1 \otimes \cdots \otimes f_n) &= f \otimes f_1 \otimes \cdots \otimes f_n, \quad n \geq 1 \end{aligned}$$

and $B_{\alpha, q}(f)$ be its adjoint with respect to $\langle \cdot, \cdot \rangle_{\alpha, q}$, that is, $B_{\alpha, q} = (B_{\alpha, q}^\dagger)^*$. $B_{\alpha, q}^\dagger$ and $B_{\alpha, q}$ are called the (α, q) -creation and (α, q) -annihilation operators, respectively.

The following proposition is a direct consequence of the definition.

Proposition A.2. (1) *The (α, q) -annihilation operator $B_{\alpha, q}$ acts on the elementary vectors as follows:*

$$\begin{aligned} B_{\alpha, q}(f)\Omega &= 0, \quad B_{\alpha, q}(f)f_1 = \langle f, f_1 \rangle \Omega, \\ B_{\alpha, q}(f)(f_1 \otimes \cdots \otimes f_n) &= L + R \end{aligned}$$

where

$$\begin{aligned} L &= \sum_{k=1}^n q^{k-1} \langle f, f_k \rangle f_1 \otimes \cdots \otimes \overset{\vee}{f}_k \otimes \cdots \otimes f_n, \\ R &= \alpha q^{n-1} \sum_{k=1}^n q^{k-1} \langle f, \bar{f}_{n-(k-1)} \rangle f_1 \otimes \cdots \otimes \overset{\vee}{f}_{n-(k-1)} \otimes \cdots \otimes f_n, \end{aligned}$$

for $n \geq 2$ where $\overset{\vee}{f}_k$ means that f_k should be deleted from tensor product.

(2) *The (α, q) -creation and the (α, q) -annihilation operators satisfy the commutation relation,*

$$B_{\alpha, q}(f)B_{\alpha, q}^\dagger(g) - qB_{\alpha, q}^\dagger(g)B_{\alpha, q}(f) = \langle f, g \rangle I + \alpha \langle f, \bar{g} \rangle q^{2N}, \quad f, g \in \mathcal{H}.$$

The readers can refer to [6] for details.

Corollary A.3. (1) *The q -annihilation operator $b_q(x)$ acts on the elementary vectors as follows:*

$$\begin{aligned} b_q(f)\Omega &= 0, \quad b_q(f)f_1 = \langle f, f_1 \rangle \Omega, \\ b_q(f)(f_1 \otimes \cdots \otimes f_n) &= \sum_{k=1}^n q^{k-1} \langle f, f_k \rangle f_1 \otimes \cdots \otimes \overset{\vee}{f}_k \otimes \cdots \otimes f_n \quad n \geq 2, \end{aligned}$$

where $\overset{\vee}{f}_k$ means that f_k should be deleted from tensor product.

(2) *The q -creation and the q -annihilation operators satisfy the q -commutation relation (q -CCR)*

$$b_q(f)b_q^\dagger(g) - qb_q^\dagger(g)b_q(f) = \langle f, g \rangle \mathbf{1} \quad f, g \in \mathcal{H}.$$

References

- [1] M. Anshelevich, Free martingale polynomials, *J. Funct. Anal.*, **201**, (2003), 228–261.
- [2] M. Anshelevich, Appell polynomials and their relatives, *Int. Math. Res. Not.*, **65**, (2004), 3469–3531.
- [3] N. Asai, M. Bożejko, and T. Hasebe, Radial Bargmann representation for the Fock space of type B, *J. Math. Phys.*, **57**, (2016), 021702 (13 pages).
- [4] N. Asai, H. Yoshida, Poisson type operators on the Fock space of type B, *J. Math. Phys.*, **60**, (2019), 011702 (9 pages).
- [5] M. Bożejko and W. Bryc, On a class of free Levy laws related to a regression problem, *J. Funct. Anal.*, **236**, no. 1, (2006), 59–77.
- [6] M. Bożejko, W. Ejsmont, and T. Hasebe, Fock space associated with Coxeter groups of type B, *J. Funct. Anal.*, **269**, (2015), 1769–1795.
- [7] M. Bożejko, B. Kümmerer, and R. Speicher, q -Gaussian processes: Non-Commutative and classical aspects, *Comm. Math. Phys.*, **185**, (1997), 129–154.
- [8] M. Bożejko and R. Speicher, An example of a generalized Brownian motion, *Comm. Math. Phys.*, **137**, (1991), 519–531.
- [9] W. Bryc and J. Wesołowski, Conditional moments of q -Meixner processes, *Probab. Theory Rel. Fields*, **131**, (2005), 415–441.
- [10] T.S. Chihara, *An Introduction to Orthogonal Polynomials*. Gordon and Breach, 1978.
- [11] W. Ejsmont, Partitions and deformed cumulants of type B with remarks on the Blitvić model, arXiv:1811.02675.
- [12] A. Hora and N. Obata, *Quantum probability and spectral analysis of graphs*. Springer-Verlag, Berlin, 2007.
- [13] R. Koekoek, P.A. Lesky and R.F. Swarttouw, *Hypergeometric orthogonal polynomials and their q -analogues*, Springer-Verlag, Berlin, 2010.
- [14] N. Saitoh and H. Yoshida, q -deformed Poisson random variables on q -Fock space, *J. Math. Phys.*, **41**, No. 8, (2000), 5767–5772.
- [15] N. Saitoh and H. Yoshida, The infinite divisibility and orthogonal polynomials with a constant recursion formula in free probability theory, *Probab. Math. Statist.*, **21**, No. 1, (2001), 159–170.
- [16] H. Yoshida, The q -Meixner distributions associated with a q -deformed symmetric Fock space, *J. Phys. A, Math. Theor.*, **44**, (2011), 165306 (8 pages).
- [17] H. Yoshida, The q -Meixner self-adjoint operators on the q -deformed Fock space, *RIMS Kôkyûroku*, **1819**, (2012), 183–192.